Technische Universität München
Department of Computer Science
Chair for Efficient Algorithms
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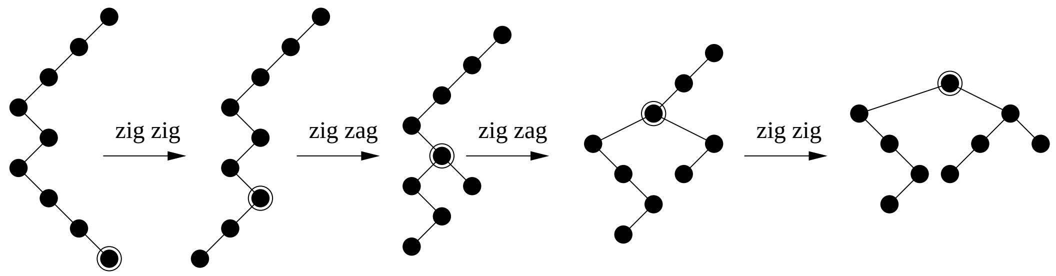
WS 1999/2000
Solutions 1
May 23, 2000

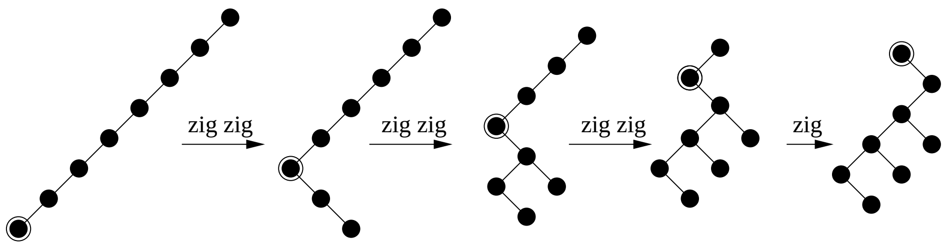
Efficient Algorithms and Data Structures II

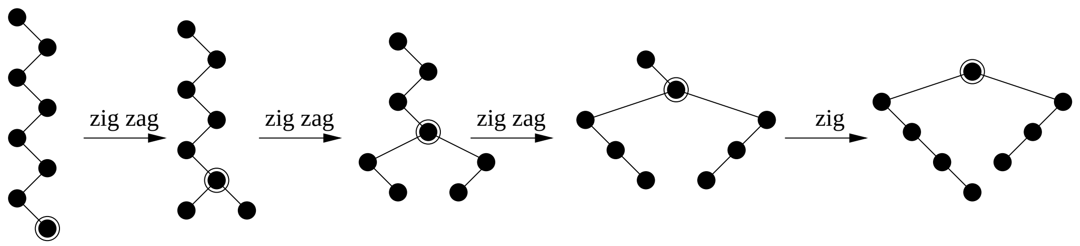
Problem 1 (10 credits)

In each case, show the tree after the circled node is splayed to the root node.

Solution.



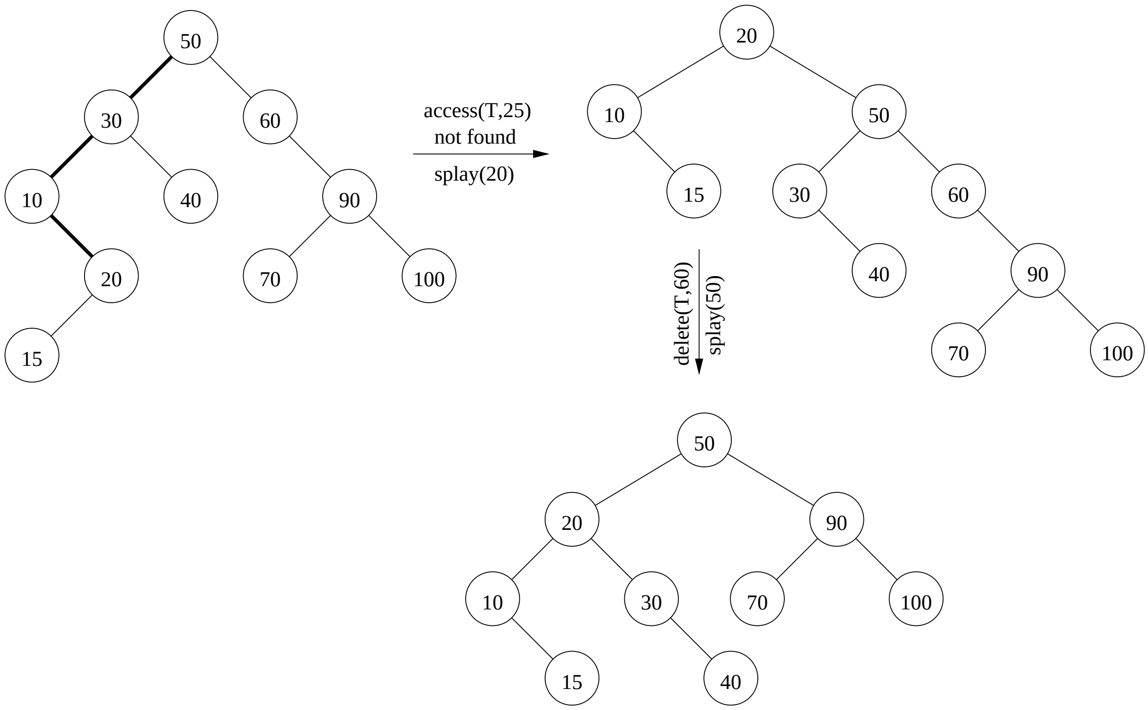




Problem 2 (8 credits)

Show the tree after the execution of access(T,25) followed by delete(T,60).

