Computations of Moduli Spaces for semiquasihomogenous Singularities and an Implentation in `SINGULAR`

Diplomarbeit

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Preface

One of the important achievements of singularity theory was the classification of some "generic" classes of hypersurface singularities via normal forms by V.I. Arnold, cf. [AVG 1985]. For more complicated classes of singularities the classification by normal forms seems to be impossible. In 1997 Greuel, Hertling, and Pfister came up with a geometric classification of semiquasihomogeneous hypersurface singularities with fixed part w.r.t. right- and contact equivalence by geometric methods, i.e., by providing a moduli space for such singularities with some invariants fixed. In this thesis we provide a Singular implementation for computing moduli spaces for right equivalence.

The structure of this thesis is as follows :
In **Chapter 1** we state the required background algebraic geometry and singularity theory. **Chapter 2** is a presentation of the theory developed in [GHP 1997] (w.r.t. right-equivalence). **Chapter 3** contains the description of algorithms for computing equations of the moduli space. **Chapter 4** is devoted to Brieskorn-Pham singularities and quasihomogeneous singularities with inner modality ≤ 4. **Chapter 5** contains the user manual of the Singular implementation. The **Appendix** contains the source code of the actual implementation.

What's New:

1. The algorithms `ArnoldAction`, `LinearizeAction`, `ModEqn`, `StabEqns` and `ZeroSet` for the construction of moduli spaces for semiquasihomogenous power series with fixed principal part w.r.t. right equivalence (Chapter 3).

2. We provide three new `Singular` libraries (`qhmoduli.lib`, `rinvar.lib`, `zeroset.lib`). The `qhmoduli.lib` contains procedures for the computation of equations for the moduli space of semiquasihomogenous hypersurface singularities with fixed principal part, where the only restrictions are time and memory. The `rinvar.lib` contains procedures for the computation of invariant rings of reductive groups (Derksen's

algorithm).

3. A structure theorem on the stabilizer of Brieskorn-Pham singularities (Theorem 4.1.83) which says that the stabilizer group of a Brieskorn-Pham polynomial can be generated by diagonal and permutation matrices.

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Chapter 1

Introduction

This chapter provides an introduction to moduli spaces and the kind of singularities treated in this thesis. The necessary tools required to construct a moduli space for them are also introduced.

1.1 Singularities, varieties, and group-actions

By $\mathbf{C}\{x_1, x_2, ..., x_n\}$ we denote the convergent power series ring in n variables, abbreviated by $\mathbf{C}\{\mathbf{x}\}$ and by $\mathbf{K}$ we denote an algebraically closed field of characteristic 0. We are concerned with two different objects, namely the singularities which we want to classify and the classification spaces. The singularities are represented by power series in $\mathbf{C}\{x_1, x_2, ..., x_n\}$ while the classification spaces are affine varieties, .i.e., they can be represented by rings of the form $\mathbf{K}[x_1, x_2, ..., x_n]/I$ where $I \subset \mathbf{K}[x_1, x_2, ..., x_n]$ is a radical ideal. In this section we introduce several fundamental definitions from algebraic geometry and singularity theory used throughout this thesis.

We need the notion of a zero set of a set of functions which will be used in an algebraic and analytic context.

Definition 1 *Let F be a set of $\mathbf{K}-$valued functions on $\mathbf{K}^n$. The **zero-set** of F is $\mathcal{V}(F) := \{\mathbf{v} \in \mathbf{K} : f(\mathbf{v}) = 0$ for all $f \in F\}$.*

1.1.1 The algebraic category

The basic objects are topological spaces homeomorphic to closed subsets of $\mathbf{K}^n$ w.r.t. the Zariski topology. Together with coordinate rings and morphisms they form the category of affine varieties.

Definition 2 *(a) $X \subset \mathbf{K}^n$ is an* ***algebraic set*** *iff there exists a set of polynomials $F \subset \mathbf{K}[x_1, x_2, ..., x_n]$ s.t. $X = \mathcal{V}(F)$.*
(b) $X \subset \mathbf{K}^n$ is a ***closed subset*** *in the Zariski topology iff X is an algebraic set.*
(c) The Zariski topology of an algebraic set $X \subset \mathbf{K}^n$ is the induced subspace topology on X w.r.t. the Zariski topology on $\mathbf{K}^n$.
(d) A function $f : X \to \mathbf{K}$, where $X \subset \mathbf{K}^n$ is an algebraic set, is called ***regular*** *iff for each $x \in X$ there exists $x \in U \subset X$ open and polynomials $g, h \in \mathbf{K}[x_1, x_2, ..., x_n]$ s.t. h is nowhere zero on U and $f = \frac{g}{h}$ on U.*

We will see below that algebraic sets correspond to ideals of $\mathbf{K}[x_1, x_2, ..., x_n]$. Since the intersection of finitely many and the union of arbitrarily many ideals is an ideal, the closed sets defined above form a topology, called the Zariski topology. An algebraic variety is an algebraic set plus the ring of regular functions on it, called the coordinate ring.

