14 Wireless Overlay Networks II

In this section we focus on constructions for sector-based spanners and planar spanners.

14.1 Sector-based spanners

The basic idea underlying the Yao graphs is to cut the space around each node into sectors of equal angle θ and to connect each node to the nearest neighbor in each of its sectors (see Figure 1). As we will see, this will give a relative neighborhood graph if θ is sufficiently small. For any pair of nodes u, v, let $C_{u,v}$ denote the sector (or cone) of u containing v.

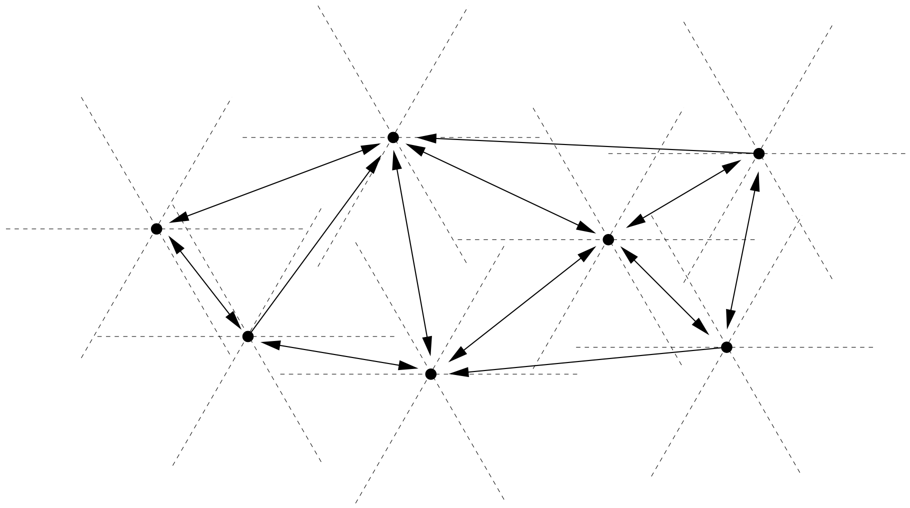

Figure 1: An example of a Yao graph.

Definition 14.1 *Consider any finite $V \subset \mathbb{R}^2$ and let $k \in \mathbb{N}$. Suppose that the space around every node $v \in V$ is cut into k sectors with angle $\theta = 2\pi/k$. Then the* Yao graph *$YG_\theta(V)$ of V consists of the following set of edges:*

$$E = \{(u, v) \mid u, v \in V \textit{ and there is no } w \in V \textit{ with } w \in C_{u,v} \textit{ and } ||uw|| < ||uv||\} .$$

We start with a basic property of Yao graphs.

Theorem 14.2 *If $\theta = 2\pi/k$ with $k > 6$, then $YG_\theta(V)$ is a RNG.*

Proof. Follows immediately from Lemma 14.4. □

The theorem immediately implies that Yao graphs with $k > 6$ are weak spanners. But they are more than that, as shown in the next theorem.

Theorem 14.3 *If $\theta = 2\pi/k$ with $k > 6$, then $YG_\theta(V)$ is a geometric spanner with stretch factor at most*

$$\frac{1}{1 - 2\sin(\theta/2)} .$$

Proof. We first need a lemma.

Lemma 14.4 *Let $p \in \mathbb{R}^2$ be a point and C be a sector originating at p. Furthermore, let q and r be two points in C with $||pq|| \leq ||pr||$. Then $||qr|| \leq ||pr|| - (1 - 2\sin(\theta/2))||pq||$.*

Proof.

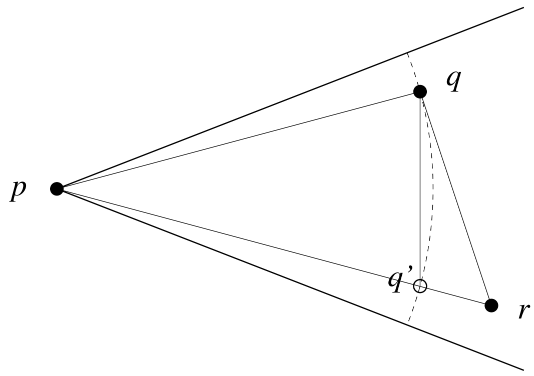

Figure 2: The sector of p that contains r.

Consider Figure 2. In this figure, q' represents the point on the line from p to r with the same distance to p as q. Applying the triangle inequality to q, q', and r, we get

$$||qr|| \leq ||qq'|| + ||q'r|| \; . \tag{1}$$

$||qq'||$ is certainly maximized if q and q' are on opposite sides of the sector. Hence,

$$||qq'|| \leq 2\sin(\theta/2) \cdot ||pq|| \; . \tag{2}$$

Moreover,

$$||q'r|| = ||pr|| - ||pq'|| = ||pr|| - ||pq|| \; . \tag{3}$$

Plugging (2) and (3) into (1) yields

$$\begin{aligned} ||qr|| &\leq 2\sin(\theta/2) \cdot ||pq|| + ||pr|| - ||pq|| \\ &= ||pr|| - (1 - 2\sin(\theta/2))||pq|| \; . \end{aligned}$$

□

Given a source-destination pair (s, t), consider the following strategy to get from s to t: Always take the edge whose other endpoint lies in the same sector as t.

