# Chapter 9

# Rice's integrals – a method for solving generalized differences

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Often in the analysis of algorithm and data structures we have the need to estimate the asymptotic growth of differences and sequences defined by recurrence equations. But often we don't know the explicit representation of the solutions. Therefor we need methods, which we can establish asymptotics without knowing the exact representation. Rice's integral is such a method. In this paper we will introduce basics in complex analysis and develope the mathematic foundations needed in theoretical computer science. The paper is based on the lecture "Analysis 4" held by Prof. W. Heise in 2003 at the TU München and an article by Flajolet et al. [FS95].

## 9.1 Basics of Complex Analysis

### 9.1.1 Complex Differentiability

In the whole of this paper, we will have to deal massivly with complex analysis. As complex analysis is often seen as a discipline of pure mathematics, most people working in computer science, even in theoretical computer science, have not heard much about it. So I will give a rough introduction to it and present the needed theorems. It is assumed that the reader has basic knowledge of real analysis, e.g., knows what a real function is, what continuity and differentiablity means, and , what complex numbers are.

First of all we consider a complex mapping on an open set $E$

$$f : E \subset \mathbb{C} \to \mathbb{C}, z = x + yi \mapsto f(z) = u(x, y) + iv(x, y) \tag{9.1}$$

This mapping is said to be continous, if the correspondig 2-dimensional mapping

$$g : E' \subset \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (u(x, y), v(x, y)) \tag{9.2}$$

is continous in the sense of multidimensional real analysis.

If $g$ is differentiable at $(x_0, y_0)$, we have the derivative

$$A := \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}_{x=x_0, y=y_0} \tag{9.3}$$

and $A$ is a linear mapping with the property

$$\forall (x,y) \in E' : g(x,y) = g(x_0,y_0) + A \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + B(x - x_0, y - y_0) \tag{9.4}$$

where $B$ is a mapping with $B(0,0) = (0,0)^T$ and $\lim\limits_{(x,y)\to(0,0)} \frac{B(x,y)}{|(x,y)|} = (0,0)^T$ which, of course, implies continuity at $(0,0)$

As a linear mapping $E \subset \mathbb{C} \to \mathbb{C}$ is in fact a complex multiplication, the natural way to introduce complex derivatives, is trying to find a complex number $f'(z_0)$, which acts the same way as $A$ in (9.4):

$$\forall z \in E : f(z) = f(z_0) + f'(z_0) \cdot ((x - x_0) + i(y - y_0)) + b((x - x_0) + i(y - y_0)) \tag{9.5}$$

where $b$ is a complex mapping with $b(0) = 0$ and $\lim\limits_{z\to 0} \frac{b(z)}{|z|} = 0$. If such a number $f'(z_0)$ exists, f is said to be complex differentiable at $z_0$.

Now we compare the matrix-vector-multiplication with the product of two complex numbers:

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 \\ a_{2,1}x_1 + a_{2,2}x_2 \end{pmatrix} \tag{9.6}$$

$$(a_1 + ia_2)(x_1 + ix_2) = (a_1x_1 - a_2x_2) + i(a_2x_1 + a_1x_2) \tag{9.7}$$

we get the correspondence $a_{1,1} = a_1 = a_{2,2}$ and $a_{1,2} = -a_2 = a_{2,1}$

From this, it is plausible that a real function $\mathbb{R}^2 \to \mathbb{R}^2$ understood as a complex function is complex differentiable, iff $u_x = v_y$ and $u_y = -v_x$. These equations are known as the Cauchy-Riemann Differential Equation (CRDE).

As a result, we have that every complex differentiable function is differentiable in a real sense, but a real differentiable function is only complex differentiable, if the CRDEs are satisfied. So complex differentiability is somewhat stronger.

**Definition 9.1.** A complex mapping $f$, as introduced in (9.1), is said to be differentiable at a point $z_0 \in E$ with derivative $f'(z_0)$, iff (9.5) holdes for some appropriate $b$.

**Definition 9.2.** The complex mapping $f$ is said to be holomorphic at a point $z_0$ on $E$, iff $f$ is complex differentiable at any point of a neighbourhood $F \subset E$ of $z_0$, which is more than just complex differentiable at $z_0$.

**Definition 9.3.** The complex mapping $f$ is said to be holomorphic on an open set $F \subset E$, iff $f$ is holomorphic at any point $z_0 \in F$.

In fact, when evaluating complex derivatives, not many changes occur; e.g. $(z^n)' = nz^{n-1}$, $(e^z)' = e^z$ or $(\sin(z))' = \cos(z)$

### 9.1.2 Integration

In real analysis we integrate over intervalls, where the integral is a limit of sums. These sums take into account the values of the function and the length of the parts of the discretisation of the intervall. The discretisation $Z$ gets "finer" in the sense, that the longest part tends to 0:

$$\int_a^b f(x)dx = \lim_{n\to\infty, |Z_n|\to 0} \sum_{k=0, x_{n,k}\in[z_{n,k}, z_{n,k+1}]}^{n} f(x_{n,k})(z_{n,k+1} - z_{n,k}) \tag{9.8}$$

where $z_{n,0} = a < z_{n,1} < \ldots < z_{n,n} < z_{n,n+1} = b$ for every $n$
In some way, we walk along the intervall from $a$ to $b$, picking some points, where we examine the function closer, and get a number from this process. Of course, it shouldn't be important, whether we are faster or slower while "walking". This is expressed by the substitution formula:

$$\int_a^b g(\gamma(s))\gamma'(s)ds = \int_c^d g(t)dt \tag{9.9}$$

where $\gamma : [a,b] \to [c,d]$ is piecewise differentable and monotonous and $g : [c,d] \to \mathbb{R}$ is piecewise continous. In $\gamma$ we have the information how fast we are on the intervall at each point of the parametrisation.
In a similar way we can integrate "along" a piecewise differentiable curve $\gamma : \mathbb{R} \to \mathbb{C}; t \mapsto \gamma(t)$; we can think of $\mathbb{C}$ as $\mathbb{R}^2$. In this situation $\gamma$ is called an integration path. An integral of a complex function is the sum of the integrals of the real and the imaginary part, so for a complex function $f$ we get[1]:

**Definition 9.4.** The integral of a complex function $f$ along a curve $\gamma$ is:

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt = \int_a^b \Re\left(f(\gamma(t)) \cdot \gamma'(t)\right) dt + i \int_a^b \Im\left(f(\gamma(t)) \cdot \gamma'(t)\right) dt \tag{9.10}$$

For example we have the curve $\gamma : [0, 2\pi] \to \mathbb{C}; t \mapsto e^{it}$ which is a circle of radius 1 and center 0 and $f(z) = \frac{1}{z}$.

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it}dt = \int_0^{2\pi} idt = 2\pi i \tag{9.11}$$

Some properties of real integrals can be copied almost literarily:
The length of a path of integration is

$$\Lambda(\gamma) = \int_a^b \left|\gamma'(t)\right| dt = \int_a^b \sqrt{(\Re\gamma'(t))^2 + (\Im\gamma'(t))^2}dt \tag{9.12}$$

For a piecewise continous $f : [a,b] \to \mathbb{C}$:

$$\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|\, dt \tag{9.13}$$

For $f$, continous on the image of a path of integration $\gamma$ and $M$ the maximum absolute value of f on that image, we have:

$$\left|\int_\gamma f(t)dt\right| \le M \cdot \Lambda(\gamma) \tag{9.14}$$

So far we have always considered curves and not their images as domain of integration. But the following result shows, that somehow only the domain is important. For this purpose we call a surjective, piecewise continous differentiable, real function $\phi$ a parameter transform, iff for all points of its domain, where $\phi$ is differentiable, the derivative is greater than 0.

**Theorem 9.1.** *If $\phi : [c,d] \to [a,b]$ is a parameter transform and $\gamma_1 : [a,b] \to \mathbb{C}$ an integration path, then $\gamma_2 = \gamma_1 \circ \phi : [c,d] \to \mathbb{C}$ is also a path of integration and for a function $f$ continous on the image of $\gamma_1$ we have:*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz \tag{9.15}$$

[1] Note, that we integrate along a curve in the first place and not the image of the curve.

