Chapter 11

Mellin transforms and asymptotics: Harmonic sums

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This survey presents a unified and essentially self-contained approach to the asymptotic analysis of a large class of sums that arise in combinatorial mathematics, discrete probabilistic models, and the average-case analysis of algorithms. It relies on the Mellin transform, a close relative of the integral transforms of Laplace and Fourier. The method applies to harmonic sums that are superpositions of rather arbitrary "harmonics" of a common base function. Its principle is a precise correspondence between individual terms in the asymptotic expansion of an original function and singularities of the transformed function. Here no theorem is proved, and even not every theorem is completely formulated. For precise presentation of the theory reader is refered to the original paper.

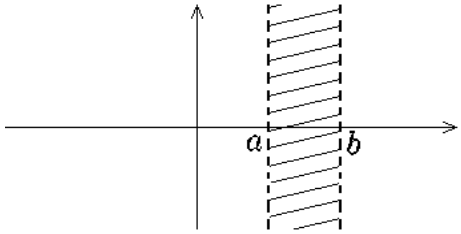
We have to deal a lot with complex variable functions and I'll remind you some basic concepts about them. The first concept about complex variable functions is holomorphic function. Function is called holomorphic in some area, if it has complex derivative in every point of this area.

The second concept is 'analitic function'. Function is analitic in some point z_0 of complex plane, if it can be expanded into Taylor series in this point, i.e. $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$. Similary, the function is called analitic in an area, if it is analitic in every point of that area. One of the central result of the complex variable function theory is the theorem, that every holomorphic in some area function is analitic there. The converse statement holds too. We'll use the word 'analitic' a lot.

Except holomorphic functions there are meromorphic functions. Meromorphic in an area function is a function, that is holomorphic there except discrete set of points that are called poles. Discrete set means a set, every point of which can be isolated from other points. For example, every finite set is discrete.

Consider a function $f(z) = \frac{1}{z(z-1)}$. It is analitic (and therefore holomorphic) in $\mathbb{C} \setminus \{0, 1\}$, but it is meromorphic in entire $\mathbb{C}$ with poles $z = 0$ and $z = 1$.

The last concept is 'open strip'. Open strip $\langle a, b\rangle = \{z = x + iy \mid a < y < b\}$ is a set of points in a complex plane that looks like:



Open strip can be infinite, if a or b is infinity, and it's obvious that $\langle -\infty, \infty \rangle = \mathbb{C}$

11.1 Mellin transform definition

Robert Hjalmar Mellin (1854-1933) was Finnish mathematitian who studied the transform which now bears his name and established its reciprocal properties. Now we are finally going to learn what Mellin transform is.

Definition 11.1. Let $f(x)$ be real function defined on $(0, +\infty)$. Then its Mellin transform is complex valued function that is defined by equality

$$\mathfrak{M}\left[f(x); s\right] = f^*(s) = \int_0^{+\infty} f(x)x^{s-1}dx$$

Of course, integral from definition usually converges not for all $s \in \mathbb{C}$, but it usually converges for all s from some open strip, which in this case is called 'fundamental' strip.

Proposition 11.1. *If $f(x) = \mathrm{O}(x^u)$ as $x \to 0$ and $f(x) = \mathrm{O}(x^v)$ as $x \to +\infty$, then the integral from Mellin transform definition converges for every $s \in \langle -u, -v \rangle$ and defines an analitic function in this strip.*

Example 11.1. If $f(x) = x^k$, then $f(x) = \mathrm{O}(x^k)$ as $x \to 0$ and $f(x) = \mathrm{O}(x^k)$ as $x \to +\infty$. Proposition states that transform of $f(x)$ (that is $f^*(s)$) exists in the open strip $\langle -k, -k \rangle$, but this strip is empty. In fact, transform of x^k simply doesn't exist, i.e. $\int_0^\infty x^k x^{s-1}dx$ doesn't converge for every $k \in \mathbb{R}$ and $s \in \mathbb{C}$. One can simply check it.

Now I present examples of functions that do have Mellin transforms.

Example 11.2. Let $f(x) = \frac{1}{1+x}$. $f(x) = \mathrm{O}(1) = \mathrm{O}(x^0)$ as $x \to 0$, and $f(x) = \mathrm{O}(x^{-1})$ as $x \to +\infty$. Now we can make use of proposition 11.1, here $u = 0$ and $v = -1$. Proposition states, that in this case Mellin transform $f^*(s) = \int_0^{+\infty} \frac{1}{1+x}x^{s-1}dx$ exists in the fundamental strip $\langle 0, 1 \rangle$. The integral can be evaluated and it occurs, that $f^*(s) = \frac{\pi}{\sin \pi s}$. But the equality holds only for $s \in \langle 0, 1 \rangle$, for other s integral simply doesn't converge. By the way, function $\frac{\pi}{\sin \pi s}$ by itself can be evaluated practically in entire $\mathbb{C}$, except, may be, integer points.