Definition 3 *Let $X \subset \mathbf{K}^n$ be an algebraic set.*
(a) The ***coordinate ring*** *of X,denoted by $\mathcal{O}(X)$, is the ring of regular functions on X.*
(b) The pair $(X, \mathcal{O}(X))$ is a ***(closed) subvariety****.*

Up to now we have considered only closed subsets of $\mathbf{K}^n$. Affine varieties need not be embedded in some $\mathbf{K}^n$ (but can always be embedded).

Definition 4 *A topological space Z together with an algebra $\mathcal{O}(Z)$ of $\mathbf{K}-$valued functions is an* ***affine variety*** *(over $\mathbf{K}$) if there exists a closed subvariety $X \subset \mathbf{K}^n$ and a homeomorphism $\phi : Z \to X$ which identifies $\mathcal{O}(Z)$ and $\mathcal{O}(X)$. In this case $\mathcal{O}(Z)$ is called the* ***coordinate ring*** *of Z*

In particular, any closed subvariety is an affine variety. We concentrate on the relation between ideals and affine varieties. Let $F \subset \mathbf{K}[x_1, x_2, ..., x_n]$ be a set of polynomials, $I = \langle F \rangle$ be the ideal generated by F and $X := \mathcal{V}(F)$. We have $X = \mathcal{V}(I)$ since $x \in X$ iff $f(x) = 0$ for $f \in F$ and $h \in I$ iff $h = \sum_{f \in F} p_f \cdot f$ for some polynomials $p_f \in \mathbf{K}[x_1, x_2, ..., x_n]$. Hilbert's Basis Theorem shows that we can restrict ourselves to finite sets of polynomials.

Theorem 5 *(Hilbert 1890) Any ideal $I \subset \mathbf{K}[x_1, x_2, ..., x_n]$ is finitely generated.*

Proof. We refer ,e.g., to [CLS 1992], Theorem 4 in Ch.2.5. ■

A commutative ring R is Noetherian iff each ideal of R is finitely generated. The Basis Theorem says that the polynomial ring $\mathbf{K}[x_1, x_2, ..., x_n]$ is Noetherian.

In the sequel we do not distinguish between a finite set F and the ideal generated by F. To any subset $X \subset \mathbf{K}^n$ we can associate a radical ideal..

Definition 6 *(a) An ideal $I \subset \mathbf{K}[x_1, x_2, ..., x_n]$ is a* ***radical ideal*** *if $f^k \in I$ for some $k > 0$ implies $f \in I$.*
(b) Let $X \subset \mathbf{K}^n$ be an arbitrary subset. The ideal associated to X is

$$\mathcal{I}(X) := \{f \in \mathbf{K}[x_1, x_2, ..., x_n] \mid \forall x \in X : f(x) = 0\}.$$

If X is a subvariety then the radical ideal $\mathcal{I}(X)$ is called the ***defining ideal*** *of X.*

Coordinate rings and quotient rings of $\mathbf{K}[x_1, x_2, ..., x_n]$ are tied together by Hilbert's Nullstellensatz, which relates the ideal $\mathcal{I}(X)$ associated to a subvariety $X = \mathcal{V}(I)$ and the radical ideal of I.

Theorem 7 *(Hilbert 1893) For any ideal $I \subset \mathbf{K}[x_1, x_2, ..., x_n]$ we have*

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

Proof. We refer ,e.g., to [CLS 1992], Theorem 6 in Ch.4.2. ■

Hilbert's Nullstellensatz justifies the notion of a defining ideal if X is a subvariety.

Proposition 8 *(a) The coordinate ring of the closed subvariety $X = \mathcal{V}(I)$ is*

$$\mathcal{O}(X) = \mathbf{K}[x_1, x_2, ..., x_n]/\sqrt{I}.$$

(b) The Zariski closure $\overline{X}$ of an arbitrary subset $X \subset \mathbf{K}^n$ is the subvariety $\mathcal{V}(\mathcal{I}(X))$.