Although the proof is simple, we won't state it here. The essence of this theorem is, that the speed, at which we follow the line defined by a curve, is not important for the value of the integral – as long as we follow the line "smooth" enough, i.e. we don't "jump" around and we don't stay to long at one point.

**Definition 9.5.** For a continous function $f$ on an open set $E \subset \mathbb{C}$ we call $F : E \to \mathbb{C}$ the antiderivative of $f$, if $F$ is holomorphic on $E$ and $F' = f$. $f$ is said to have local antiderivatives, if for each $z_0 \in E$ there exists a neighbourhood $G \subset E$ of $z_0$, so that there exists a holomorphic $F$ with $F' = f$ on $G$.

### 9.1.3 Holomorphic functions and the Cauchy Integral Theorem

Holomorphic functions do have many nice properties. The proofs are often sophisticated and technical, so we will again omit them.

**Definition 9.6.** A complex function $f$ is said to be analytic at a point $z_0$ if there exists a neighbourhood $G$ of $z_0$, for example an open circle, such that $f$ has a powerseries expansion, which converges for some radius $R$. $f$ is said to be analytic on an open set, if it is analytic on every point of this set.

**Theorem 9.2.** *For a continous function $f$ on a domain[2] $E$ the following statements are equivalent:*

1. *$f$ is holomorphic*
2. *$f$ has local antiderivatives*
3. *$f$ is differentiable in the real sense and obeys the CRDEs*
4. *$f$ is analytic*

Since powerseries can be differentiated without any loss in their convergent domain, this shows, that holomorphic function are arbitrarily often differentiable – at least in a local sense; but since we can cover domains with circles we can connect the domains of definition of these derivatives and get one derivative for the whole domain of a holomorphic function. This is an outstandig property of holomorphic functions: in real analysis a function can be differentiable but doesn't have to be for a second time; in complex analysis a function, which is differentiable on an open set, is arbitrarily often differentiable.
Since powerseries are more or less Taylor series, holomorphic functions are completly defined by only one point – if you know all higher order derivatives at just one point. Another interesting property similar to that is, when you know the function at countable many points, which cluster at one point, then the holomorphic functions is also defined uniquely. All this has the consequence, that whenever we have a holomorphic function on a certain domain, then possible holomorphic extensions are again unique – so restrictions of functions to a domain which is too small for our purpose is no problem because we just extend the function to wherever we want, at least if no poles hinder us.

**Theorem 9.3.** *Let $E$ be a convex domain and $Z \subset E$ a set of points without cluster points. If $f : E \to \mathbb{C}$ is continous and holomorphic on $E \setminus Z$ then for every closed (meaning $\gamma(a) = \gamma(b)$) path of integration $\gamma : [a, b] \to \mathbb{C}$ the following holds:*

$$\int_\gamma f(z)dz = 0 \tag{9.16}$$

[2] a domain is a connected, open set; those, who have never heard of connectivity, can imagine this as a set, in which from every point to every other point there exists a path – despite this is called path connectivity and is stronger than sole connectivity, it is enough to get an idea of it.

This is the Cauchy integral theorem[3] and is something like the fundametal theorem of integral and differential calculus: if you have two paths from $x$ to $y$ you can put these two together to form a closed path, by altering the direction of one of those paths; so their integral is zero. Since the alternation means a change in sign[4], this means that the integrals along the two paths for each of them have the same value, and the integral itself does only depend on the starting and end point of integration.
At least if the function is holomorphic "enough" and the domain is convex – the later can be extend to so called simple connected domains, witch is roughly speaking connected and "without holes".

### 9.1.4 Cauchy integral formula and residue calculus

Next we will state the Cauchy integral formulas[5] and formulate the residue theorem, which will be the central tools used in solving Rice's integrals.

**Theorem 9.4.** *Let $U$ be an open disc in a domain $E$ with $\overline{U} \subset E$[6] and we denote a positivly oriented boundary of $U$ with $\partial U$[7]. Let $f : E \to \mathbb{C}$ be holomorphic, then the following statement hold:*

$$\forall z \in U : \quad f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{9.17}$$

$$\forall z \in U : \quad f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{9.18}$$

$$\forall z \in E \setminus \overline{U} : \quad 0 = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{9.19}$$

There are some generalisations of the CIT and CIF for general paths (not only circles and points can be encircled more than once). Their mathematical exact presentation would need some unnecesary complicated definitions. The main result is however, that the circle in the CIF can be deformated "continously" arbitrarily as long as the path doesn't cross the singularity $z$. We can even encircle $z$ more than once, but the integral will than be an integer multiple of integral, whose path does encircle $z$ only once, according to the number how often and in what direction $z$ is encircled.
Untill now we had only considered polynomial singularities with the CIF. But the concept of integrating around singularities can be extended.
As stated above every function that is holomorphic can be expressed as a powerseries. There exists a generalisation of this concept, which is called Laurent series. Laurent series have the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{9.20}$$

It can be shown that under certain conditions $f$ can be expressed within some disc partialy containing $E$ and whose boundary is in $E$ entirely as a Laurent series. Here $f$ must be defined on a domain $E$ for every point $z_0$, for which a path in $E$ exists that encircles $z_0$.
In the following we consider holomorphic functions $f$ which have isolated singularities; this means roughly speaking, that the domain of definition doesn't have any holes beside some points and these points do not cluster.

[3] further denoted with CIT
[4] this can easily be checked by considering the substition formula
[5] further denoted with CIF
[6] $\overline{U}$ indicates the closure of U in the standard topology on $\mathbb{C}$
[7] a positivly oriented boundary of a disc is a path (this means a special curve and not a set) starting from one point at the boundary then encircles the disc counterclockwise untill it returns for the first time to the starting point – this means it is a closed path

**Definition 9.7.** The coefficient at $n = -1$ of a Laurent series of a holomorphic function $f$ with isolated singularities of type (9.20) is called the residue at $z_0$: $\operatorname*{Res}_{z_0}(f) = a_{-1}$

It can be shown[8] that there exists an $\epsilon$, such that a circle $U_\epsilon(z_0)$ with radius $\epsilon$ small enough and center $z_0$ has the property $\overline{U_\epsilon(z_0)} \setminus \{z_0\} \subset E$ and that

$$\operatorname*{Res}_{z_0}(f) = \frac{1}{2\pi i} \int_{\partial U_\epsilon(z_0)} f(z)dz \tag{9.21}$$

Taking into account the CIT this yields the residue theorem[9]:

**Theorem 9.5.** *Let $E$ be an open set and $U$ an open disc with $E \subset U$. Let, for some $n \in \mathbb{N}_0$, $f$ be holomorphic on $E \setminus \{z_1, \ldots, z_n\}$. Then*

$$\frac{1}{2\pi i} \int_{\partial U} f(z)dz = \sum_{k=0}^{n} \operatorname*{Res}_{z_k}(f) \tag{9.22}$$

This is an extension of the CIT. It can even be extended to the case, where $U$ is not a circle in $U$ and even if $U$ is not bounded, as long as the singularities are isolated in the encircled domain. This finishes our short introduction to complex analysis.

# 9.2 Other mathematical formulas

## 9.2.1 The Gamma function

The Gamma function is a generalisation of the factorial to complex numbers – one of its definitions is

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{9.23}$$

It satisfies the relations:

$$\forall n \in \mathbb{N}_0 : \Gamma(n) = (n-1)! \quad \forall x \in \mathbb{C} \setminus -\mathbb{N}_0 : \Gamma(x+1) = x\Gamma(x) \tag{9.24}$$

A direct consequence is:

$$\prod_{i=0}^{n} (s-i) = \frac{\Gamma(s+1)}{\Gamma(s-n)} \tag{9.25}$$

The Gamma function is holomorphic on its domain $\mathbb{C} \setminus -\mathbb{N}_0$; at the negativ integers it diverges.
An estimate for the growth of the Gamma function is the so called Stirling formula:

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \left(1 + O(\frac{1}{n})\right) \tag{9.26}$$

The following formula, which is a direct consequence of the Stirling formula is the standard estimate:

$$\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} = n^\alpha \left(1 + O(\frac{|a|^2}{n})\right) \tag{9.27}$$

It can be shown that

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \ln(n)\right) =: \gamma \approx 0.577 \tag{9.28}$$

[8] in fact, this is a direct consequence of the structure of the Laurent series and the fact that $\forall n \in \mathbb{Z} \setminus \{-1\} : \int_\gamma z^n = 0$ if $\gamma$ is a closed path

[9] again the proof is quiet obvious and the idea simple, if you are used to complex analysis, but if exactly formulated very technical

This $\gamma$ is known as Eulers constant.
Using this, another result is:

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} - \sum_{k=1}^{\infty}\left(\frac{1}{x+k} - \frac{1}{k}\right) \tag{9.29}$$

The proofs of all the statements above are wonderful perls of mathematical analysis. But as with all perls, you have to dive deep to find them, so the proofs are far from trivial and we omit them all. The Gamma function does have many other nice properties – but we won't need them.