Example 11.3 (Gamma function). Now we consider the function $f(x) = e^{-x}$. $f(x) = \mathrm{O}(1) = \mathrm{O}(x^0)$ as $x \to 0$, and $\forall M{>}0$ $f(x) = \mathrm{O}(x^{-M})$ as $x \to +\infty$. Again, after using the proposition 11.1 we obtain, that Mellin transform of $f(x)$ exists in the open strip $\langle 0, M \rangle$ for every $M > 0$. It means, that the fundamental strip of this Mellin transform is $\langle 0, +\infty \rangle$. Now let's evaluate the transform. $f^*(s) = \int_0^{+\infty} e^{-x}x^{s-1}dx$. This integral is called Gamma function and notation is $\Gamma(s)$. There is a well known functional equation on gamma function which states that $s\Gamma(s) = \Gamma(s+1)$. This equation allows us to evalute gamma function not only in the right half of complex plane, but also in every other point of complex plane. For example, $(-\frac{1}{2})\Gamma(-\frac{1}{2}) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$, so $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$. The problem is only with nonpositive integers.

Statement $0\Gamma(0) = \Gamma(1)$ doesn't allow us to evaluate gamma function in zero, it demonstrates, that there is a pole of gamma function in zero. Every negative integer is a pole of gamma function for the same reason.

Gamma function occurs frequently in Mellin transforms. But now we look on the last example of Mellin transform before going to learn basic properties of Mellin transform.

Example 11.4 (Transform of step function).

$$H(x) = \begin{cases} 1, x \in (0,1) \\ 0, x \in (1,+\infty) \end{cases}$$

As in the previous example and for the same reasons, fundamental strip of the transformed function is $\langle 0,+\infty\rangle$. Here the transform can be simply evaluated. $H^*(s) = \int_0^{+\infty} H(x)x^{s-1}dx = \int_0^1 x^{s-1}dx = \frac{1}{s}$. As in the all previous examples, we see that transformed function is defined not only in the fundamental strip. Here fundamental strip is $\langle 0,+\infty\rangle$, but $\frac{1}{s}$ may be evaluated in $\mathbb{C} \setminus \{0\}$. Fundamental strip is only the place where an integral converges.
If we had considered another step function $\overline{H}(x) = 1 - H(x)$, we would have obtained the transform $\overline{H}^*(s) = -\frac{1}{s}$ with fundamental strip $\langle -\infty, 0\rangle$.

11.2 Mellin transform basic properties

All basic properties of Mellin transform can be simply obtained by means of such methods as integration by parts and change of variable. Here they are summarized in a table.

	$f(x)$	$f^*(s)$	$\langle\alpha,\beta\rangle$	
(1)	$x^\nu f(x)$	$f^*(s+\nu)$	$\langle\alpha-\nu,\beta-\nu\rangle$	
(2)	$f(x^\rho)$	$\frac{1}{\rho}f^*(\frac{s}{\rho})$	$\langle\rho\alpha,\rho\beta\rangle$	$\rho > 0$
(3)	$f(\frac{1}{x})$	$-f^*(-s)$	$\langle-\beta,-\alpha\rangle$	
(4)	$f(\mu x)$	$\frac{1}{\mu^s}f^*(s)$	$\langle\alpha,\beta\rangle$	$\mu > 0$
(5)	$\sum_k \lambda_k f(\mu_k x)$	$\left(\sum_k \frac{\lambda_k}{\mu^s}\right) f^*(s)$		
(6)	$f(x)\log x$	$\frac{d}{ds}f^*(s)$	$\langle\alpha,\beta\rangle$	
(7)	$\Theta f(x)$	$-sf^*(s)$	$\langle\alpha',\beta'\rangle$	$\Theta = x\frac{d}{dx}$
(8)	$\frac{d}{dx}f(x)$	$-(s-1)f^*(s-1)$	$\langle\alpha'+1,\beta'+1\rangle$	
(9)	$\int_0^x f(t)dt$	$-\frac{1}{s}f^*(s+1)$		

The most interesting are the fourth and the fifth properties. The fourth one is called 'separation property' and the fifth property is its generalisation. If the sum $\sum_k \lambda_k f(\mu_k x)$ is finite, fifth property is obvious because of linearity of Mellin transform. But if the sum is infinite, the fifth property holds only if function $f(x)$ and series $\sum_k \frac{\lambda_k}{\mu^s}$ satisfy some additional conditions. Anyway, we'll use this property in this paper for infinite sums without paing attention to the problem. All studied functions are good enough and fifth property holds for them.