Let the path obtained by this rule be $p = (s = v_0, v_1, \ldots, v_\ell = t)$. The path indeed ends at t, because for $k > 6$ we are guaranteed to have

$$||v_i t|| > ||v_{i+1} t||$$

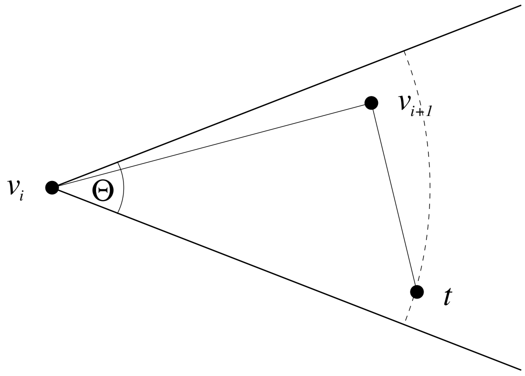

Figure 3: Figure illustrating that for $\theta < \pi/3$, v_{i+1} is closer to t than v_i.

for all i (see Figure 3).

Using Lemma 14.4, it holds that

$$\sum_{i=0}^{\ell-1} ||v_{i+1}t|| \leq \sum_{i=0}^{\ell-1} (||v_i t|| - (1 - 2\sin(\theta/2))||v_i v_{i+1}||) \ .$$

Rearranging the terms yields

$$\begin{aligned} \sum_{i=0}^{\ell-1} ||v_i v_{i+1}|| &\leq \frac{1}{1 - 2\sin(\theta/2)} \sum_{i=0}^{\ell-1} (||v_i t|| - ||v_{i+1}t||) \\ &= \frac{1}{1 - 2\sin(\theta/2)} \cdot ||st|| \ , \end{aligned}$$

which proves the theorem. □

Combining this with Theorem 13.9 yields the following result.

Corollary 14.5 *If $\theta = 2\pi/k$ with $k > 6$, then $YG_\theta(V)$ is a (c, δ)-power spanner for every $\delta \geq 1$ with*

$$c \leq \left(\frac{1}{1 - 2\sin(\theta/2)}\right)^{\delta} \ .$$

A much better result was shown by Li et al. [6] for $\delta \geq 2$. We strengthen their result to any $\delta \geq 1$.

Theorem 14.6 *If $\theta = 2\pi/k$ with $k > 6$, then $YG_\theta(V)$ is a (c, δ)-power spanner for every $\delta \geq 1$ with*

$$c \leq \frac{1}{1 - (2\sin(\theta/2))^{\delta}} \ .$$

Proof. Let $c = \frac{1}{1-(2\sin(\theta/2))^{\delta}}$. We construct a path p from u to v. If $(u, v) \in YG_\theta(V)$ then we set p to (u, v). Otherwise, there must exist another node w in the same sector as v with $(u, w) \in YG_\theta(V)$. Then p is the concatenation of (u, w) and a path from w to v. Since $k > 6$, it follows from Lemma 14.4

that $||wv|| < ||uv||$. Hence, each node appears at most once in the path p. We prove by induction on the number of its edges that p has a cost of at most $c||uv||^\delta$.

If $(u, v) \in YG_\theta(V)$ then $||p||^\delta = ||uv||^\delta < c||uv||^\delta$. So assume that the claim is true for any path with ℓ edges. Then consider a path p from u to v with $\ell + 1$ edges, which is the concatenation of an edge (u, w) and a path p' from w to v with ℓ edges. Let $\sigma = 2\sin(\theta/2)$ and $x \geq 0$ be chosen so that $||uv|| = (1 + x)||uw||$. From the induction hypothesis it holds that $||p'||^\delta \leq c||wv||^\delta$. Hence,

$$
\begin{aligned}
||p|| &\leq ||uw||^\delta + c||wv||^\delta \\
&\overset{\text{Lemma 1.20}}{\leq} ||uw||^\delta + \frac{1}{1-\sigma^\delta}(\sigma||uw|| + (||uv|| - ||uw||))^\delta \\
&= \left(1 + \frac{1}{1-\sigma^\delta}(\sigma + x)^\delta\right) ||uw||^\delta
\end{aligned}
$$

To prove the theorem, we have to show that

$$||p|| \leq c||uv||^\delta = \frac{1}{1-\sigma^\delta}(1+x)^\delta ||uw||^\delta$$