### 9.2.2 Zeta functions and Modified Bell Polynomials

The so called incomplete Hurwitz $\zeta$ function is:

$$\zeta_n(r,\beta) = \sum_{i=0}^{n-1} \frac{1}{(i+\beta)^r} \tag{9.30}$$

$\zeta_n(r,1)$ defines the generalized harmonic numbers $\zeta_n(r)$ and their limit $(n \to \infty)$ is the famous Riemann $\zeta$ function.
From (9.29) it follows, that

$$\zeta_{n+1}(1,\beta) = \ln(n) - \frac{\Gamma'(\beta)}{\Gamma(\beta)} + O(\frac{1}{n}) \tag{9.31}$$

The modified Bell polynomials $\mathbf{L_m} = \mathbf{L_m}(x_1, x_2, \dots, x_m)$ are defined as

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) = 1 + \sum_{m=1}^{\infty} \mathbf{L_m} t^m \tag{9.32}$$

It is rather technical than difficult to proof that in general

$$\mathbf{L_m}(x_1, x_2, \dots) = \sum_{1m_1+2m_2+\dots=m} \frac{1}{m_1!m_2!\dots}\left(\frac{x_1}{1}\right)^{m_1}\left(\frac{x_2}{2}\right)^{m_2}\dots \tag{9.33}$$

and to get an idea of them, we have

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) = 1 + x_1 t + \left(\frac{x_2}{2} + \frac{x_1^2}{2}\right)t^2 + \\ \left(\frac{x_3}{3} + \frac{x_1 x_2}{2} + \frac{x_1^3}{6}\right)t^3 + \left(\frac{x_4}{4} + \frac{x_1 x_3}{3} + \frac{x_2^2}{8} + \frac{x_2 x_1^2}{4} + \frac{x_1^4}{24}\right)t^4 + \dots \tag{9.34}$$

## 9.3 Motivation and Basic Integrals

First we will introduce generalized differences for a sequence $\{f_k\}_{k\in\mathbb{N}_0}$:

$$\Delta f_n = f_{n+1} - f_n \qquad \Delta^n f_0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} f_k = (-1)^n D_n[f] \tag{9.35}$$

The differences $D_n$ arise often in the average case analysis of some data structures as search trees or tries. As a naive bound we get:

$$|D_n[f]| \le 2^n \max_{0\le k\le n} |f_n| \tag{9.36}$$

But for many sequences $f_n$ that come across in data structure analysis this bound is way to rough and polynomial bounds can be found – this phenomenon is called exponential cancelation.
In the analysis of recurrent sequences generating functions are often used to simplify and solve the relations. For example the exponential generating is

$$f(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} \tag{9.37}$$

and Poisson generating function is defined by

$$\hat{f}(z) = \sum_{n=0}^{\infty} f_n e^{-z} \frac{z^n}{n!} \tag{9.38}$$

We consider the transform $f_n \mapsto g_n = D_n[f]$. Substitution in the exponetial generating function or the Poisson generating function respectively yields the equations

$$g(z) = e^z f(-z) \quad \hat{g}(z) = e^{-z} \hat{f}(-z) \tag{9.39}$$

So it can be supposed, that when these transforms induce drastical simplifications of recurences or difference equations, high order differences as $D_n$ may play a significant role.
We assume in the following, that a holomorphic function $\phi(x)$ interpolates the values of the sequence $f_n$, which means $\forall k \in \mathbb{N}_0 : f_k = \phi(k)$.

**Lemma 9.1.** *Let $\phi$ be a holomorphic function in a domain that contains the half-line $[n_0, \infty[$ and $\mathcal{C}$ is a positivly oriented closed path in the domain of $\phi$, which encircles $[n_0, n]$ and does not include any of the integers $0, 1, \ldots, n_0 - 1$ nor a point, where $\phi$ is not holomorphic. Then the following holds*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = \frac{(-1)^n}{2\pi i} \int_{\mathcal{C}} \phi(s) \frac{n!}{s(s-1)\ldots(s-n)} ds \tag{9.40}$$

*Proof.* We apply the residue theorem (9.22). Since the only points where the integrand is not holomorphic are the integers $n_0, n_0 + 1, \ldots, n$, we only have to consider these integers. Let $k$ be such an integer; the residues can be evaluated according to the CIF (9.17). Then we have:

$$\operatorname*{Res}_{s=k} \phi(s) \frac{n!}{s(s-1)\ldots(s-n)} =$$

$$\operatorname*{Res}_{s=k} \frac{1}{s-k} \left( \phi(s) \frac{n!}{s(s-1)\ldots(s-k+1)(s-k-1)\ldots(s-n)} \right) = \phi(k) \frac{(-1)^{n-k} n!}{k!(n-k)!} \tag{9.41}$$

Simple summation while taking the sign into account yields the proposition. □

For the further discussion the definition of polynomial growth will be important:

**Definition 9.8.** A function $\phi$ in an unbounded domain $\Omega$ is said to have polynomial growth, if for some $r$ the formula $|\phi(s)| = O(|s|^r)$ holds as $s \to \infty$ in $\Omega$. $r$ is called the degree of $\phi$

If the function in (9.40) is of polynomial growth in the half-plain $\Re(s) > n_0 - \epsilon$ for some $\epsilon > 0$ and $n$ is sufficiently large, then we have for some $n_0 > c > \max\{n_0 - \epsilon, n_0 - 1\}$ the representation

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = -\frac{(-1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) \frac{n!}{s(s-1)\ldots(s-n)} ds \tag{9.42}$$

Take as contour of integration the path

$$\gamma_j : \left[-j, j+\frac{1}{2}\right] \to \mathbb{C}; \begin{cases} c+ix & -j \leq x \leq j \\ c+je^{-2\pi i(\frac{1}{4}-x)} & j \leq x \leq j+\frac{1}{2} \end{cases}$$

where $j > n$. This path is negatively oriented (so the sign changes) and we can apply (9.40). Since for any allowed $j$ (9.40) holds, we consider the limit $j \to \infty$. The first part of the path yields the integral in (9.42) and the second can be estimated as $O(|j|^{-n-1+r+1})$ using formula (9.14), since the integrand and therefor its maximum has asymtotical growth $O(|j|^{-n-1+r})$ and the length of the path is $\frac{2\pi j}{2}$. So if $n$ is sufficiently large the second part of the integral vanishes.

# 9.4 Integrals of Functions with Poles and Representation of Sumes

## 9.4.1 Rational functions

**Theorem 9.6.** *Let $\phi$ be a rational function holomorphic in a domain that contains the half-line $[n_0, \infty[$. If $n$ is big enough we have*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = -(-1)^n \sum_s \operatorname*{Res}_s \left( \phi(s) \frac{n!}{s(s-1)\dots(s-n)} \right) \tag{9.43}$$

*where the sum is taken over all poles of $\phi$ and over $0, 1, \dots, n_0 - 1$*

*Proof.* First we use Lemma 9.1 and take as path of integration a circle of radius $R$ big enough to encircle all poles. When $R \to \infty$ and $n > r$ the integral on the right side of (9.40) tends to 0 by a similar argument used for (9.42). Applying once again the residue theorem (9.22), we find that the sum on the righthand side of (9.43) minus its lefthand side is 0, which directly yields (9.43) □

As a next step we try to express the residues. For this purpose we will need the incomplete Hurwitz zeta function and the modified Bell polynomials introduced in (9.30) and (9.33). As every rational function can be expressed as a linear combination of terms of the form $A(x-a)^{-r}$, where $r \in \mathbb{N}_0$, we only have to consider functions $\phi$ of this type.