Example 11.5 (Zeta function). Here we'll use the fifth property to introduce zeta function. Consider the function

$$g(x) = \frac{e^{-x}}{1-e^{-x}} = e^{-x} + e^{-2x} + e^{-3x} + \cdots$$

The series converges for every $x > 0$. Now we apply the fifth property. $\lambda_k = 1$, $\mu_k = k$ and $f(x) = e^{-x}$, so

$$g^*(s) = \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) \mathfrak{M}\left[e^{-x}; s\right] = \zeta(s)\Gamma(s)$$

Series $\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$ converges for every $s \in \langle 1, +\infty\rangle$. Fundamental strip of the transform is $\langle 1, +\infty\rangle$ too, and why it is so we'll discuss later.

Now we'll summarize in a table a number of Mellin transforms, some of them were obtained earlier, some of them can be obtained by means of Mellin transform basic properties.

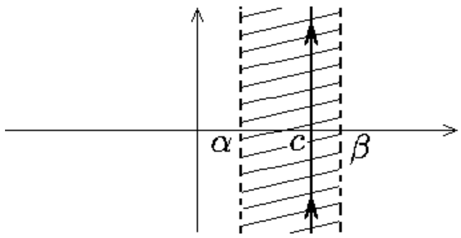
$f(x)$	$f^*(s)$	$\langle\alpha, \beta\rangle$
e^{-x}	$\Gamma(s)$	$\langle 0, +\infty\rangle$
$e^{-x} - 1$	$\Gamma(s)$	$\langle -1, 0\rangle$
e^{-x^2}	$\frac{1}{2}\Gamma(\frac{1}{2}s)$	$\langle 0, +\infty\rangle$
$\frac{e^{-x}}{1-e^{-x}}$	$\zeta(s)\Gamma(s)$	$\langle 1, +\infty\rangle$
$\frac{1}{1+x}$	$\frac{\pi}{\sin \pi s}$	$\langle 0, 1\rangle$
$\log(1+x)$	$\frac{\pi}{s \sin \pi s}$	$\langle -1, 0\rangle$
$H(x) \equiv 1_{0<x<1}$	$\frac{1}{s}$	$\langle 0, +\infty\rangle$
$x^\alpha (\log x)^k H(x)$	$\frac{(-1)^k k!}{(s+\alpha)^{k+1}}$	$\langle -\alpha, +\infty\rangle$ $k \in \mathbb{N}$

Here the most interesting is in the first two lines, we see two different functions having the same Mellin transform. The only diffence is in fundamential strips of the transforms. And now it's a good moment to formulate a theorem about reconstruction of initial function having only its Mellin transform.

Theorem 11.1. *Let f(x) have Mellin transform $f^*(s)$ with fundamental strip $\langle\alpha, \beta\rangle$. Let $\alpha < c < \beta$ and $f^*(c+it)$ is integrable. Then the equality*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}dx = f(x)$$

holds almost everywhere.



The picture presents a patch of integration used in the theorem.

11.3 Singularities

Definition 11.2. Laurent expansion of function $\phi(s)$ in point s_0 is an equality:

$$\phi(s) = \sum_{k \geq -r}^{+\infty} c_k (s - s_0)^k$$

Here $c_{-r} \neq 0$. If $r > 0$, then s_0 is called a pole of order r. If $r = 1$, then pole is called simple. If $r = 2$, pole is double.

If $r \leq 0$, then function is analitic in s_0, because Laurent series in this case degenerates into Taylor series.

Example 11.6. Consider the function $\frac{1}{s^2(s+1)}$, it has two poles on the complex plane. Double pole is at $s_0 = 0$ and simple pole is at $s_0 = -1$:

$$\frac{1}{s^2(s+1)} = \frac{1}{s+1} + 2 + 3(s+1) + \cdots \qquad s_0 = -1$$
$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + 1 - s + \cdots \qquad s_0 = 0$$

Definition 11.3. A singular element (s.e.) of $\phi(s)$ at s_0 is an initial sum of Laurent expansion truncated at terms of O(1) or smaller.

Example 11.7. We consider the same function $\phi(s) = \frac{1}{s^2(s+1)}$ as in the previous example. Singular elements at $s_0 = 0$ are:

$$\left[\frac{1}{s^2} - \frac{1}{s}\right], \left[\frac{1}{s^2} - \frac{1}{s} + 1\right], \ldots$$

Here we can trancate the Laurant expansion wherever we want, but we are to include terms with negative degree of s. The same is about singular elements in $s_0 = 1$. They are:

$$\left[\frac{1}{s+1}\right], \left[\frac{1}{s+1} + 2\right], \left[\frac{1}{s+1} + 2 + 3(s+1)\right], \ldots$$

We are to include all items with negative degree of $s + 1$, other items we can include if we want, but usually there is no need to do it.