(Recall that $||uv|| = (1 + x)||uw||$.) To show this, consider the function

$$f(x) = \frac{1}{1-\sigma^\delta}(1+x)^\delta - \frac{1}{1-\sigma^\delta}(\sigma + x)^\delta - 1 \ .$$

It holds that

$$f'(x) = \frac{\delta}{1-\sigma^\delta}\left((1+x)^{\delta-1} - (\sigma+x)^{\delta-1}\right)$$

$f'(x)$ is obviously at least 0 for all $\sigma \leq 1$, $\delta \geq 1$, and $x \geq 0$. Hence, $f(x)$ attains its minimum at $x = 0$. Since

$$f(0) = \frac{1}{1-\sigma^\delta} - \frac{\sigma^\delta}{1-\sigma^\delta} - 1 = 0 \ ,$$

it follows that, indeed, $||p|| \leq c||uv||^\delta$ for all $\delta \geq 1$. □

There is also a simple distributed protocol for the Yao graph. Again, suppose that every node $u \in V$ knows its neighborhood $N(u)$ in the UDG and the current positions of the nodes in $N(u)$. Every node aims at maintaining a connection to the closest node in every sector. Let $E(u)$ be the current set of connections of node u. Then the protocol in Figure 4 can be used. This protocol has the following performance:

Theorem 14.7 *The Yao protocol self-stabilizes in one round. In the stable state, the outdegree of every node is at most k.*

Proof. Follows directly from the protocol. □

The drawback of the Yao graph is that, although its out-degree is at most k, its in-degree may be as high as $n - 1$ (consider, for example, the disk in Figure 10 with one node in its center and all other nodes on its border). Various sub-graphs of the Yao graph have been suggested to remove this drawback. We will present two of them here.

Protocol Yao:

For every node $u \in V$ repeatedly do:

1) for every node $w \in E(u)$:
   if there is a node $v \in N(u)$ in w's sector with $||uv|| < ||uw||$ then
   remove w from $E(u)$
2) for every sector C of u:
   if C has at least one node in $N(u)$ but no node in $E(u)$ then
   add the node w in C of smallest distance to u to $E(u)$

Figure 4: A self-stabilizing protocol for Yao graphs.

Definition 14.8 *The* sparsified Yao graph $SpYG_\theta(V)$ *is a sub-graph of* $YG_\theta(V)$ *with edge set*

$$E = \{(u, v) \in E(YG_\theta(V)) \mid \textit{for all } w \in V \textit{ with } (w, v) \in E(YG_\theta(V)) \textit{ and } w \in C_{v,u}\textit{: } ||vw|| > ||vu||\} .$$

In words, for every sector of every node v, the sparsified Yao graph only keeps the shortest of all edges into v. Hence, the sparsified Yao graph has an in-degree of at most k and an outdegree of at most k, and therefore a degree of at most $2k$.

Definition 14.9 *The* symmetric Yao graph $SyYG_\theta(V)$ *is a sub-graph of* $YG_\theta(V)$ *with edge set*

$$E = \{(u, v) \in E(YG_\theta(V)) \mid (v, u) \in E(YG_\theta(V))\} .$$

In words, the symmetric Yao graph only keeps an edge (u, v) if not only v is the nearest neighbor of u in $C_{u,v}$ but also u is the nearest neighbor of v in $C_{v,u}$. Hence, the symmetric Yao graph has a degree of at most k. Obviously,

$$SyYG_\theta(V) \subseteq SpYG_\theta(V) \subseteq YG_\theta(V)$$

and Figure 5 shows that there are cases in which the edge sets of the different graphs are proper subsets of each other. Thus, it suffices to prove connectivity for $SyYG_\theta(V)$ in order to prove connectivity for both variants of the Yao graph.

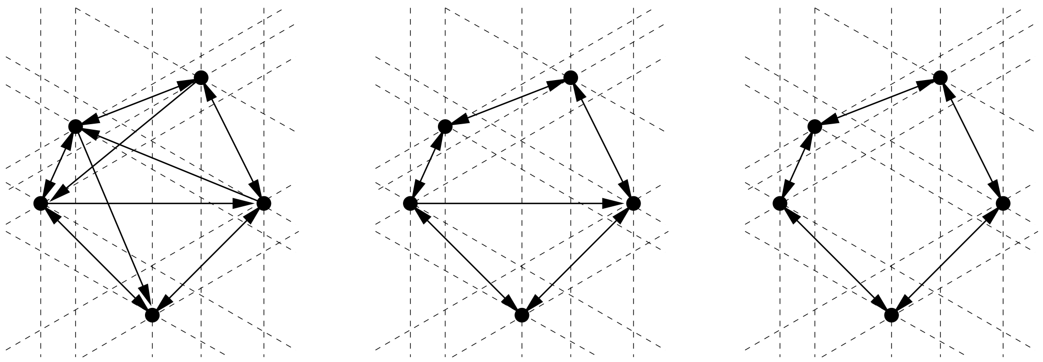

Figure 5: The Yao graph, the sparsified Yao graph, and the symmetric Yao graph of a point set.