**Lemma 9.2.** *When $\alpha \in \mathbb{C} \setminus \mathbb{N}_0$, then*

$$I_n(\alpha) = (-1)^n n! \operatorname*{Res}_{s=\alpha} \left( \frac{1}{(s-\alpha)^r} \frac{1}{s(s-1)\dots(s-n)} \right) \tag{9.44}$$

*has the following asymptotic*

$$I_n(\alpha) = -\Gamma(-\alpha) n^\alpha \frac{(\ln n)^{r-1}}{(r-1)!} \left( 1 + O\left(\frac{1}{\ln n}\right) \right) \tag{9.45}$$

In the following we use the symbol $\langle \phi(s) \rangle_{k,\alpha}$ to denote the $k$th coefficient in the Laurent series at a certain point $\alpha \in \mathbb{C}$.

*Proof.*

$$
\begin{aligned}
I_n(\alpha) = &- n!\left\langle (s-\alpha)^r \frac{1}{(-s)(1-s)\dots(n-s)}\right\rangle_{-1,\alpha} = \\
&- n!\left\langle \frac{1}{(-s)(1-s)\dots(n-s)}\right\rangle_{r-1,\alpha} = \\
&- n!\left\langle \frac{1}{(-\alpha-s)(1-\alpha-s)\dots(n-\alpha-s)}\right\rangle_{r-1,0} = \\
&- n!\left\langle \exp\left(-\ln\left(\prod_{j=0}^{n}(j-\alpha-s)\right)\right)\right\rangle_{r-1,0} = \\
&- n!\left\langle \exp\left(-\sum_{j=0}^{n}\ln(j-\alpha-s)\right)\right\rangle_{r-1,0}
\end{aligned}
\tag{9.46}
$$

Since we have $\ln(j-\alpha-x) = \ln\left((j-\alpha)\left(1+\frac{-x}{j-\alpha}\right)\right) = \ln(j-\alpha)+\ln\left(1+\frac{-x}{j-\alpha}\right)$, we can apply the series expansion $\ln(1+x) = \sum\limits_{m=1}^{\infty}(-1)^{m+1}\frac{x^m}{m}$ to get

$$
\begin{aligned}
&= -n!\exp\left(-\sum_{j=0}^{n}\ln(j-\alpha)\right)\left\langle \exp\left(\sum_{j=0}^{n}\left(\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{s}{j-\alpha}\right)^m\right)\right)\right\rangle_{r-1,0} = \\
&\quad - n!\frac{1}{(-\alpha)(1-\alpha)\dots(n-\alpha)}\left\langle \exp\left(\sum_{m=1}^{\infty}\left(\sum_{j=0}^{n}\frac{1}{(j-\alpha)^m}\frac{s^m}{m}\right)\right)\right\rangle_{r-1,0} \overset{(9.30)}{=} \\
&\quad - n!\frac{1}{(-\alpha)(1-\alpha)\dots(n-\alpha)}\left\langle \exp\left(\sum_{m=1}^{\infty}\zeta_{n+1}(m,-\alpha)\frac{s^m}{m}\right)\right\rangle_{r-1,0} \overset{(9.33)}{=} \\
&\quad - \frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)}\mathbf{L_{r-1}}(\zeta_{n+1}(1,-\alpha),\zeta_{n+1}(2,-\alpha),\dots\zeta_{n+1}(r-1,-\alpha)) \overset{(9.27)}{=} \\
&- \frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)}\mathbf{L_{r-1}}(\ln n - \frac{\Gamma'(-\alpha)}{\Gamma(-\alpha)}+O(1/n),\zeta_{n+1}(2,-\alpha),\dots\zeta_{n+1}(r-1,-\alpha)) =
\end{aligned}
\tag{9.47}
$$

Since the incomplete Hurwitz zeta function fullfills $\zeta_n(r,\beta) = O(1)$ for $n\to\infty$ and $r\in\mathbb{N}\setminus\{1\}$ and since beside the first coefficients of the modified Bell polynomials $\mathbf{L_m}$ all coefficent are of degree smaller than $m$ all other coefficient can be neglected.

$$
\begin{aligned}
&= -\frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)}\frac{1}{(r-1)!}(\ln n - \frac{\Gamma'(-\alpha)}{\Gamma(-\alpha)}+O(1/n))^{r-1} = \\
&\quad - \frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)}\frac{(\ln n)^{r-1}}{(r-1)!}\cdot\left(1+O\left(\frac{1}{\ln n}\right)\right) \overset{(9.31)}{=} \\
&\quad - \Gamma(-\alpha)n^\alpha\left(1-O\left(\frac{1}{n}\right)\right)\frac{(\ln n)^{r-1}}{(r-1)!}\cdot\left(1+O\left(\frac{1}{\ln n}\right)\right) = \\
&\quad - \Gamma(-\alpha)n^\alpha\frac{(\ln n)^{r-1}}{(r-1)!}\cdot\left(1+O\left(\frac{1}{\ln n}\right)\right)
\end{aligned}
\tag{9.48}
$$

□

As a first example we analyze for an $m\in\mathbb{N}$ the asymtotic growth of the sum

$$
S_n(m) = \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k^m}
\tag{9.49}
$$

Here we can use the function $\phi(s) = \frac{1}{s^m}$ to interpolate the sequence and we set $n_0 = 1$. We have only one pole not in $\{n_0, n_0+1, \ldots, n\}$, that is 0, which is of the order $m+1$. We have to modify our calculations yielding (9.2), since the pole is in $\mathbb{N}_0$:

$$
\begin{aligned}
S_n(m) = -\operatorname*{Res}_{s=0}\left(\frac{1}{s^{m+1}}\frac{n}{s-n}\frac{n-1}{s-n+1}\cdots\frac{2}{s+2}\frac{1}{s+1}\right) = \\
-\operatorname*{Res}_{s=0}\left(\frac{1}{s^{m+1}}\left(\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\cdots\left(1-\frac{s}{n}\right)\right)^{-1}\right) = \\
-\left\langle\left(\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\cdots\left(1-\frac{s}{n}\right)\right)^{-1}\right\rangle_{m,0}
\end{aligned}
\tag{9.50}
$$

Similar to the calculations taken out in (9.46) and (9.47) with the generalized harmonic numbers $\zeta_n(k)$ this is

$$
-\left\langle \exp\left(\sum_{k=1}^{\infty}\zeta_n(k)\frac{s^k}{k}\right)\right\rangle_{m,0} \tag{9.51}
$$

Now we can use once again the modified Bell polynomials and the facts, that $\zeta_n(k) = \zeta(k) + O\left(\frac{1}{n^{k-1}}\right)$ for $k \geq 2$ and $\zeta_n(1) = \ln(n) + \gamma + O(1/n)$, which follows from (9.31) with $\beta = 1$ and $\Gamma'(1) = \gamma\Gamma(1)$ to get:

$$
\begin{aligned}
-S_n(m) = \sum_{1m_1+2m_2+\ldots=m}\frac{1}{m_1!m_2!\ldots}\left(\frac{\zeta_n(1)}{1}\right)^{m_1}\left(\frac{\zeta_n(2)}{2}\right)^{m_2}\left(\frac{\zeta_n(3)}{3}\right)^{m_3}\cdots = \\
\left(1+O\left(\frac{1}{n}\right)\right)\sum_{1m_1+2m_2+\ldots=m}\frac{1}{m_1!m_2!\ldots}\cdot \\
\left(\ln(n)+\gamma+O\left(\frac{1}{n}\right)\right)^{m_1}\left(\frac{\zeta(2)}{2}\right)^{m_2}\left(\frac{\zeta(3)}{3}\right)^{m_3}\cdots
\end{aligned}
\tag{9.52}
$$

Since the $\zeta(k)$ are constants we have for a polynomial $P_m$ of degree $m$ the asymtotics

$$
-S_n(m) = P_m(\ln(n)) + O\left(\frac{(\ln(n))^m}{n}\right) \tag{9.53}
$$

Using the values of the $\zeta$ function, we get for the first values of $m$

$$
-S_n(1) = \ln(n) + \gamma + O\left(\frac{1}{n}\right) \tag{9.54}
$$

$$
-S_n(2) = \frac{1}{2}(\ln(n))^2 + \gamma\ln(n) + \frac{\gamma}{2} + \frac{\pi^2}{12} + O\left(\frac{\ln(n)}{n}\right) \tag{9.55}
$$