Proof. (a) We refer ,e.g., to [Hartshorne 1997], Theorem 3.2 in Ch.I.
(b) We refer ,e.g., to [CLS 1992], Proposition 1 in Ch.4.4. ■

Hence subvarieties of $\mathbf{K}^n$ are given as quotient rings $\mathbf{K}[x_1, x_2, ..., x_n]/I$ where I is a radical ideal. A $\mathbf{K}-$algebra R which is isomorphic to the quotient ring $\mathbf{K}[x_1, x_2, ..., x_n]/I$ for some radical ideal I is called an **affine algebra over $\mathbf{K}$**. We obtain a bijection

$$\{\text{affine } \mathbf{K} - \text{algebras}\} \simeq \{\text{subvarieties of } \mathbf{K}^n\}.$$

In order to obtain a category we need the notion of a morphism between varieties.

Definition 9 *(a) Let X and Y be subvarieties of $\mathbf{K}^n$ and $\mathbf{K}^m$ respectively. A* ***morphism of subvarieties*** *$\phi : X \to Y$ is a tuple $(\phi_1, \phi_2, ..., \phi_m)$ of regular functions $\phi_1, \phi_2, ..., \phi_m$ on X s.t. $\phi(X) \subset Y$.*
(b) Let X and Y be affine varieties. A ***morphism of affine varieties*** *$\phi : X \to Y$ is a continuous map s.t. $\phi^*(f) := f \circ \phi \in \mathcal{O}(X)$ for all $f \in \mathcal{O}(Y)$. The $\mathbf{K}-$algebra homomorphism ϕ^* is called the* ***pullback*** *of ϕ.*

Proposition 10 *For affine varieties X, Y we have*

$$Mor(X, Y) \simeq Hom_{\mathbf{K}-alg}(\mathcal{O}(Y), \mathcal{O}(X)).$$

Proof. We refer ,e.g., to [Kraft 1985], Ch. AI.2, Theorem 2.2. ■

Note that subvarieties are defined as the zero set of finitely many polynomials and morphisms of subvarieties are given by tuples of polynomials. One can perform several geometric operations on subvarieties by manipulating ideals of polynomial rings. Therefore they are perfectly suited for computer algebra.

Definition 11 *The category of affine varieties has affine varieties as objects and morphisms of affine varieties as morphisms.*

The notion of schemes is very useful for the construction of quotients of affine varieties w.r.t. algebraic groups. Let R be a commutative ring. The (prime) spectrum of R is $spec(R) := \{P \subset R : P \text{ a prime ideal}\}$. The closed sets of the Zariski topology of $spec(R)$ are given by $\mathcal{V}(I) := \{P \in Spec(R) : I \subset P\}$. Let R be a commutative $\mathbf{K}-$algebra and set $X := spec(R)$. The pair $(X, \mathcal{O}_X)$ is an **affine scheme of finite type over K** where $\mathcal{O}_X$ is the so called structure sheaf of X. For our purposes it suffices to note that the global sections of $\mathcal{O}_X$ are isomorphic to R. For the precise definition of schemes and sheaves we refer, e.g., to Ch. II of [Hartshorne 1997].

Remark 1 *We refer to Ch. II of [Hartshorne 1997] for proofs of the equivalences below.*
(a) There are equivalences of categories, defined by the covariant functor

$$t : (\textit{affine varieties over } \mathbf{K}) \stackrel{\simeq}{\to} (\textit{reduced affine schemes of finite type over } \mathbf{K})$$

and the contravariant functor

$$spec : (\textit{affine } \mathbf{K}-\textit{algebras}) \stackrel{\simeq}{\to} (\textit{reduced affine schemes of finite type over } \mathbf{K})$$

where an affine $\mathbf{K}-$algebra is a finitely generated reduced $\mathbf{K}-$algebra, cf. [Hartshorne 1997]. In the sequel we often define an affine variety X by

$X = spec(R)$ for some affine $\mathbf{K}-$algebra R. This should be seen as $X := t^{-1}(spec(R))$. In this way we avoid the definition of the action of a group G on nonclosed points of X.
(b) We obtain an equivalence of categories

$$(\textit{affine varieties over } \mathbf{K}) \stackrel{\simeq}{\rightarrow} (\textit{affine } \mathbf{K}-\textit{algebras})$$

as follows. The coordinate ring $\mathcal{O}(Z)$ of an affine variety satisfies $\mathcal{O}(Z) \simeq \mathbf{K}[x_1, x_2, ..., x_n]/I$ for some radical ideal I. The zero-set $\mathcal{V}(I)$ is a closed subvariety which corresponds to Z. Conversely, any radical ideal I of an affine $\mathbf{K}-$algebra R defines a closed subvariety $t^{-1}(spec(R/I)) =: X' \subset X := t^{-1}(spec(R))$.

Some morphism occurring in the practical computation of equations of the moduli space have Zariski closed image, which turns out to be important.