Theorem 14.10 ([3]) *For all non-degenerate node sets V and $k > 6$, $SyYG_\theta(V)$ is connected.*

Proof. We prove the theorem by induction over the distances between the pairs of nodes, starting with the pair of lowest distance, say (u, w). In this case, w must be the nearest neighbor in a sector of u and u must be the nearest neighbor in a sector of w. Hence, $(u, w) \in E(SyYG_\theta(V))$, and therefore u and w are connected.

Now, suppose that we already know for the k pairs with lowest distance that each of them is connected. Then let (u, w) be the next pair to be considered. We distinguish between three cases:

1. There is a node $v \in C_{u,w}$ that is closer to u then w: Since $k > 6$, it holds in this case that $||uv|| < ||uw||$ and $||vw|| < ||uw||$. Hence, according to our induction hypothesis, u and v and v and w must be connected. Thus, also u and w must be connected.

2. There is a node $v \in C_{w,u}$ that is closer to w then u: Using the same arguments as for the first case, it follows that also in this case u and w must be connected.

3. None of the two cases above hold: Then w must be the nearest neighbor of u in $C_{u,w}$ and u must be the nearest neighbor of w in $C_{w,u}$. Hence, $(u, w) \in E(SyYG_\theta(V))$, and therefore u and w are connected.

Thus, all pairs of nodes in V must be connected, and therefore $SyYG_\theta(V)$ must be connected. □

Unfortunately, the symmetric Yao graph is not a good power spanner for any $\delta \geq 1$, which implies that it is not even a good weak spanner.

Theorem 14.11 ([1]) *The symmetric Yao graph is not a (c, δ)-power spanner for any constant c and any $\delta \geq 1$.*

However, the sparsified Yao graph is a good weak spanner.

Theorem 14.12 ([1]) *If $k > 6$, then the sparsified Yao graph is a weak c-spanner with $c = \frac{2}{1-2\sin(\theta/2)}$.*

Proof. Let $G = (V, E)$ be the sparsified Yao graph and $G_Y = (V, E_Y)$ be the underlying Yao graph. Consider any two nodes $u, w \in V$. Our goal is to show that there is a path from u to w in G that is inside a disk with center u of radius $||uw||/(1 - 2\sin(\theta/2))$. For any sector, define the Yao-neighbor v of u as the (unique) node v in that sector with $(u, v) \in E_Y$. Then we know:

- if a node u has no directed edge in a sector, then either the sector is empty or there is a Yao-neighbor v incident to an edge $(u', v) \in E$ where u' is in another sector of u with $||u'v|| < ||uv||$. Furthermore, $||uu'|| < ||uv||$ because $\theta < \pi/3$.

- Every node u has at least one neighbor v because the sparsified Yao graph is connected.

We recursively construct the path $p(u, w)$ as follows: if $(u, w) \in E$ then $p(u, w) = (u, w)$. Suppose that w is not u's Yao-neighbor in $C_{u,w}$ but u has an edge to its Yao-neighbor, w_1, in $C_{u,w}$. Then $p(u, w)$ is defined as

$$p(u, w) = u \circ p(w_1, w)$$

where $u \circ p(w_1, w)$ is the concatenation of u with the node sequence for the path $p(w_1, w)$ from w_1 to w that still needs to be determined.

Suppose that u does not have a connection to its Yao-neighbor w_1 in $C_{u,w}$. Then we know that there exists an edge $(v_1, w_1) \in E$ where v_1 is outside of the sector $C_{u,w}$ and $||uv_1|| < ||uw_1||$. Furthermore, we have $||uw_1|| \leq ||uw||$. Then we replace w by w_1 and repeat the arguments above. This iteration ends when u has a connection to a Yao-neighbor w_m or v_m. Because every node has at least one neighbor in E, this process terminates. If it terminates with w_m, then we define $p(u, w)$ as

$$p(u, w) = u \circ p(w_m, v_{m-1}) \circ p(w_{m-1}, v_{m-2}) \circ \ldots \circ p(w_2, v_1) \circ p(w_1, w)$$

where $p(w_1, w)$ may just be equal to w if $w_1 = w$. Otherwise, we define $p(u, w)$ as

$$p(u, w) = u \circ v_m \circ p(w_m, v_{m-1}) \circ \ldots \circ p(w_2, v_1) \circ p(w_1, w)$$