Moreover for $m = 1$ we get the exact result

$$
\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k} = -S_n(1) = \zeta_n(1) \tag{9.56}
$$

The above asymptotic equation can be generalized for $m \notin \mathbb{N}$, as we will see later. Another example is the sequence

$$
T_n = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k^2+1} \tag{9.57}
$$

This sequence obeys the recurrence

$$
T_0 = 1 \quad T_1 = \frac{1}{2} \quad T_n = \frac{n}{n^2+1}((2n-1)T_{n-1} - (n-1)T_{n-2}) \tag{9.58}
$$

which seams hard to solve or even estimate by conventional methods.
Since the sequence underlaying this differences can be interpolated by $\phi(s) = (1+s^2)^{-1} = \frac{1}{s-i} + \frac{1}{s+i}$, this allows us to directly applay (9.2) using trigonometric identities and the fact that $|\Gamma(z)| = |\Gamma(\overline{z})|$ to get

$$\Gamma(-i)n^i\left(1+O\left(\frac{1}{n}\right)\right) + \Gamma(i)n^{-i}\left(1+O\left(\frac{1}{n}\right)\right) =$$
$$\left(\Gamma(-i)e^{i\ln(n)} + \Gamma(i)e^{-i\ln(n)}\right)\left(1+O\left(\frac{1}{n}\right)\right) = \rho\cdot\cos(\ln(n)+\theta) + o(1) \quad (9.59)$$

for some $\theta$ and $\rho = 2\,|\Gamma(i)| = 2\sqrt{\pi/\sinh(\pi)} \approx 1.04313$.
This example shows, that complex poles introduce periodic behavior in the asymtotics of a sequence.

### 9.4.2 Meromorphic functions

Meromorphic functions are generalisations of rational function. Meromorphic functions are holomorphic on an certain domain except isolated singularities.

**Theorem 9.7.** *Let $\phi$ be a function holomorphic in a domain that contains the half-line $[n_0, \infty[$. If $n$ is big enough we have*

1. *If $\phi$ is meromorphic on $\mathbb{C}$ and of polynomial growth, then*

$$\sum_{k=n_0}^{n}\binom{n}{k}(-1)^k\phi(k) = -(-1)^n\sum_s \operatorname*{Res}_s\left(\phi(s)\frac{n!}{s(s-1)\dots(s-n)}\right) \quad (9.60)$$

*where the sum is taken over all poles of $\phi$ and over $0, 1, \dots, n_0 - 1$*

2. *If $\phi$ is meromorphic on the half-plane defined by $\Re(s) \geq d$ for some $d < n_0$ and of polynomial growth in this set, then*

$$\sum_{k=n_0}^{n}\binom{n}{k}(-1)^k\phi(k) = -(-1)^n\sum_s \operatorname*{Res}_s\left(\phi(s)\frac{n!}{s(s-1)\dots(s-n)}\right) + O(n^d) \quad (9.61)$$

*where the sum is taken over all poles but $n_0, n_0+1, \dots, n$*

*Proof.* Since in both cases the function is meromorphic, the number of poles are countable. So in the first case we can find positively oriented, concentric circles $\gamma_j$ whose radii tend to $\infty$ and do not come across any pole. In the second case we can find for any $\epsilon > 0$ a $d < d' < d + \epsilon$, such that the pathes defined by $[d' - iR_j, d' + iR_j]$ and the half circle with center $d'$ and radius $R_j$ don't cross a pole and $R_j$ tends to infinity. Since $d'$ is arbitrarily close to $d$, we can assume $d = d'$. Of course, if the theorem is used in practice, other types of paths can be used, when they have the essential properties stated and used here.
Now we integrate along these curves and get using the residue theorem similar to theorem 9.6 the results above. Since $\phi$ is of polynomial growth[10] we can use the arguments at (9.42) to get the asymtotics of the integrals over the "infinite" paths. In the first case we get 0 for $n$ big enough to overwhelm the polynomial growth. In the second case the $O(n^d)$ comes from the integral along the parallel to the imaginary

[10] in fact, it is only necessary to have polynomial growth on the union of the paths of integration – so we don't have to bother about poles or even infinitly many poles on our compact set, because we just circumnavigate them; in the second case the polynomial growth is even only needed beside a compact set, because the path of integration is not allowed to cross a pole and then holomorphic functions do attain their maximum on a compact set, so they can be estimated by a constant, which is trivially of polynomial growth

axes, as this argument illustrates (It's not a proof – it's just a plausibility argument; we asume $|\phi(s)| = O(|s|^r)$ for an integer $r$):

$$\left|\frac{(-1)^n}{2\pi i}\int_{d-i\infty}^{d+i\infty}\phi(s)\frac{n!}{s(s-1)\dots(s-n)}ds\right| \overset{(9.13)}{\leq}$$

$$\frac{1}{2\pi}\int_{d-i\infty}^{d+i\infty}\left|\phi(s)\frac{n!}{s(s-1)\dots(s-n)}\right| ds \leq$$

$$\frac{1}{2\pi}\int_{d-i\infty}^{d+i\infty}|s|^r\left|\frac{n!}{s(s-1)\dots(s-n)}\right| ds =$$

$$\frac{1}{2\pi}\int_{d-i\infty}^{d+i\infty}\frac{|s|^r}{|s(s-1)\dots(s-r+1)\cdot(s-r)(s-r-1)|}|(n(n-1)\dots(n-d+1))|$$

$$\left|\frac{(n-d)(n-d-1)\dots(r+2-d)}{(n-s)(n-s-1)\dots(r+2-s)}\right||(r+2-d)(r+1-d)(r-d)\dots 2\cdot 1|^{-1}\, ds \leq$$

$$O(n^d)\int_{d-i\infty}^{d+i\infty}\frac{1}{|s|^2}ds = O(n^d) \quad (9.62)$$

This rough approximation holds for $r+1-d<0$. For the other case, this argument is not applicable, although I think, that $O(n^d)$ holds even in this case. □

As our next example we want to analyze the recurrence relation

$$f_n = a_n + 2\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^n}f_k \quad (9.63)$$

Further we will assume that $a_0 = a_1 = 0$; this is without loss of generality. Then we use exponential generating function introduced in (9.37) to get

$$f(z) = \sum_{n=0}^{\infty} f_n\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(a_n + 2\sum_{k=0}^{n}\frac{1}{2^n}\binom{n}{k}f_k\right)\frac{z^n}{n!} =$$

$$\sum_{n=0}^{\infty}a_n\frac{z^n}{n!} + \sum_{n=0}^{\infty}\left(2\sum_{k=0}^{n}\frac{1}{2^n}\binom{n}{k}f_k\frac{z^n}{n!}\right) =$$

$$a(z) + 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{2^{n-k}(n-k)!}\frac{f_k}{2^k k!}z^{k+(n-k)}\right) =$$

$$a(z) + 2\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{f_n}{n!}\left(\frac{z}{2}\right)^n\right) = a(z) + 2e^{z/2}f\left(\frac{z}{2}\right) \quad (9.64)$$

This easily translates into the Poisson generating function via multiplication the whole equation by $e^{-z}$ to get

$$\hat{f}(z) = \hat{a}(z) + 2\hat{f}\left(\frac{z}{2}\right) \quad (9.65)$$

so for the coefficients $\hat{f}_n = n!\langle\hat{f}(z)\rangle_{n,0}$ we have

$$\hat{f}_n = \hat{a}_n + 2\frac{1}{2^n}\hat{f}_n \Rightarrow \hat{f}_n = \frac{\hat{a}_n}{1-2^{1-n}} \quad (9.66)$$

Since the equality

$$f_n = \sum_{k=0}^{n}\binom{n}{k}\hat{f}_n \quad (9.67)$$

holds, we get the identity respecting $a_0 = a_1 = 0$ and hence $\hat{a}_0 = \hat{a}_1 = 0$

$$f_n = \sum_{k=0}^{n}\binom{n}{k}\frac{\hat{a}_k}{1-2^{1-k}} = \sum_{k=2}^{n}\binom{n}{k}\frac{\hat{a}_k}{1-2^{1-k}} \quad (9.68)$$