Definition 11.4. Let $\phi(s)$ be meromorphic in some area Ω with $\mathfrak{S}$ including all the poles of $\phi(s)$ in Ω. A singular expansion of $\phi(s)$ in Ω is a formal sum of singular elements of $\phi(s)$ at all points of $\mathfrak{S}$. Notation: $\phi(s) \asymp E$.

Example 11.8.

$$\frac{1}{s^2(s+1)} \asymp \left[\frac{1}{s+1}\right]_{s=-1} + \left[\frac{1}{s^2} - \frac{1}{s}\right]_{s=0} + \left[\frac{1}{2}\right]_{s=1} \qquad s \in \mathbb{C}$$

It is a singular expansion of function $\phi(s) = \frac{1}{s^2(s+1)}$ in all complex plane. $\mathfrak{S} = \{-1, 0, 1\}$. There is no pole at point $s_0 = 1$, but we can include a singular element at it in a singular expansion, if we want. $\mathfrak{S}$ is to include all the poles of the function, but it also may include any other points. However there is usually no sense in it.

Singular expansion is only a formal sum, we are not trying to evaluate it or even to simplify. This sum only shows us which poles does function have and what singular elements are there.

Example 11.9 (Singular expansion of gamma function). I'll remind you that gamma function is defined by the equality

$$\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1} dx$$

Integral converges for $s \in \langle 0, +\infty \rangle$, but functional equation $s\Gamma(s) = \Gamma(s+1)$ allows to make a continuation of the gamma function to all complex plane except nonpositive integers. (It has been already noticed in example 11.3) Now we are going to obtain a singular expansion of gamma function in $\mathbb{C}$.
First af all, functional equation on gamma function implies

$$\Gamma(s) = \frac{\Gamma(s+m+1)}{s(s+1)(s+2)\dots(s+m)}, \quad m \in \mathbb{N} \cup \{0\}$$

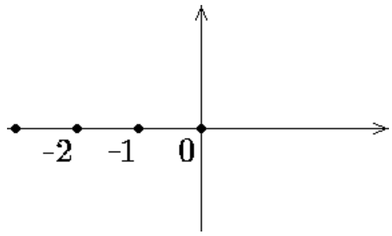
It demonstrates again, that gamma function has poles in nonpositive integers, but now we can learn much more about them. After making a kind of substitution of $-m$ for s we obtain

$$\Gamma(s) \sim \frac{(-1)^m}{m!} \frac{1}{s+m}, \quad \text{as } s \to -m$$

This means that left side divided by right side tends to 1 as s tends to $-m$. But we can understand it as that the right side is a singular element of gamma function in point $s = -m$. Now we know singular elements in all the poles of gamma function and thus we can write a singular expansion:

$$\Gamma(s) \asymp \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}, \quad m \in \mathbb{C}$$

As it was already said, we are not trying to evaluate the sum, we only look on it and see what poles does Gamma function have, and what singular elements are there. For example we see that all poles of gamma function are simple.



Here the arrangement of gamma function poles is demonstrated at the picture.

11.4 Direct mapping

We have already seen in proposition 11.1, that asymptotics of function $f(x)$ in zero results on the leftmost boundary of the fundamental strip of Mellin transform $f^*(s)$. The same is with the asimptotics in infinity. It results on the rightmost boundary of the fundamental strip. Direct mapping is a theorem about what information can we obtain about Mellin transform, if we know a more detailed asymptotics of inital function $f(x)$ in zero and infinity.

Theorem 11.2 (Direct mapping). *Let $f(x)$ have a transform $f^*(s)$ with nonemty fundamental strip $\langle \alpha, \beta \rangle$. Let*

$$f(x) = \sum_{(\xi,k)\in A} c_{\xi,k} x^{\xi} (\log x)^k + \mathrm{O}(x^{\gamma}), \quad x \to 0$$

where $k \in \mathbb{N} \cup \{0\}$, $-\gamma < -\xi \leq \alpha$. Then $f^*(s)$ is continuable to a meromorphic function in the strip $\langle -\gamma, \beta \rangle$, where it admits the singular expansion

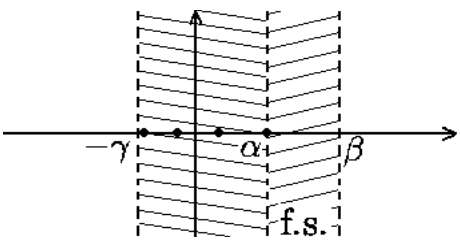
$$f^*(s) \asymp \sum_{(\xi,k)\in A} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}}, \quad s \in \langle -\gamma, \beta \rangle$$

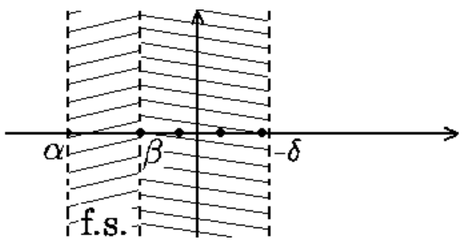
If asymptotic expansion is given at infinity, then the similar result holds. The only difference is, that meromorphic continuation is to the right of the fundamental strip and there is an additional minus sign in singular expansion of transformed function. Look for explanation in two tables given below.
Singular expansion of transformed function presented in the theorem may seem to be confusing, but in real life there are no logarithms in asymptotic expansions. If there is one, it comes in the first degree. In this cases k equals 0 or 1, which makes singular expansion much more simple.
Let's put in a table some information that we know about connection between asymptotic expansion of initial function and properties of transformed one.