Now, notice that all nodes v_i, w_i are inside the disk of radius $||uw||$ around u. We continue to refine the paths $p(w_i, v_{i-1})$ as we did for $p(u, w)$ above. Consider any fixed pair (w_i, v_{i-1}). Let $r_0 = ||uw||$, $||uw_i|| = ||uv_{i-1}|| - d_0$, and $\sigma = 2\sin(\theta/2)$. Then it follows from Lemma 14.4 that

$$\begin{aligned} ||w_i v_{i-1}|| &\leq ||uv_{i-1}|| - (1-\sigma)(||uv_{i-1}|| - d_0) \\ &\leq r_0 - (1-\sigma)(r_0 - d_0) \end{aligned}$$

and w_i has a distance of at most $r_0 - d_0$ from u. Hence, the nodes identified for $p(w_i, v_{i-1})$ are inside the disk of radius $||w_i v_{i-1}||$ around w_i and therefore inside the disk of radius

$$r_0 + (r_0 - (1-\sigma)(r_0 - d_0)) - d_0$$

around u. Continuing the refinements recursively, it follows that all nodes in the ith recursion are inside the disk of radius $\sum_{j=0}^{i} r_i - \sum_{j=0}^{i} d_i$ around u where r_i is recursively defined as

$$r_i = r_{i-1} - (1-\sigma)(r_{i-1} - d_{i-1}) \ .$$

It holds that

$$\begin{aligned} \sum_{i\geq 0} r_i - \sum_{i\geq 0} d_i &\leq r_0 \sum_{i\geq 0} \sigma^i + (1-\sigma)\left(\sum_{i\geq 0} d_i\right)\left(\sum_{i\geq 0} \sigma^i\right) - \sum_{i\geq 0} d_i \\ &= r_0 \sum_{i\geq 0} \sigma^i = \frac{1}{1-\sigma} \cdot ||uw|| \end{aligned}$$

Thus, there is a path from u to w that is completely inside the disk of radius $1/(1-\sigma)||uw||$ around u, which proves the theorem. □

Though the sparsified Yao graph is not a relative neighborhood graph like the original Yao graph, it is easy to check that when restricting to the UDG of V, the proof of Theorem 14.12 is still correct for all pairs u, w with $||uw|| \leq 1$. Hence, it follows from the proof of Theorem 13.4 that the sparsified Yao graph is also a weak c-spanner of the UDG of V. Thus, Theorem 13.10 implies that it is also a power spanner of the UDG of V for every $\delta \geq 2$ and therefore useful for wireless ad hoc networks.

Next, we present a distributed protocol for the sparsified Yao graph. Again, we assume that each node $u \in V$ knows its neighborhood $N(u)$ and the current positions of the nodes in $N(u)$. Node u also keeps track of the following sets:

- $E(u)$: set of edges that u currently has to nodes in $N(u)$.
- $R(u)$: set of nodes in $N(u)$ requesting a connection to u.

Then the nodes work as given in Figure 6.

Protocol SparseYao:

For every node $u \in V$ repeatedly do:
  1) for every node $w \in N(u)$:
    if w is the closest node in $C_{u,w}$ and it is possible (from u's view) that $(u, w) \in SpYG_\theta(V)$ then
      u sends a connection request to w (causing w to add u to $R(w)$)
    else remove w from $E(u)$ if it is there
  2) for every node $w \in R(u)$:
    if w is closest to u in w's sector among all nodes in $R(u)$ then
      u sends w permission to connect (causing w to add u to $E(w)$)
    else u notifies w that it gave permission to another node w' (s.t. w removes u from $E(w)$)

Figure 6: A self-stabilizing protocol for the sparsified Yao graph.

Theorem 14.13 *The SparseYao protocol self-stabilizes in one round.*

Proof. Suppose that all neighbor sets are up-to-date. Then every node u knows the correct nodes w in step 1) that it has edges to in the Yao graph. Thus, at the end of step 1), every node w knows all the nodes v that have w as the closest node in one of their sectors. Hence, in step 2), every node u can select incoming edges according to the specifications of the sparsified Yao graph and will inform their starting points, which makes sure that at the end of step 2), all nodes w are added to $E(u)$ with (u, w) being an edge in $SpYG_\theta(V)$. On the other hand, $E(u)$ will contain only those nodes w with $(u, w) \in SpYG_\theta(V)$ because in step 1, all nodes w for which no request is sent are removed from $E(u)$, and in step 2, all nodes w that rejected the request are removed from $E(u)$. $\square$

14.2 Planar spanners

The most well known class of planar spanners are the Delaunay graphs. The Delaunay graph of a set of points in $\mathbb{R}^2$ is equivalent to their Delaunay triangulation and the dual of their Voronoi diagram. Since the Delaunay triangulation of any point set in $\mathbb{R}^2$ is planar, the Delaunay graph is planar. In the following, let $\triangle(uvw)$ be the triangle formed by the nodes u, v, and w and $\bigcirc(uvw)$ be the unique circle through u, v, and w.