To proof (9.67) we first show:

$$\hat{f}_n = n!\langle \hat{f}\rangle_{n,0} = n!\left\langle \sum_{n=0}^{\infty} f_n e^{-z}\frac{z^n}{n!}\right\rangle_{n,0} = n!\left\langle \sum_{n=0}^{\infty} f_n \sum_{k=0}^{\infty}(-1)^k\frac{z^k}{k!}\frac{z^n}{n!}\right\rangle_{n,0} =$$

$$n!\left\langle \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\frac{1}{j!}f_{i-j}\frac{z^j z^{i-j}}{(i-j)!}\right\rangle_{n,0} = n!\left\langle \sum_{n=0}^{\infty}\sum_{k=0}^{i}\frac{1}{k!}\frac{1}{n-k!}(-1)^k f_{n-k}z^n\right\rangle_{n,0} =$$

$$(-1)^n\left\langle \sum_{n=0}^{\infty}\sum_{k=0}^{i}\frac{n!}{k!(n-k)!}(-1)^k f_k z^n\right\rangle_{n,0} = (-1)^n\sum_{k=0}^{i}\binom{n}{k}(-1)^k f_k \quad (9.69)$$

Next we examine the righthand side of (9.67):

$$\sum_{k=0}^{n}\binom{n}{k}\hat{f}_k = \sum_{k=0}^{n}\binom{n}{k}(-1)^n\sum_{j=0}^{k}\binom{n}{j}(-1)^j f_j = \sum_{l=0}^{n}\sum_{m=l}^{n}(-1)^{l+m}\binom{n}{m}\binom{m}{l}f_l \quad (9.70)$$

Since we want this sum to be $f_n$, we have to show, that the inner sum is $\delta_{l,n}$[11]

$$\sum_{m=l}^{n}(-1)^{l+m}\binom{n}{m}\binom{m}{l} = \sum_{m=l}^{n}(-1)^{l+m}\frac{n!}{m!(n-m)!}\frac{m!}{l!(l-m)!} =$$

$$\sum_{m=l}^{n}(-1)^{l+m}\frac{n!}{(n-m)!(l-m)!l!} = \frac{n!}{l!}\sum_{m=l}^{n}(-1)^{l+m}\frac{1}{(n-m)!(l-m)!} =$$

$$\frac{n!}{l!}\sum_{k=0}^{n-l}(-1)^{2l+k}\frac{1}{(n-l-k)!k!} = \frac{n!}{l!(n-l)!}\sum_{k=0}^{n-l}(-1)^k\frac{(n-l)!}{(n-l-k)!k!} =$$

$$\binom{n}{l}\sum_{k=0}^{n-l}\binom{n-l}{k}(-1)^k(1)^{n-l-k} = \binom{n}{l}(1+(-1))^{n-l} = \binom{n}{l}\delta_{l,n} = \delta_{l,n} \quad (9.71)$$

Since (9.63) appears in the analysis of tries, the asymptotics of

$$U_n = \sum_{k=2}^{n}\binom{n}{k}\frac{\hat{a}_k}{1-2^{1-k}} \quad (9.72)$$

are of great interest. The $\hat{a}_k$ are usualy simple; for $n \geq 2$ we get for the $a_n = n-1$, which appears taking a closer look to tries, $\hat{a}_n = (-1)^n$. So we can apply theorem 9.7 with $f_k = (2^{k-1}-1)^{-1}$. We have infinitly many poles at $\chi_k = 1 + \frac{2\pi ik}{\ln 2}$.
We choose as path of integration circles centered at the origin, that avoid the poles and let the radius tend to $\infty$. The whole analysis is carried out in Knuth's "Art of computer programming"; we don't carry it out here, but the result is:

$$U_n = \frac{n}{\ln 2}\left(\ln(n) + \gamma - 1 - \frac{\ln 2}{2} + \sum_{k\in\mathbb{Z}\setminus\{0\}}\Gamma\left(-1-\frac{2\pi ik}{\ln 2}\right)e^{2\pi ik\ln(n)/\ln 2}\right) + O(1) \quad (9.73)$$

Since the $\Gamma$ function decreases rapidly along the imaginary axes the effects of the sum can almost be neglected and we get the simplyfied asymptotic

$$n\log_2(n) + nP(\log_2(n)) + O(1) \quad (9.74)$$

More generally regular spaced poles introduce disturbances, which asymptoticaly behave like Fourier series in $\ln(n)$, as seen her with $P$.

[11] This is the widly used Dirac $\delta$ symbol

Another example is the sum $V_n = \sum_{k=1}^{n-1} \binom{n}{k} \frac{B_k}{2^k-1}$, which arises in the analysis of Patricia tries. Since $B_k = 0$ for $k \in 2\mathbb{N}+1$, the sign is not necessary, and using $B_k = -k\zeta(1-k)$ for $k \in 2\mathbb{N} \cup \{1\}$, we have to analyze the integral

$$V_n = \frac{(-1)^n}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{n!}{(s-1)(s-2)\dots(s-n)} \frac{\zeta(1-s)}{2^s-1} ds \tag{9.75}$$

The path of integration is the infinite rectangle from $\frac{1}{2}-\infty$ to $\frac{1}{2}+\infty$ and from $n-\frac{3}{4}+\infty$ to $n-\frac{3}{4}-\infty$; but it can be shown that the integral of the second path is identical 0 for each $n$ and so the rectangle can be extended to $\infty$ therefor is a variant of the second part of theorem 9.7, where the residues are not yet evaluated.
Now we have to consider the double pole at 0 (from the $\zeta$ function and from $(2^s-1)^{-1}$) and all the simple poles at $\chi_k = 2\pi ik/\ln(2)$, which yields analog to the example befor

$$V_n = \frac{\ln(n)}{\ln(2)} - \frac{1}{2} - \frac{1}{\ln(2)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \zeta(1-\chi_k)\Gamma(1-\chi_k)e^{2\pi ik\ln(n)/\ln(2)} + O(1) \tag{9.76}$$

There are also some other examples, where Rice's integrals can be used succesfully; for example digital trees or quad trees used for multidimensional searching.
In the last example the extrapolating function was just given to us, but you can imagine that this would otherwise be a difficult task – especially with such not everyday occuring numbers as the Bernoulli numbers. If we have coefficients which are sums or products of other sequence $\alpha_k$ and we can interpolate the elements of this sequence by $\alpha(s)$, then we can use

$$A_n = \prod_{k=1}^{n} \alpha_k \Rightarrow A(s) = \prod_{k=1}^{\infty} \frac{\alpha(k)}{\alpha(k+s)} \quad A_n = \sum_{k=1}^{n} \alpha_k \Rightarrow A(s) = \sum_{k=1}^{\infty} (\alpha(k) - \alpha(k+s)) \tag{9.77}$$

### 9.4.3 Functions with Algebraic and Logarithmic Singularities

Now we turn to general algebraic and logarithmic functions. The problem with these function is that they can not to defined on $\mathbb{C}$, even not with some pointwise exceptions, but only with some uncountable exceptions. For example the complex extension of the logarithm and the root are defined on $\mathbb{C} \setminus ]-\infty, 0]$[12]. For this reason we can not simply integrate around the points, where the functions are not defined, because every circle would be "slashed", but we have to use the so called Hankel contours to get our integration done.
Because of the difficulties that arise with this class of functions, we will only consider examples; but the first in more detail. Since we are familiar with the sequence $\frac{1}{k^m}$ from (9.49), where $m$ was an integer, we try to generalise this to arbitrary $\lambda$, as we indicated before.