$f(x)$	$f^*(s)$
Order at 0: $\mathrm{O}(x^{-\alpha})$	Leftmost boundary of f.s. at $\Re(s) = \alpha$
Order at $+\infty$: $\mathrm{O}(x^{-\beta})$	Rightmost boundary of f.s. at $\Re(s) = \beta$
Expansion till $\mathrm{O}(x^{\gamma})$ at 0	Meromorphic continuation till $\Re(s) = -\gamma$
Expansion till $\mathrm{O}(x^{\delta})$ at $+\infty$	Meromorphic continuation till $\Re(s) = -\delta$

The next table contains information about connections between terms in asymptotic expansion of initial function and singularities of its Mellin transform.

$f(x)$	$f^*(s)$
Term $x^a(\log x)^k$ at 0	Pole with s.e. $\frac{(-1)^k k!}{(s+a)^{k+1}}$
Term $x^a(\log x)^k$ at $+\infty$	Pole with s.e. $-\frac{(-1)^k k!}{(s+a)^{k+1}}$
Term x^a at 0	Pole with s.e. $\frac{1}{s+a}$
Term $x^a \log x$ at 0	Pole with s.e. $-\frac{1}{(s+a)^2}$





Information contained in two tables is visualized on these two pictures.

Example 11.10. We have already obtained the singular expansion of gamma function by means of functional equation on it. Now we'll obtain the same result, but by applying the direct mapping theorem.
It was shown in example 11.3, that function $f(x) = e^{-x}$ has Mellin transform $f^*(s) = \Gamma(s)$. Asymptotic expansion of $f(x)$ at 0 is as follows:

$$f(x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = \sum_{j=0}^{M} \frac{(-1)^j}{j!} x^j + \mathrm{O}(x^{M+1})$$

Theorem states, that $f^*(s)$ is meromorphicaly continuable to $\langle -M-1, +\infty \rangle$. (And we do know it already) The asymptotic expansion of continued function is:

$$f^*(s) \asymp \sum_{j=0}^{M} \frac{(-1)^j}{j!} \frac{1}{s+j} \qquad s \in \langle -M-1, +\infty \rangle$$

The last holds for every positive M, so we can rewrite it in the following way:

$$f^*(s) \asymp \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \frac{1}{s+j} \qquad s \in \mathbb{C}$$

and this is an asymptotic expansion in entire $\mathbb{C}$. This expansion we have already seen in example 11.9.

Example 11.11. In example 11.5 we intoduced the Mellin pair

$$g(x) = \frac{e^{-x}}{1-e^{-x}}; \quad g^*(s) = \zeta(s)\Gamma(s)$$

with fundamental strip $\langle 1, +\infty \rangle$. The asymptotic expansion of $g(x)$ at 0 is

$$g(x) = \frac{e^{-x}}{1-e^{-x}} = \sum_{j=-1}^{+\infty} B_{j+1} \frac{x^j}{(j+1)!}$$

where B_j are so-called Bernoulli numbers, we can suppose that this asymptotic expansion is a definition of Bernoully numbers. $B_0 = 1$, $B_1 = -1/2$. As in the previous example, asymptotic expansion is complete, i.e. we can write, that $g(x) = \ldots + \mathrm{O}(x^M)$ for any positive M we want. So, direct mapping theorem states, that transform $g^*(s)$ has meromorphic continuation to strip $\langle -\infty, +\infty \rangle = \mathbb{C}$. The singular expansion there is

$$g^*(s) = \zeta(s)\Gamma(s) \asymp \sum_{j=-1}^{+\infty} \frac{B_{j+1}}{(j+1)!} \frac{1}{s+j}, \qquad s \in \mathbb{C}$$

If we compare it with singular expansion of gamma funcion

$$\Gamma(s) = \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \frac{1}{s+j}, \qquad m \in \mathbb{C}$$

we can extract the singular expansion of zeta function

$$\zeta(s) = \left[\frac{1}{s-1}\right]_{s=1} + \sum_{j=0}^{+\infty} \left[(-1)^j \frac{B_{j+1}}{j+1}\right]_{s=-j}$$