Definition 14.14 *For any $V \subset \mathbb{R}^2$, the* Delaunay graph $Del(V)$ *of V consists of all edges (u, v) that have a node $w \in V$ for which $\bigcirc(uvw)$ does not contain any other node of V.*

For an example of a Delaunay graph see Figure 7. It is known [2, 4] that the Delauney graph is a geometric c spanner with $c = \frac{2\pi}{3\cos(\pi/6)} \approx 2.42$, but the Delaunay graph is difficult to maintain locally. Therefore, several variants of it have been proposed. The most well-known variant is the Gabriel graph.

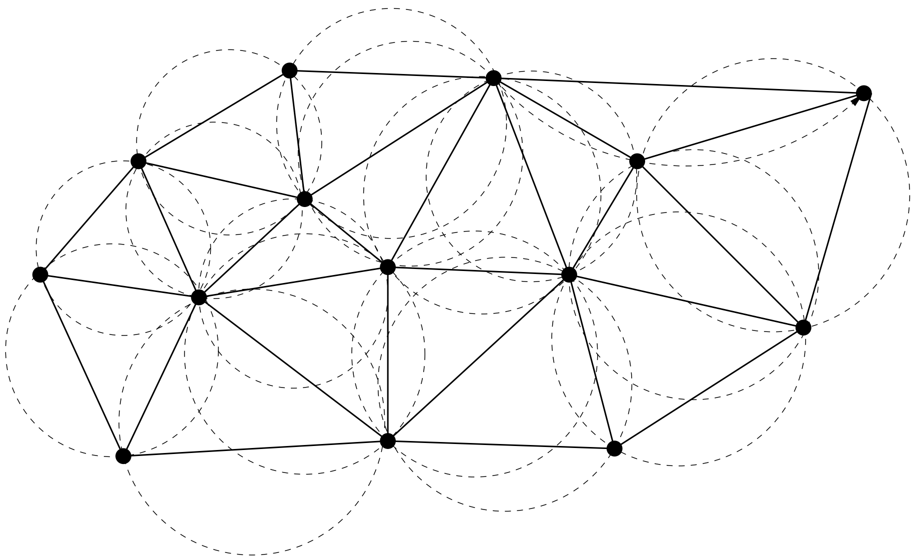

Figure 7: An example of a Delaunay graph.

Definition 14.15 *For any* $V \subset \mathbb{R}^2$*, the* Gabriel graph $GG(V)$ *of* V *consists of all edges* (u, w) *with the property that there is no node* $v \in V$ *with*

$$||uv||^2 + ||vw||^2 < ||uw||^2$$

In words, the Gabriel graph of V consists of all edges $\{u, w\}$ with the property that the open sphere through u and w with diameter $||uw||$ does not contain any other node in V. An example of a Gabriel graph is given in Figure 8. The Gabriel graph has the following interesting properties:

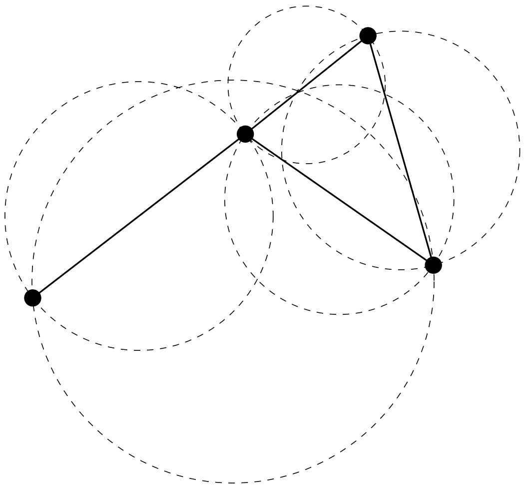

Figure 8: A Gabriel graph.

Theorem 14.16 *For any* $V \subset \mathbb{R}^2$*, the Gabriel graph of* V *is a relative neighborhood graph and a subgraph of the Delaunay graph of* V*.*

Proof. The Gabriel graph is a relative neighborhood graph because for all pairs (u, w) with $(u, w) \notin GG(V)$ there must be a node $v \in V$ with $||uv||^2 + ||vw||^2 < ||uw||^2$, which implies that $||uv|| < ||uw||$ and $||vw|| < ||uw||$.

It is also a subgraph of the Delaunay graph because for every edge (u, w) in $GG(V)$ it must hold that there is no node v in the open sphere of diameter $||uw||$ though u and w. If $|V| > 2$, we can then just extend the sphere into any direction until it hits a node v. In this case, there is no other node in $\bigcirc(uvw)$ which implies that (u, w) is also an edge in $Del(V)$. $\square$

Unfortunately, Theorem 13.8 implies that the Gabriel graph is not a geometric spanner. With better techniques one can even create a counterexample with stretch factor $\Omega(\sqrt{n})$ [6]. But Theorem 13.13 implies that the Gabriel graph is a weak 2-spanner, and even more importantly, it is an optimal power spanner for every $\delta \geq 2$.

