Optimization problems

$\Pi = \langle \mathcal{I}, \mathcal{S}ol, val, goal \rangle$ such that

- $\mathcal{I} \subseteq \Sigma^*$ is the set of instances;
- For every instance $I \in \mathcal{I}$, $\mathcal{S}ol(I)$ denotes the set of solutions of I and is nonempty.
- For every instance I and solution $x \in \mathcal{S}ol(I)$, $val(I, x)$ denotes a positive integer. The function $val(\cdot, \cdot)$ is called the objective function.
- $goal \in \{\text{MIN}, \text{MAX}\}$.

Optimization problems (cont.)

An optimization problem belongs to the class $\mathcal{NPO}$, iff

- $\mathcal{I}$ is recognizable in polynomial time.
- The size of the solutions is polynomially bounded in the length of I, *i.e.*, there exists a polynomial p such that

$$|x| \le p(|I|) \qquad \text{for all } I \in \mathcal{I} \text{ and } x \in \mathcal{S}ol(I).$$

- The question "*Is* $x \in \mathcal{S}ol(I)$*?*" is decidable in polynomial time.
- The function val is computable in polynomial time.

Approximation algorithms

An approximation algorithm for an optimization problem $\Pi = \langle \mathcal{I}, \mathcal{S}ol, val, goal \rangle$ is an algorithm $\mathcal{A}$ which computes for each instance $I \in \mathcal{I}$ (in polynomial time) a solution $x_{\mathcal{A}} \in \mathcal{S}ol(I)$. We denote the value of the objective function for the solution obtained by $\mathcal{A}(I) := val(I, x_{\mathcal{A}})$.

The approximation algorithm $\mathcal{A}$ has performance ratio $1 + \rho$

$$:\Leftrightarrow \quad \frac{1}{1+\rho} \leq \frac{\mathcal{A}(I)}{\mathsf{OPT}(I)} \leq 1 + \rho \quad \text{for all instances } I \in \mathcal{I}.$$

Optimization Classes

$\mathcal{APX}$:= optimization problems which have a polynomial time approximation algorithm with performance ratio $1 + \rho$ for some $\rho > 0$.

$\mathcal{PAS}$:= optimization problems which have a polynomial time approximation algorithm with performance ratio $1 + \rho$ for all $\rho > 0$ gibt.

$\mathcal{FPAS}$:= optimization problems which have an approximation algorithm with running time $poly(n, 1/\rho)$ and performance ratio $1 + \rho$ for all $\rho > 0$.