We see, that there are no poles in nonpositive integers, so we have obtained a result, that zeta function in meromorphic in entire $\mathbb{C}$ with the only pole in $s_0 = 1$. Since singular expansion is a sum of initial sums of Laurent expansions, we can extract information about values of zeta function in nonpositive integers.

$$\zeta(-m) = (-1)^m \frac{B_{m+1}}{m+1}, \qquad m \in \mathbb{N} \cup \{0\}$$

By the way, $B_{2k+1} = 0$ for $k \in \mathbb{N}$, so zeta function is zero in even negative integers, $\zeta(0) = -1/2$, and

$$\zeta(-2m+1) = -\frac{B_{2m}}{2m}, \qquad m \in \mathbb{N}$$

Example 11.12. There was another Mallin pair

$$f(x) = \frac{1}{x+1}; \quad f^*(s) = \frac{\pi}{\sin \pi s}$$

with fundamental strip $\langle 0, 1 \rangle$. Asymptotic expansion at 0

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n, \quad x \to 0$$

implies posibility of continuation of transformed function to the left of fundamental strip, namly to $\langle -\infty, 1 \rangle$. Singular expansion there is

$$f^*(s) \asymp \sum_{n=0}^{+\infty} \frac{(-1)^n}{s+n}, \quad s \in \langle -\infty, 1 \rangle$$

If we consider asymptotic expansion at $+\infty$

$$f(x) = \frac{1/x}{1+1/x} = \sum_{n=0}^{+\infty} (-1)^{n-1} x^{-n}, \quad x \to +\infty$$

we learn about meromorphic continuation of transformed function to the right of fundumantal strip. Singular expansion to the right of fundamental strip is

$$f^*(s) \asymp -\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{s-n}, \quad s \in \langle 0, \infty \rangle$$

Minus sign, as it has been already said, comes from the fact, that we consider the asymptotic expansion at infinity.
Now two singular expansions we can conbine into one and it gives us singular expansion of transformed function in entire $\mathbb{C}$

$$f^*(s) \equiv \frac{\pi}{\sin \pi x} \asymp \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{s+n}, \quad s \in \mathbb{C}$$

As we see, expansion is right, but we could have obtained it much easier.

11.5 Converse mapping

Direct mapping theorem was a method of obtaining information about transformed function given the initial function. But usually it is not very interesting. We are interested in information about initial function and not about its transform. So now we are ready to introduce converse mapping theorem. It will be given exact formulation, but further we will apply the theorem without checking if exemined function satisfies all conditions.

Theorem 11.3 (Converse mapping). *Let f(x) be continious on $(0, +\infty)$ function, that has a transform $f^*(s)$ with nonempty fundammental strip $\langle \alpha, \beta \rangle$. Let $f^*(s)$ be meromorphically continuable to $\langle -\gamma, \beta \rangle$ with a fininte number of poles there, and be analytic on $\Re(s) = -\gamma$. Let $f^*(s) = \mathrm{O}(|s|^{-r})$ with $r > 1$ when $|s| \to +\infty$ in $\langle -\gamma, \beta \rangle$. If*

$$f^*(x) \asymp \sum_{(\xi,k) \in A} d_{\xi,k} \frac{1}{(s-\xi)^k}, \quad s \in \langle -\gamma, \beta \rangle$$

Then an asymptotic expansion of $f(x)$ at 0 is

$$f(x) = \sum_{(\xi,k) \in A} d_{\xi,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} (\log x)^{k-1} \right) + \mathrm{O}(x^\gamma)$$

Converse mapping theorem is a theorem, that derives asymptotics of initial function from singularities of transformed. Next table includes some explanation of the theorem.

$f^*(f)$	$f(x)$
Pole at ξ	Term in asymptotic expansion $\approx x^{-\xi}$
left of f.s.	expansion at 0
right of f.s.	expansion at $+\infty$
Simple pole	
left: $\frac{1}{s-\xi}$	$x^{-\xi}$ at 0
right: $\frac{1}{s-\xi}$	$-x^{-\xi}$ at $+\infty$
Multiple pole	logarifmic factor
left: $\frac{1}{(s-\xi)^{k+1}}$	$\frac{(-1)^k}{k!}x^{-\xi}(\log x)^k$ at 0
right: $\frac{1}{(s-\xi)^{k+1}}$	$-\frac{(-1)^k}{k!}x^{-\xi}(\log x)^k$ at $+\infty$

Example 11.13. Consider a function $\phi(s) = \frac{\Gamma(s)\Gamma(\nu-s)}{\Gamma(\nu)}$. It is analytic in strip $\langle 0, \nu\rangle$. We know the singular expansion of gamma function an we can use this knwoledge to obtain the singular expansion of $\phi(s)$ in the strip $\langle -\infty, \nu\rangle$:

$$\phi(s) \asymp \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \frac{\Gamma(\nu+j-1)}{\Gamma(\nu)} \frac{1}{s+j}, \quad s \in \langle -\infty, \nu\rangle$$

