Technische Universität München
Department of Computer Science
Chair for Efficient Algorithms
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Solutions 7
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Efficient Algorithms and Data Structures II

Problem 1 (10 credits)

Five men and five women are attending a dance. Katie will dance only with Garry or Harry, Barbara will dance only with Fred or Ignatz, Carol will dance only with Harry or Jack, Donna will dance only with Fred or Gary, and Emma will dance only with Gary or Ignatz.
Is it possible for all ten people to dance the last dance so that each woman dances with someone she finds acceptable?
[From Graph Theory and its Applications by J. Gross & J. Yellen]

Solution.

Consider a bipartite graph $G = (L \cup R, E)$, with L as the set of women, and R as the set of men. The edges $e \in E$ represent the women's choices. Obviously, theach node $v \in L$ has a degree of two, whereas in R, three nodes have a degree of two, one has a degree of three (Gary), and one has a degree of one (poor Jack). Now for a perfect matching in G to exist, Hall's condition

$$\forall X \subseteq L: \ |N(X)| \geq X$$

must hold. Now consider any subset $X \subseteq L$. Clearly, there are $2|X|$ edges leaving X, which have to find endpoints in $N(X)$. $N(X)$ can contain at most one node with a degree of three, the remaining nodes have a degree of at most two. Hence, $N(X)$ consists of at least $1 + \frac{2|X|-3}{2} = 1 + |X| - 3/2 = |X| - 1/2$ nodes (under the worst case assumption that the degree-3 node is indeed member of $N(X)$). As there are no half nodes, we have etsablished $|N(X)| \geq X$.

Problem 2 (10 credits)

Suppose that there are four jobs to be performed and five workers who are qualified for every job. The time in hours each worker needs to do each job is given in the table below. Is it possible to assign each job to a different worker so that each worker starts at the same time, and all four jobs are completed within four hours?

	Job1	Job2	Job3	Job4
W1	3	7	5	8
W2	6	3	2	3
W3	3	5	8	6
W4	5	8	6	4
W5	6	5	7	3

[From Graph Theory and its Applications by J. Gross & J. Yellen]

Solution.

Obviously, we need only consider entries with value at most 4, resulting in

	Job1	Job2	Job3	Job4
W1	3			
W2		3	2	3
W3	3			
W4				4
W5				3

Now construct a bipartite graph $G = (L \cup R, E)$ with $L = \{W_1, \ldots, W_5\}$ and $R = \{J_1, \ldots, J_4\}$. Throw in an edge for each entry of the modified table above. We now want to know whether there is a right-complete matching for G, i.e., every node of R is matched (but of course not every node of L, but $|R| = 4$ of them).
This is not possible. For instance, consider $\{J_2, J_3\} \subset R$. Both of them have just one edge to $W_2 \in L$. Hence, Hall's condition is not fullfilled.

Problem 3 (20 credits)

Suppose that in a given town there are n girls and n boys. Each girl and each boy want to get married, so they rank the members of the opposite sex according to their preference. They submit their preference list to the matchmaker, who then decides who is to marry whom, and finally a town-wide wedding takes place.

A marriage is said to be *unstable* if there is at least one girl and one boy (call them Alice and Bob) such that Alice prefers Bob to her fiance and Bob prefers Alice to his fiancee. In such a case, Alice and Bob will have a tendency to leave their current mates to get engaged to each other. We call the pair Alice-Bob a *dissatisfied pair* under this marriage. A marriage in which there are no dissatisfied pairs is called a *stable marriage*.

In this problem, we describe a greedy algorithm that the matchmaker runs to guarantee that the resulting marriage is stable.

a) Consider the following sketch for the matchmaker algorithm, also called "boy proposes, girl disposes" proposal scheme:
While there are still boys who are not engaged, pick an unattached boy (call him Bob). Bob then proposes to the most desirable girl on his list who has not already rejected him (call this girl Alice). Alice will accept Bob's proposal if she is currently unattached or if her current mate (say Fred) is less desirable to her than Bob. In the latter case, poor Fred is jilted and he reverts back to the unattached state.
This process continues until all the boys get engaged.
Prove that the above algorithm terminates and outputs a marriage.

b) For the algorithm described above, prove that the marriage that results from it is stable.

c) Suppose that the matchmaker now uses the same algorithm but switches around the roles of the boys and girls, so that now it is the girls who propose marriage to the boys.
Show, by giving a counterexample, that this new scheme will not always give the same marriage as the one described in part a.

[From MIT: Goldwasser & Micali]

Solution.

a) Note that each boy proposes to at most n girls (until he reaches the end of his list). Hence, there are at most n^2 proposals all in all. Further, by definition of the protocol, in each round at least one boy proposes to at least one girl, so the reservoir of initially n^2 proposals is reduced by at least one proposal per round. We can concude that the algorithm terminates after at most n^2 rounds.

   Next, note that after any round, the numbers of enganged women and men are equal. Once a woman is engaged, she remains so until the end of the protocol (not necessarily with the same boy, though). Now assume that after termination of the algorithm, one boy (Bob) and one woman (Carol) are not engaged. Is this possible? No. If Carol really is not engaged, this implies that no-one ever has proposed to her. But Bob would have done so some time, even if Carol were at the bottom of his list. Hence, Carol would have remained engaged until the end. This contradicts our assumption.

b) Suppose that after the end of the algorithm, Bob&Carol and Ted&Alice are engaged. The marriage is unstable if Bob prefers Alice over Carol, and Alice prefers Bob over Ted. In this case, Bob would have proposed to Alice before Carol. There are two possibilities:

   1. Alice has accepted Bob's proposal. Then, why aren't they engaged any more? Alice has dumped Bob for some other. Since a girl only dumps in order to get engaged with someone she likes more than her current mate, we have that Alice prefers Ted over Bob.
   2. Alice has not accepted Bob's proposal. This can only be true if she prefers her current mate over Bob. Since Ted is Alice's mate at the end of the algorithm, this again implies that she prefers Ted over Bob.

c) We will give a small example. Suppose that there are men Bob and Ted, and there are women Alice and Bob. Assume the following preference lists:

$$\begin{aligned}
&\text{Bob} &&: (\text{Alice, Carol}) \\
&\text{Ted} &&: (\text{Carol, Alice}) \\
&\text{Alice} &&: (\text{Ted, Bob}) \\
&\text{Carol} &&: (\text{Bob, Ted})
\end{aligned}$$

Now if the men propose, then Bob proposes to Alice, and Ted proposes to Carol. Obviously, both pairs become engaged, the men are happy, and the women are grumbling. On the other hand, if the women propose, then Alice proposes to Ted, and Carol proposes to Bob. Again, both pairs become engaged, but now the women are happy.