Completion of Linear Differential Systems to Involution*

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Abstract. In this paper we generalize the involutive methods and algorithms have been devised for polynomial ideals to differential ones generated by a finite set of linear differential polynomials in the differential polynomial ring over a zero characteristic differential field. Given a ranking of derivative terms and an involutive division, we formulate the involutivity conditions which form a basis of involutive algorithms. We present an algorithm for computation of a minimal involutive differential basis. Its correctness and termination hold for any constructive and noetherian involutive division. As two important applications we consider posing of an initial value problem for a linear differential system providing uniqueness of its solution and Lie symmetry analysis of nonlinear differential equations. In particular, this allows to determine the structure of arbitrariness in general solution of linear systems and thereby to find the size of symmetry group.

1 Introduction

Among the properties of systems of analytical partial differential equations (PDEs) which may be investigated without their explicit integration there are compatibility and formulation of an *initial-value problem* providing existence and uniqueness of the solution. The classical Cauchy-Kowalevsky theorem establishes a certain class of quasilinear PDEs which admit posing such an initial-value problem. The main obstacle in investigating other classes of PDE systems of some given order q is existence of *integrability conditions*, that is, such relations for derivatives of order $\leq q$ which are differential but not pure algebraic consequences of equations in the system.

An *involutive* system of PDEs has all the integrability conditions incorporated in it. This means that prolongations of the system do not reveal integrability conditions. Extension of a system by its integrability conditions is called *completion*. The concept of involutivity was invented hundred years ago by E.Cartan [1] in his investigation of the Pfaff type equations in total differentials. For these purposes he used the exterior calculus developed by himself. The Cartan approach was generalized by Kähler[2] to arbitrary systems of exterior differential equations. The underlying completion procedure [3] was implemented in [4,5].

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