This $\phi(s)$ is a Mellin transform of some function $f(x)$, which can be obtained by applying of theorem 11.1

$$f(x) = \frac{1}{2\pi i} \int_{\nu/2-i\infty}^{\nu/2+i\infty} \phi(s) x^{-s} ds$$

Singularities of $\phi(s)$ in strip $\langle -\infty, \nu\rangle$ encode asymptotics for $f(x)$ at 0

$$f(x) = \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \frac{\Gamma(\nu+j-1)}{\Gamma(\nu)} x^j, \quad x \to 0$$

One could have remembed binomial theorem and noticed, that function $\overline{f}(x) = (1 + x)^{-\nu}$ has the same expansion at 0. This means, that difference $\varpi(x) = f(x) - \overline{f}(x)$ decase to zero faster, than any power of x, i.e. $\varpi(x) = O(x^M)$, $\forall M > 0$. In fact, $\varpi(x) \equiv 0$, and we have indirectly obtained a new Mellin pair

$$f(x) = (1+x)^{-\nu}, \quad f^*(s) = \frac{\Gamma(s)\Gamma(\nu-s)}{\Gamma(\nu)}$$

Example 11.14. Consider a function $\phi(s) = \Gamma(1-s)\frac{\pi}{\sin \pi s}$. It is analitic in strip $\langle 0, 1\rangle$. We know singular expansion of every factor in this function and thus we can write singular expansion

$$\phi(s) \asymp \sum_{0}^{+\infty} (-1)^n n! \frac{1}{s+n}, \quad s \in \langle -\infty, 1\rangle$$

Applying of converse mapping theorem yields an asymptics for initial function $f(x)$

$$f(x) \sim \sum_{n=0}^{+\infty} (-1)^n n! x^n, \quad x \to 0$$

Symbol '$\sim$' is written instead of '$=$' to make an emphasis, that expansion is only asymptotic, i.e. series from the right side doesn't converge for any $x > 0$, but we can write, that $f(x) = \sum_{n=0}^{M} (-1)^n n! x^n + \mathrm{O}(x^M)$ as $x \to 0$ for any positive M. In fact, $f(x) = \int_0^{+\infty} \frac{e^{-t}}{1+xt} dt$.

11.6 Harmonic sums

Definition 11.5. A sum of the type $G(x) = \sum_k \lambda_k f(\mu_k x)$ is called harmonic sum with base function $g(x)$, frequencies μ_k and amplitudes λ_k. Series $\Lambda(s) = \sum_k \frac{\lambda_k}{\mu^s}$ is called the Dirichlet series.

Now we are going to discuss when the fifth basic property of Mellin transform holds for infinite harmonic sums. Next proposition is not formulated fully, base function and Dirichlet series are to satisfy some certain conditions about speed of growth, but all this is skipped here for simplicity.

Proposition 11.2. *The Mellin transform of the harmonic sum $G(x) = \sum_k \lambda_k f(\mu_k x)$ is defined in the intersection of the the fundamental strip of $g(x)$ and the domain of absolute convergence of the Dirichlet series $\Lambda(s) = \sum_k \frac{\lambda_k}{\mu^s}$ which is of the form $\Re(s) > \sigma_a$ (or $\Re(s) < \sigma_a$) for some real σ_a. In addition, $G^*(s) = \Lambda(s) g^*(s)$.*

In next (and last) two examples we'll derive two well known asymptotics. One is an asymptotic of harmonic numbers, and the second is Stirling's formula. But here we'll obtain complete asymptotics, which is not known so well.

Example 11.15 (Harmonic numbers). Consider the function

$$h(x) = \sum_{k=1}^{+\infty} \left[\frac{1}{k} - \frac{1}{k+x} \right] = \sum_{k=1}^{+\infty} \frac{1}{k} \frac{x/k}{1 + x/k}$$

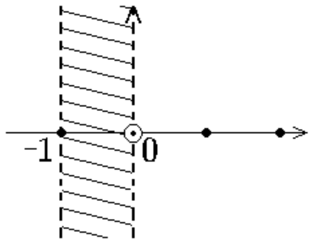
It is usuall harmonic sum with frequencies $\mu_k = 1/k$, amplitudes $\lambda_k = 1/k$ and base function $g(x) = \frac{x}{1+x}$. Function $h(x)$ is connected with harmonic numbers in very simple way: $h(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H_n$ for any $n \in \mathbb{N}$. Now we are going to evaluate Mellin transform of $G(x)$. To do this, we are to evaluate the Dirichlet series $\Lambda(s)$ and the tranform of base functon $g(x)$. $\Lambda(s) = \sum_k \frac{\lambda_k}{\mu^s} = \sum_{k=1}^{+\infty} k^{-1+s} = \zeta(1-s)$. Transform of base function is $g^*(s) = -\frac{\pi}{\sin \pi s}$ with fundamental strip $\langle -1, 0 \rangle$, which is the result of applying of the first base property of Mellin transform. Notice, that the Dirichlet series $\Lambda(s) = \zeta(1-s)$ converges absolutely in the strip $\langle -1, 0 \rangle$
Now we are ready to write the transform of $h(x)$, which is

$$h^*(s) = \Lambda(s) g^*(s) = -\zeta(1-s) \frac{\pi}{\sin \pi s}, \quad s \in \langle -1, 0 \rangle$$