**Theorem 9.8.** *For any nonintegral $\lambda$, the sum*

$$S_n(\lambda) = \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^{-\lambda} \tag{9.78}$$

*has an asymptotic expansion in descending powers of* $\ln(n)$ *of the form*

$$-S_n(\lambda) = (\ln(n))^\lambda \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma^{(j)}(1)}{j!\Gamma(1+\lambda-j)} \frac{1}{(\ln(n))^j} \tag{9.79}$$

[12] People having done some complex analysis know, that there are some means to make this restriction a little bit more flexible, but you will never get rid of a malicious slash, which cuts into the complex plane

*Proof.* As seen in (9.50) we can represent the sum as an integral in an analoge way

$$S_n(\lambda) = \frac{1}{2\pi i}\int_{\mathcal{C}} \omega_n(s)\frac{ds}{s^{\lambda+1}} \text{ with } \omega_n(s) = \left(\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\dots\left(1-\frac{s}{n}\right)\right)^{-1} \tag{9.80}$$

Here $\mathcal{C}$ may be the vertical line $\Re(s) = \frac{1}{2}$ as in (9.42). Despite all, these integrals would be hard to evaluate; since we are only interested to encircle the poles (here only $1, 2, \dots, n$) we can deformate the path of integration as we like unless we don't encounter any new pole or the slash. For this reason we introduce the Hankel contour:

$$\begin{aligned}
\mathcal{C} &= \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 \\
\mathcal{C}_1 &= \left\{ s \middle| |s| = R \wedge \left( |\Im(s)| \geq \frac{1}{\ln(n)} \vee \Re(s) > 0 \right) \right\} \\
\mathcal{C}_2 &= \left\{ s \middle| s = \frac{i-t}{\ln(n)} \wedge t \geq 0 \wedge |s| \leq R \right\} \\
\mathcal{C}_3 &= \left\{ s \middle| s = \frac{e^{\theta i}}{\ln(n)} \wedge \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\} \\
\mathcal{C}_4 &= \left\{ s \middle| s = \frac{-i-t}{\ln(n)} \wedge t \geq 0 \wedge |s| \leq R \right\}
\end{aligned}$$

This is a circle of radius $R > n$ around the origin, that circumvents the slash by leaving some space around it; but as $n$ increases the slash is left less space, so that if $n$ tends to $\infty$ any point in the domain of $s^{-\lambda}$ will be encircled.
Now we split the integral in three parts $S_n(\lambda) = J_1 + J_2 + J_3$; we will see that for the asymptotic $J_1$ and $J_2$ can be neglected.

1. Let $J_1$ be the part, that belongs to the outer "circle" $\mathcal{C}_1$. Of course $s^{-\lambda-1}$ is of polynomial growth, so we can once more use our estimate for polynomial growth, to see that $J_1$ is $O(R^{-n-\lambda})$ and in the end, when $R \to \infty$, we get $J_1 = 0$

2. Now we will estimate the parts of $\mathcal{C}_2$ and $\mathcal{C}_4$ with $\Re(s) < -\frac{1}{\sqrt{\ln(n)}} =: -t_0$. For simplification, we forget about the term $s^{-\lambda-1}$ and asume we integrate along the negative real axes till we reach $-t_0$[13]. This simplifications do not touch the character of our estimate, but make it simpler. Additionaly, we mirror everything on the imaginary axes, to get:

$$\mu(n) := \int_{t_0}^{\infty} \frac{dt}{(1+\frac{t}{1})\dots(1+\frac{t}{n})} \tag{9.81}$$

As next step we split the new path into the intervalls $[t_0, 1]$, $\left[1, n^{1/3}\right]$ and $\left[n^{1/3}, \infty\right]$, to get the integrals $\mu_1(n)$, $\mu_2(n)$ and $\mu_3(n)$

[13]We asume $\lambda + 1 \geq 0$ and then it is quiet obvious, that we can neglect the powers of $s$ for the asymptotic analysis, because they would make the asymptotic only smaller; but for example if $\lambda$ are negative integers we get the Stirling numbers of second kind, which have normaly exponential growth – our derivation can not be applayed in this case

(a)

$$\mu_1(n) = \int_{t_0}^{1} \frac{dt}{(1+t/1)\dots(1+t/n)} = \int_{t_0}^{1} \frac{n!dt}{(1+t)(2+t)\dots(n+t)} =$$
$$\int_{t_0}^{1} \frac{\Gamma(n+1)\Gamma(t+1)}{\Gamma(n+t+1)} dt \overset{(9.26)}{=}$$
$$\int_{t_0}^{1} \frac{\sqrt{2\pi}(n+1)^{n+1-1/2}e^{-n-1}\sqrt{2\pi}(t+1)^{t+1-1/2}e^{-t-1}}{\sqrt{2\pi}(n+t+1)^{n+t+1-1/2}e^{-n-t-1}} \left(1+O\left(\frac{1}{n}\right)\right) dt \leq$$
$$\int_{t_0}^{1} \frac{\sqrt{2\pi}(t+1)^{t+1-1/2}}{e(n+t+1)^t} \left(1+O\left(\frac{1}{n}\right)\right) dt = O(1) \int_{t_0}^{1} \frac{(t+1)^{t+1-1/2}}{(n+t+1)^t} dt =$$
$$O(1) \int_{t_0}^{1} \frac{1}{(n)^t} dt \quad (9.82)$$

The last few transformation can be carried out because $t \in [t_0, 1]$, so is small compared to $n$ and can be ignored or estimated by a constant respectivly.

$$O(1) \int_{t_0}^{1} \frac{1}{(n)^t} dt = O(1) \left(\frac{-1+n^{1-t_0}}{n\ln(n)}\right) = O(e^{-t_0 \ln(n) + \ln(\ln(n))})$$
$$= O(e^{-1/2\sqrt{\ln(n)}}) \quad (9.83)$$

(b) In the next part we use, that $1 \leq t$

$$\mu_2(n) = \int_{1}^{n^{1/3}} \frac{n!}{(1+t)\dots(n+t)} dt =$$
$$\int_{1}^{n^{1/3}} 1 \cdot 2 \cdot \frac{3}{2+t} \frac{4}{3+t} \dots \frac{n}{n-1+t} \frac{1}{(1+t)(n+t)} dt \leq$$
$$\int_{1}^{n^{1/3}} O(1) \frac{1}{(1+t)(n+t)} dt = O\left(\frac{\ln(n^{1/3}+n)}{n-1}\right) = O(n^{-2/3}) \quad (9.84)$$

(c) Similar to the estimates done in the formula above we get, for $n$ large enough

$$\mu_3(n) = \int_{n^{1/3}}^{\infty} \frac{1}{(1+t/1)\dots(1+t/n)} dt \leq$$
$$\int_{n^{1/3}}^{\infty} \frac{1}{e^{(1/2)t}} dt = O(e^{-(1/2)n^{1/3}}) \quad (9.85)$$

In the whole we have the result $J_2 = O(e^{(1/2)\sqrt{\ln(n)}})$, so it is of smaller order than any negativ power of $\ln(n)$

3. Now we have to estimate $J_3$; this is the integral along the contour of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$, for which $\Re(s) \geq -t_0$, denoted in the following by $\mathcal{C}^0$.
Again we use Stirlings formula (9.26) to get the asymptotic:

$$\omega_n(s) = n^s \Gamma(1-s) \left(1 + O\left(\frac{\ln(n)}{n}\right)\right) \quad (9.86)$$

Since $s$ is very small on $\mathcal{C}^0$ the part with $O\left(\frac{\ln(n)}{n}\right)$ can be estimated easily by $O(e^{(1/2)\sqrt{\ln(n)}})$ and this yields

$$J_3 = J^0 + O(e^{(1/2)\sqrt{\ln(n)}}) \text{ with } J^0 = \frac{1}{2\pi i} \int_{\mathcal{C}^0} n^s \Gamma(1-s) \frac{ds}{s^{\lambda+1}} \quad (9.87)$$

Now we use the transformation $z = s\ln(n)$ with $\mathcal{D}^0$ is the image under this transform of $\mathcal{C}^0$ and have

$$J^0 = (\ln(n))^\lambda \frac{1}{2\pi i}\int_{\mathcal{D}^0} e^z \Gamma\left(1 - \frac{z}{\ln(n)}\right)\frac{dz}{z^{\lambda+1}} \tag{9.88}$$

By the transform we get $|z| = O(\sqrt{\ln(n)})$ on $\mathcal{D}^0$; this is the reason, we can expand the Gamma function around 1 in a power series and after changing summation and integration, we get

$$J^0 = (\ln(n))^\lambda \sum_{m=0}^{\infty}(-1)^m \frac{\Gamma^{(m)}(1)}{m!}\frac{1}{(\ln(n))^m}\frac{1}{2\pi i}\int_{\mathcal{D}^0} e^z z^{m-\lambda-1}dz \tag{9.89}$$