Definition 12 *Let $\phi : V \to W$ be a morphism of affine varieties.*
*(a) ϕ is **closed** if $\phi(V') \subset W$ is closed for any closed subset $V' \subset V$.*
*(b) ϕ is **finite** iff $\mathcal{O}(V)$ is a finitely generated $\mathcal{O}(W)-$module via ϕ^*.*

Proposition 13 *Let $\phi : V \to W$ be a morphism of affine varieties.*
(a) ϕ is a closed embedding iff $\phi^* : \mathbf{K}[W] \to \mathbf{K}[V]$ is surjective.
(b) If ϕ is finite then ϕ is closed.
(c) If ϕ is finite then the set $\phi(w)^{-1}$ is finite.

Proof. (a) We refer ,e.g., to [Hartshorne 1997], Ch. II. (b),(c) We refer, e.g., to [Kraft 1985], Ch. AI.4.1. ■

Group actions of algebraic groups will play an important role in the construction of the moduli space.

Definition 14 *(a) An affine variety G is an **algebraic group** iff there exist morphisms $\mu : G \times G \to G, \iota : G \to G$ and an element $e \in G$ s.t. G is a group w.r.t. multiplication μ, inverse ι and neutral element e.*
*(b) A **linear algebraic group** is a closed subset (in the Zariski-topology) of $GL_n(\mathbf{K})$ together with the inherited group structure.*
*(c) Let G be an algebraic group and X be an affine variety. A morphism $\alpha : G \times X \to G$ is a **group-action** iff $\alpha(e, x) = x$ and $\alpha(\sigma, \alpha(\tau, x)) = \alpha(\sigma\tau, x)$ for all $\sigma, \tau \in G$ and $x \in X$.*

1.1.2 Singularities and the complex category

The objects to classify are singularities defined by semiquasihomogeneous power series. We provide only the definition of a hypersurface singularity and omit the definition of a complex space which can be found, e.g., in [GF 1974]. In this section we use the Euclidean topology on $\mathbf{C}^n$.

Let $X \subset \mathbf{C}^n$ be a subspace and $x \in X$. The **space germ** (X, x) is the equivalence class of open neighborhoods of x where two open neighborhoods U, V of x are equivalent iff there exists an open neighborhood $W \subset X$ of x s.t. $U \cap W = V \cap W$. Let $\widetilde{f} : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a holomorphic map of space germs. This means that there exist small neighborhoods $0 \in U \subset \mathbf{C}^n$ and $0 \in V \subset \mathbf{C}$ and a holomorphic map f s.t. $f : U \to V$ represents $\widetilde{f}$. If $f' : U' \to V'$ is another holomorphic map, where $0 \in U' \subset \mathbf{C}^n$ and $0 \in V' \subset \mathbf{C}$ are open neighborhoods, and there exists a neighborhood $0 \in W \subset U \cap U'$ s.t. $f_{|W} = f'_{|W}$ then f' is another representant of $\widetilde{f}$. The holomorphic maps between $(\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ correspond to convergent power series, i.e., to elements from $\mathbf{C}\{x_1, x_2, ..., x_n\}$.

Definition 15 *Let $U \subset \mathbf{C}^n$ be a nonempty open set and $f : U \to \mathbf{C}$ be a holomorphic map.*
*(a) f has a **singular point** at $p \in U$ iff $\frac{\partial f}{\partial x_i}(p) = 0$ for $1 \leq i \leq n$.*
*(b) A **hypersurface singularity** is a germ of the form $(\mathcal{V}(f), p)$ s.t. $f(p) = 0$ together with the local $\mathbf{C}-$algebra $\mathbf{C}\{x_1, x_2, ..., x_n\}/\langle f \rangle$, where $\mathcal{V}(f) := \{u \in U : f(u) = 0\}$.*
*(c) The singularity of f is **isolated** iff exists a small neighborhood $p \in U' \subset U$ s.t. p is the only singular point of f in U'.*
*(d) The ideal $j(f) := \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right\rangle_{\mathbf{C}\{x_1 - p_1, x_2 - p_2, ..., x_n - p_n\}}$ is called the **Jacobi ideal** of f at p.*
*(e) The **Milnor algebra** of f at p is $M_{f,p} := \mathbf{C}\{x_1 - p_1, x_2 - p_2, ..., x_n - p_n\}/j(f)$.*
*(f) The **Milnor number** of f at p is $\mu(f, p) := \dim_{\mathbf{C}}(M_{f,p})$.*
If $p = 0$ we abbreviate $M_{f,p}$ by M_f and $\mu(f, p)$ by $\mu(f)$.

W.l.o.g. we restrict ourselves to maps with isolated singular points.