Before trying to write a singular expansion, we are to study zeta function some more. We know that $\zeta(s) \sim \frac{1}{s-1}$, it's a begining of Laurent expansion of zeta function at 1, but it's not enough for us. We want to know a coeficient of the zero degree in the Laurant expansion. Let it be γ, so we can write $\zeta(s) = \frac{1}{s-1} + \gamma + \cdots$ as $s \to 1$. This γ is so-called Euler constant, which is approximatly equal to 0.577215664901532860606512090082402431042159. Keeping all of this in mind, we can write singular expansion of $h^*(s)$:

$$h^*(s) \asymp \left[\frac{1}{s^2} - \frac{\gamma}{s} \right] - \sum_{k=1}^{+\infty} (-1)^k \frac{\zeta(1-k)}{s-k}, \quad s \in \langle -1, +\infty \rangle$$

Fundamental strip of the transform and its poles to the right of the fundamental strip are presented at the picture:



Double pole at zero is marked with a big circle.
Now we can finally apply converse mapping theorem and we obtain wishful asympotics:

$$H_n \sim \log n + \gamma + \sum_{k\geq 1} \frac{(-1)^k B_k}{k} \frac{1}{n^k} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \cdots$$

Example 11.16 (Stirling's formula). In this example we are to use the product decomposition of gamma function, that looks like

$$\log \Gamma(x+1) + \gamma x = \sum_{n=1}^{+\infty} \left[\frac{x}{n} - \log\left(1 + \frac{x}{n}\right)\right]$$

Let's denote the right side as $\ell(x)$. This is a harmonic sum with amplitudes $\lambda_n = 1$, frequencies $\mu_n = 1/n$ and a base function $g(x) = x - log(1+x)$. The Dirichlet series is $\Lambda(s) = \sum_{n=1}^{+\infty} \lambda_n \mu_n^{-s} = \sum_{n=1}^{+\infty} n^s = \zeta(-s)$. Transform of $g(x)$ can be evaluated by means of the first and the ninth basic properties of Mellin transform. The result is $g^*(s) = -\frac{\pi}{s \sin \pi s}$ with fundamental strip $\langle -2, -1\rangle$. As in the previous example, the Dirichlet series $\Lambda(s) = \zeta(-s)$ converges absolutely in the fundamental strip of transform of base function. So, fundamental strip of transform of $\ell(s)$ is $\langle -2, -1\rangle$ too.

$$\ell^*(s) = -\zeta(-s)\frac{\pi}{s \sin \pi s}, \quad s \in \langle -2, -1\rangle$$

We want to obtain an asymptotics in infinity, so we look on meromorphic continuation to the right of the fundamental strip. Laurant expansion of zeta function at zero is $\zeta(s) = \frac{1}{2} - \frac{1}{2}\log(2\pi) + \mathrm{O}(s)$, so singular expansion of $\ell^*(s)$ to the right of fundamental strip is

$$\ell^*(s) \asymp \left[\frac{1}{(s+1)^2} + \frac{1-\gamma}{(s+1)}\right] + \left[\frac{1}{2s^2} - \frac{\log\sqrt{2\pi}}{s}\right] + \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}\zeta(-n)}{n(s+n)}, \quad s \in \langle -2, +\infty\rangle$$

Now we can apply converse mapping theorem and derive an asymptotics of function $\ell(s)$, which is

$$\ell(x) = \Big[x\log x - (1-\gamma)x\Big] + \left[\frac{1}{2}\log x + \log\sqrt{2\pi}\right] + \sum_{n=1}^{+\infty} \frac{B_{2n}}{2n(2n-1)} \frac{1}{x^{2n-1}}, \quad x \to +\infty$$

Here every item from singular expansion was converted to an item in asymptotic expansion without any simplification, but now we do some, keeping in mind, that $\Gamma(x+1) = x!$, so

$$\log(x!) = \log\left(x^x e^{-x}\sqrt{2\pi x}\right) + \sum_{n=1}^{+\infty} \frac{B_{2n}}{2n(2n-1)} \frac{1}{x^{2n-1}}, \quad x \to +\infty$$

This dazzling formula is named after Stirling and this is a good reason to finish the paper right here.