Theorem 14.17 ([6]) *For every $\delta \geq 2$, the Gabriel graph is an optimal power spanner.*

Proof. Consider any pair of nodes $u, w \in V$ and let p be their optimal energy path. Consider an arbitrary edge $\{x, y\}$ in p. Suppose that there is a node $v \in S_{x,y}$. Then, by the Theorem of Pythagoras, $||xv||^\delta + ||vy||^\delta < ||xy||^\delta$ for every $\delta \geq 2$. Hence, replacing the edge $\{x, y\}$ in p by $\{x, v\}$ and $\{v, y\}$ would reduce its energy, which would contradict our assumption that p is an optimal energy path. Hence, there cannot be a node in $S_{x,y}$, and therefore $\{x, y\}$ must be an edge in the Gabriel graph. Thus, p must be a path in the Gabriel graph, which proves the theorem. $\square$

There is a simple protocol for the Gabriel graph. Again, suppose that every node $u \in V$ knows its neighborhood $N(u)$ in the UDG and the current positions of the nodes in $N(u)$. Let $E(u)$ be the current set of connections of node u. Then the protocol in Figure 9 can be used. It has the following performance:

Protocol Gabriel:

For every node $u \in V$ repeatedly do:
- 1) for every node $w \in E(u)$:
  - if there is a node $v \in N(u)$ with $||uv||^2 + ||vw||^2 < ||uw||$ then
    - remove w from $E(u)$
- 2) for every node $w \in N(u) \setminus E(u)$:
  - if there is no node $v \in N(u)$ with $||uv||^2 + ||vw||^2 < ||uw||$ then
    - add w to $E(u)$

Figure 9: A self-stabilizing protocol for Gabriel graphs.

Theorem 14.18 *The Gabriel protocol self-stabilizes in one round.*

Proof. Follows directly from the protocol. $\square$

Unfortunately, the outdegree of a Gabriel graph can be as high as $n - 1$ (see Figure 10). Also, since the Gabriel graph is not a geometric spanner, one may ask whether there are locally constructible planar graphs that are geometric spanners. To investigate the latter issue, we define the following classes of graphs.

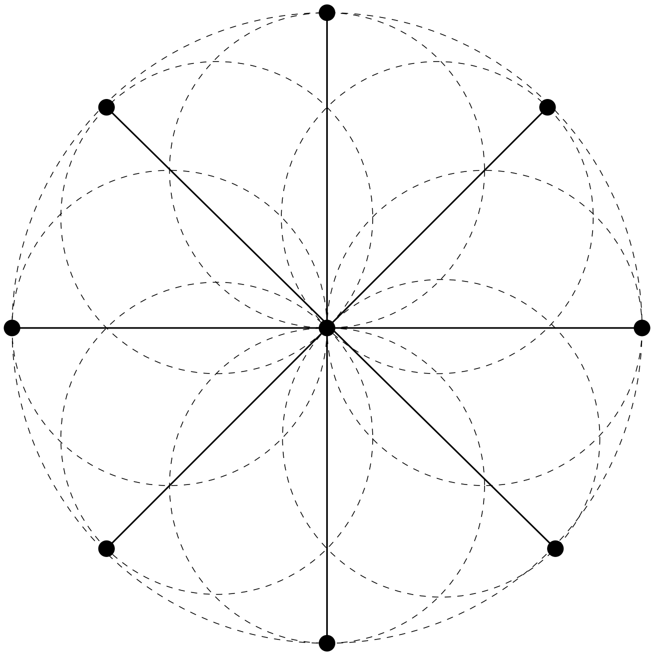

Figure 10: Gabriel graph for the unit disk with one node in its center and all other nodes on its border.

Definition 14.19 *A triangle $\triangle(uvw)$ satisfies the* k-localized Delaunay property *if the interior of the disk $\bigcirc(uvw)$ does not contain any node of V that is a k-neighbor of u, v, or w in $UDG(V)$ and $(u,v),(v,w),(w,u) \in UDG(V)$. Such a triangle is called a* k-localized Delaunay triangle.

Definition 14.20 *The* k-localized Delaunay graph *over V, denoted by $LDel^{(k)}(V)$, has exactly all Gabriel edges and the edges of all k-localized Delaunay triangles.*

Let the *constrained Delaunay graph* of a point set V be defined as $UDel(V) = Del(V) \cap UDG(V)$. The following facts are known about k-localized Delaunay graphs (e.g., [5]).