Now we have to estimate the remaining integrals; this can be done by the so called Laplace method, where we extend the contour to $-\infty$ to get $\mathcal{L}$. We will only present the result here:

$$\frac{1}{2\pi i}\int_{\mathcal{L}} e^z z^{m-\lambda-1}dz = \frac{1}{\Gamma(1-m+\lambda)} \tag{9.90}$$

Now, taking together all the parts, we have prooven the theorem. □

As an application, we can examine the sum

$$X_n = \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{\sqrt{1+k^2}} \tag{9.91}$$

We have the local behvior $(s \pm i)^{-1/2}$ at the "problem points", so we get an asymptotic growth of $\sqrt{\ln(n)}$. Similar to (9.59) we have for some $\rho$ and $\theta_0$

$$X_n = \rho\sqrt{\ln(n)}\cos(\ln(n)+\theta_0) + O((\ln(n))^{-1/2}) \tag{9.92}$$

Other examples and direct applications are

$$-S_n(-1/2) = \frac{1}{\sqrt{\pi\ln(n)}} - \frac{\gamma}{2\sqrt{\pi(\ln(n))^3}} + O((\ln(n))^{-5/2}) \tag{9.93}$$

$$-S_n(1/2) = 2\sqrt{\frac{\ln(n)}{\pi}} + \frac{\gamma}{\sqrt{\pi\ln(n)}} + O((\ln(n))^{-3/2}) \tag{9.94}$$

In general the coefficients are rational expressions of terms as $\gamma$, $\Gamma(-\lambda)$ and $\zeta(2)$, $\zeta(3)$, ...
For the rest of this part, we will only state some more examples, where the method of Rice's integrals (perhaps with use of the Hankel contour) can be applied succesfully.

**Theorem 9.9.** *For the logarithmic differences we have the asymptotics*

$$Y_n = \sum_{k=1}^{n}\binom{n}{k}(-1)^k\ln(k) = \ln(\ln(n))+\gamma+\frac{\gamma}{\ln(n)}-\frac{\pi^2+6\gamma^2}{12(\ln(n))^2}+O\left(\frac{1}{(\ln(n))^3}\right) \tag{9.95}$$

The method can also be used for entire functions, which have no poles at all (despite the artificial ones introduced by the kernel ...). For example for

$$Z_n = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!} \tag{9.96}$$

which obeys the recurrence $Z_{n+2} = (2-2/n)Z_{n+1} + (1-1/n)Z_n$ can be extrapolated by the entire function $1/\Gamma(s)$, and after carrying out Rice's method it reveals for some constants $c$ and $\theta$

$$Z_n = cn^{-1/4}\sin(2n^{1/2}+\theta) + o(n^{-1/4}) \tag{9.97}$$

We have seen many cases were heavy use of complex analysis can resolve the asymptotics of recurrences, generalized differences and sums. We summarize all this in the table below.

Some types of singularities and the asymptotics they introduce in the corresponding difference

| singularity | asymptotics |
|---|---|
| singularity of $\phi(s)$ at $s_0 = \sigma_0 + i\tau$ | approximately $n^{s_0} = n^{\sigma_0}e^{i\tau_0\ln(n)}$ |
| simple pole: $(s-s_0)^{-1}$ | $-\Gamma(-s_0)n^{s_0}$ |
| multiple pole: $(s-s_0)^{-r}$ | $-\Gamma(-s_0)n^{s_0}\frac{(\ln(n))^{r-1}}{(r-1)!}$ |
| algebraic singularity: $(s-s_0)^{\lambda}$ | $-\Gamma(-s_0)n^{s_0}\frac{(\ln(n))^{-\lambda-1}}{\Gamma(-\lambda)}$ |
| logarithmic singularity: $(s-s_0)^{\lambda}(\ln(s-s_0))^r$ | $-\Gamma(-s_0)n^{s_0}\frac{(\ln(n))^{-\lambda-1}}{\Gamma(-\lambda)\ln(\ln(n))^r}$ |

## 9.5 Mellin Transforms and Rice's Integrals

The Mellin transform of a function and its inverse have the form

$$\phi(z) = \int_0^\infty t^{z-1}f(t)dt \qquad f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-z}\phi(z)dz \tag{9.98}$$

If we take a closer look to the integral (9.42) and take into account the asymptotic of $\omega_n(s)$ (9.86)[14] we get:

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\phi(s)\frac{(-1)^n n!}{s(s-1)\dots(s-n)}ds \approx \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\phi(s)\Gamma(-s)n^s ds \tag{9.99}$$

If we would change the sign of the variable and then compare the result with the inverse Mellin transform we observe a definite analogy. Without stating it formally, in cases, were we want to evaluate Rice's integrals and we get stuck with it, it can be worth a try to evaluate the corresponding inverse Mellin transform to get an idea about the size of the asymptotics.
But the similarity can be stated formally as the Poisson-Mellin-Newton cycle.

**Theorem 9.10.** *The coefficients of a Poisson generating function are expressible as a Rice's integral of a Mellin transform of the Poisson generating function.*

$$\{f_n\} \xrightarrow{\textit{Poisson GF}} \hat{f}(t) = \sum_{n=0}^{\infty} f_n e^{-t}\frac{t^n}{n!} \xrightarrow{\textit{Mellin transform}} \hat{f}^*(s) = \int_0^\infty \hat{f}(t)t^{s-1}dt \xrightarrow{\textit{Rice's integral}} \{f_n\} \tag{9.100}$$

*Proof.* We take a closer look to the Mellin transform and state a Newton series to have

$$\hat{f}^*(s) = \int_0^\infty \hat{f}(t)t^{s-1}dt = \sum_0^\infty \frac{f_n}{n!}\int_0^\infty e^{-t}t^{s+n-1}dt \overset{(9.23)}{=} \Gamma(s)\left(f_0 + f_1\frac{s}{1!} + f_2\frac{s(s+1)}{2!} + \dots\right) \tag{9.101}$$

[14]don't forget about the additional s in the denominater, so we realy get $\Gamma(-s)$ and not $\Gamma(1-s)$, after changing sign

By differencing[15] we get to the following formula

$$f_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\hat{f}^*(-s)}{\Gamma(-s)} \tag{9.102}$$

But these differences are exactly of the Rice type, so we conclude the corresponding equations

$$f_n = \frac{(-1)^n}{2\pi i} \int_{\mathcal{C}} \left( \frac{\hat{f}^*(s)}{\Gamma(-s)} \frac{n!}{s(s-1)\dots(s-n)} \right) ds \tag{9.103}$$

$$\hat{f}^*(s) = \int_0^\infty \left( e^{-t} \sum_{n=0}^{\infty} f_n \frac{t^n}{n!} \right) t^{s-1} dt \tag{9.104}$$

This are the relations, which were stated. □

For example if we carry out the Mellin transform of the Poisson generating function (9.65) we have $\hat{f}^*(s) = \frac{\hat{a}^*(s)}{1-2^{1+s}}$ and from the formulas above we get the result

$$f_n = \sum_{k=0}^{n} \binom{n}{k} \frac{\hat{a}^*(-k)}{\Gamma(k)} \frac{(-1)}{1-2^{1-k}} \tag{9.105}$$

This is formally the same result as if we had carried out the Rice's method.
There are some other examples as digital search trees, were the Poisson-Mellin-Newton cycle can be applied. This is a hint that the formal result we get from the cycle can be developed to an actual result by the Rice's integrals. The cycle is also an explanation for some other phenomena, but this would lead too far here.

## 9.6 Summary

Despite complex analysis seems to be part of pure mathematics, it can be applied for finding asymptotics of solutions of difference equations and generalized differences, which are needed in the analysis of algorithms. By the method of Rice's integrals we can tackle the average case analysis of tries, digital search trees, multidimensional searching and other datastructures and algorithms of great practical use. In most cases a detailed calculation of the asymptotics is in fact much too complex, but you can get a first estimate of the growth by comparing the problem with the examples outlined here and the table given at the end of the section befor the last.

[15] normaly we would differenciate, but here we have not a series in powers of $s$ like $s^n$ but a series in $s(s+1)(s+2)\dots$; since differencing leads to success with $s(s-1)(s-2)\dots$, we have to play a little bit with the sign and get a result