Solution.



Problem 3 (10 credits)

Prove that the chromatic number of an interval graph equals its clique number.

Solution.

Let $\chi(G)$ denote the chromatic number of graph G, that is, at least $\chi(G)$ colors are required to color G properly, and let $\omega(G)$ denote the clique number of G, that is, no clique in G has more than $\omega(G)$ nodes.

Obviously, $\chi(G) \geq \omega(G)$, as to properly color a clique on $\omega(G)$ nodes one needs at least as many colors.

Now for showing an upper bound on $\chi(G)$, we exploit the fact that in this case G is an interval graph. In an interval graph a node v can be represented by an interval I_v on the real axis. Two nodes u and v are adjacent iff $I_u \cap I_v \neq \emptyset$. A clique number of $\omega(G)$ implies that there are never more than $\omega(G)$ intervals sharing a single point on the real axis.

Now consider the following coloring algorithm. Sort the intervals by starting point and color them in this order. A new interval is colored with the smallest possible free color. If more than one interval starts at the same point, assign the smallest free colors in arbitrary order. Clearly, this algorithm produces a proper coloring. Furthermore, if a new interval is to be assigned a color, we know that at most $\omega(G) - 1$ other intervals are "active" at this point (an interval is active at some point if this point is part of the interval), for otherwise we would have a clique of size $\omega(G) + 1$ now. Analogously, if ℓ new intervals are to be colored at the same time, $1 \leq \ell \leq \omega(G)$, we know that there are at most $\omega(G) - \ell$ intervals active, so we can assign colors from $\{1, \ldots, \omega(G)\}$ to every new interval. Obviously, this algorithm produces a proper coloring with at most $\omega(G)$ colors, stating $\chi(G) \leq \omega(G)$. This finishes the proof.

Problem 4 (10 credits)

Show that the chromatic index of odd complete graphs, K_n, where $n = 2m+1$ with $m \geq 1$, is equal to n. In other words, show that $\chi'(K_n) = n$.

Solution.

Let us first state a lower bound for $\chi'(K_n)$. If we are to assign a color to some set of edges, these edges must not be incident, that is, no two edges may share an endpoint. Therefore, finding a set of edges which can be colored identically is nothing but determining a matching. If we now find a maximum matching (w.r.t. to cardinality), then we have found as many edges as possible which may share a single color, since the addition of any other edge would lead to the loss of the no-incident-edges property. The basic idea of this part of the proof is now to find one maximum matching after the other, to color the edges belonging to the i-th matching with color i, and then to remove the edges of the current matching. This is repeated until there are no edges left, that is, all edges are colored.
Obviously, the size of a maximum matching in K_n, n odd, is $(n-1)/2$, and the number of edges in K_n is $\binom{n}{2} = n(n-1)/2$. Therefore, we need to find at least

$$\left\lceil \frac{n(n-1)/2}{(n-1)/2} \right\rceil = n$$

matchings until all edges are removed, which implies that $\chi'(K_n) \geq n$.
Note that this is true in general. If $m^*(G)$ denotes the size of a maximum matching in some graph $G = (V_G, E_G)$, then one needs at least

$$\left\lceil \frac{|E_G|}{m^*(G)} \right\rceil$$

colors to properly edge-color G.
Now we show that $\chi'(K_n) \leq n$ for n odd. Draw the K_n s.t. its nodes labeled with $0, \ldots, n-1$ lie clockwise on a circle. See Figure **??** for an example with K_7.

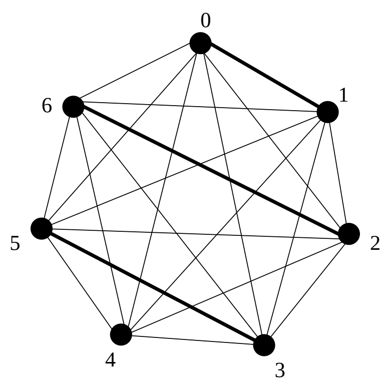


Figure 1: K_7

Now consider the edge between nodes 0 and 1. This edge is parallel to all edges whose endpoint node labels sum up to $1 (\bmod 7)$, which are represented by bold lines in the figure. Since these edges are not incident, we can assign the same color to all of them, say, color 1.

Analogously, edges whose endpoint node labels sum up to, say, $3(\text{mod}7)$ (these are $(1,2)$, $(0,3)$, and $(6,4)$) are not incident, so we can color all of them with color 3.
In general, we have n sets $S_0, \ldots, S_{n-1}$, where S_k contains edges whose endpoint node labels sum up to $k(\text{mod}n)$, and which are not incident. It is rather easy to see why they cannot be incindent. Consider some edge $(i,j) \in S_k$ with $i+j = k(\text{mod}n)$. Now for another edge to be incident to (i,j), it must share an endpoint with (i,j). W.l.o.g., assume that this endpoint is node i. Therefore, this edge must be of the form (i,x) with $x \neq j$ and $i + x = k(\text{mod}n)$ (of course, it has to be in the same set), and consequently $(i+j) = (i+x) \pm a \cdot n \Leftrightarrow x = j \pm a \cdot n$ for some integer $a \geq 1$. But this is impossible since $j \in \{0, \ldots, n-1\}$, and therefore either $x < 0$ or $x > n-1$. Consequently, all edges contained in some S_k are not incident, and we have constructed a proper edge-coloring for K_n with n colors, implying $\chi'(G) \leq n$. This finishes the proof.