Theorem 14.21 *Localized Delaunay graphs have the following properties:*

1. $UDel(V) \subseteq LDel^{(k)}(V)$ *for all* $k \geq 1$.
2. $LDel^{(k+1)}(V) \subseteq LDel^{(k)}(V)$ *for all* $k \geq 1$.
3. $LDel^{(2)}(V)$ *is a planar graph.*
4. $LDel^{(1)}(V)$ *is not always planar.*

Proof. The first two items follow immediately from the definition of localized Delaunay graphs.

Suppose that it is possible to choose some V so that $LDel^{(2)}(V)$ is not a planar graph. Then $LDel^{(2)}(V)$ must contain two edges that cross each other. Let these edges be (v,v') and (w,w'). Since $||vv'|| \leq 1$ and $||ww'|| \leq 1$ and the edges cross each other, there must be nodes $u \in \{v,v'\}$ and $u' \in \{w,w'\}$ so that $||uu'|| \leq 1$. But then $\{v,v',w,w'\} \subseteq N_2(u)$ and $\{v,v',w,w'\} \subseteq N_2(u')$

which implies that choosing (v, v') and (w, w') would violate the definition of $LDel^{(2)}(V)$. Hence, $LDel^{(2)}(V)$ is always planar.

It remains to show that there is a set V so that $LDel^{(1)}(V)$ is not planar. For this consider the example in Figure 11. In this example, $\bigcirc(u, v, w)$ does not contain any node in $N(u)$, $N(v)$, and $N(w)$, and $\bigcirc(x, y, z)$ does not contain any node in $N(x)$, $N(y)$, and $N(z)$. Hence, (v, w) and (x, z) are edges in $LDel^{(1)}(V)$, but they intersect.

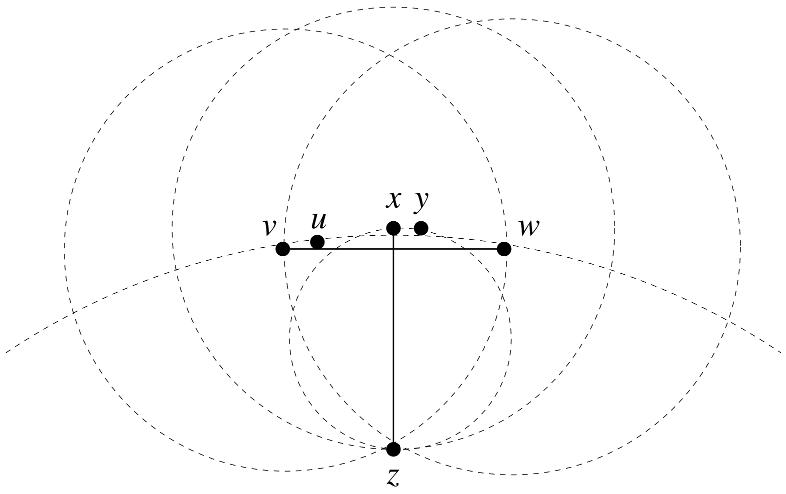

Figure 11: $LDel^{(1)}(V)$ is not planar.

□

Since $UDel(V)$ is a geometric c-spanner with $c \approx 2.42$ it follows that $LDel^{(2)}(V)$ is a geometric c-spanner with $c \approx 2.42$, and it is also planar. Also, $LDel^{(2)}(V)$ can be easily maintained locally. See Figure 12 for a self-stabilizing protocol. In this protocol we assume that every node $u \in V$ knows its 2-neighborhood

$$N^2(u) = \cup_{v \in N(u)} N(v)$$

and the current positions of the nodes in $N^2(u)$.

Protocol LDel2:

For every node $u \in V$ repeatedly do:
- 1) for every node $w \in E(u)$:
  - if $\{u, w\}$ is not a Gabriel edge and does not belong to a k-localized Delaunay triangle then
    - remove w from $E(u)$
- 2) for every node $w \in N(u) \setminus E(u)$:
  - if $\{u, w\}$ is a Gabriel edge or belongs to a k-localized Delaunay triangle then
    - add w to $E(u)$

Figure 12: A self-stabilizing protocol for the 2-localized Delaunay graph.

Theorem 14.22 *The LDel2 protocol self-stabilizes in one round.*

Proof. Follows directly from the protocol. □

However, as mentioned above, Gabriel graphs and therefore all graphs of the localized Delaunay graph family have the problem that the degree may be very high (see Figure 10). This problem can be solved by constraining a Delaunay graph in the same way Yao graphs are constrained to sparsified Yao graphs: cut the space around each node into $k > 6$ sectors of equal angle, and accept only the connection of the closest node with an incoming edge in the original graph. Similar to the proof of the sparsified Yao graph, this gives a sparsified Delaunay graph that is still a weak spanner. Other constructions have been proposed that can even maintain a Euclidean $O(1)$-spanner but at the cost of requiring an algorithm that may need a long time to stabilize at some solution [7].

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