Theorem 16 *f has an isolated singularity at p iff $\mu(f, p) < \infty$.*

Proof. We refer ,e.g., to [AVG 1985]. ■

The elements $f_1, f_2, ..., f_k \in R$, where R is a commutative ring with 1, form a **regular sequence** if f_{i+1} is not a zero divisor in $R/ < f_1, ..., f_{i-1} >$. This notion turns out to be useful in several proofs of properties of $G^{\mathbf{w}}_{f_0}$ and

E_{f_0}. If R is local (i.e., has only one maximal ideal) then any permutation of $f_1, f_2, ..., f_k$ is a regular sequence. In general, this is not true for rings which are not local.

Corollary 17 *Any permutation of the partial derivatives* $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}$ *forms a regular sequence in* $\mathbf{C}\{x_1 - p_1, x_2 - p_2, ..., x_n - p_n\}$.

Proof. Since $\dim(M_{f,p}) < \infty$ and $\mathbf{C}\{x_1 - p_1, x_2 - p_2, ..., x_n - p_n\}$ is a local ring this is a standard fact in commutative algebra. ■

In the sequel we assume that $p = 0$, i.e., we consider hypersurface singularities of the form $(\mathcal{V}(f), 0)$ for some $f \in \mathbf{C}\{x_1, x_2, ..., x_n\}$. For the construction of a moduli space we fix the following equivalence relation.

Definition 18 *(a) By* $Aut(\mathbf{C}\{x_1, x_2, ..., x_n\})$ *we denote the ring of local* $\mathbf{C}-$*algebra homomorphisms of* $\mathbf{C}\{x_1, x_2, ..., x_n\}$*, i.e.,* $\mathbf{C}-$*algebra homomorphisms of* $\mathbf{C}\{x_1, x_2, ..., x_n\}$ *preserving the maximal ideal* $\langle x_1, x_2, ..., x_n\rangle$.
(b) Two power series $f, g \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ *are* ***right equivalent****, written* $f \sim_r g$*, iff there exists* $\varphi \in Aut(\mathbf{C}\{x_1, x_2, ..., x_n\})$ *s.t.* $f = \varphi(g)$.
(c) Two power series $f, g \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ *are* ***contact equivalent*** *iff there exists* $\varphi \in Aut(\mathbf{C}\{x_1, x_2, ..., x_n\})$ *and a unit* $u \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ *s.t.* $f = u \cdot \varphi(g)$.

Right equivalence is probably the finest meaningful equivalence relation of hypersurface singularities. For power series $f, g \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ we may have $(\mathcal{V}(f), 0) \simeq (\mathcal{V}(g), 0)$ (i.e., there exists a local $\mathbf{C}-$algebra automorphism ϕ of $\mathbf{C}\{x_1, x_2, ..., x_n\}$ s.t. $\phi(\langle f\rangle) = \langle g\rangle)$ but $f \nsim_r g$. Conversely, $f \sim_r g$ implies $(\mathcal{V}(f), 0) \simeq (\mathcal{V}(g), 0)$.

Definition 19 *The power series* f *is* $k-$***determined*** *w.r.t.* $\sim_r$ *iff for all* $g \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ *we have* $g^{(k)} = f^{(k)}$ $g \sim_r f$ *where* $g^{(k)}$ *denotes the* $k-$*jet.*

Theorem 20 *If* $f \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ *has an isolated singularity at* 0 *then* f *is* $\mu(f) + 1-$*determined.*

Proof. We refer ,e.g., to [AVG 1985]. ■

Hence any power series f with an isolated singularity is right equivalent to a polynomial, more precisely we have $f \sim_r f^{(\mu(f)+1)}$.

1.2 Semi-quasihomogeneous power series and unfoldings

We define the notion of semiquasihomogeneous power series and unfoldings of these power series and mention that the Milnor number of a semiquasihomogeneous power series is already determined by its principal part. Semiquasihomogeneous power series with fixed part are the objects under consideration.

Let $\mathbf{w} = (w_1, w_2, ..., w_n) \in \mathbf{N}^n$ be a tuple of positive integers, called weights. For $\boldsymbol{\alpha} \in \mathbf{N}^n$ we denote the monomial $\prod_{i=1}^n x_i^{\alpha_i}$ by $\mathbf{x}^{\boldsymbol{\alpha}}$.

Definition 21 *(a) The monomial $\mathbf{x}^{\boldsymbol{\alpha}} \in \mathbf{C}[x_1, x_2, ..., x_n]$ is **quasihomogeneous of type** $(d; \mathbf{w})$ if $|\mathbf{x}^{\boldsymbol{\alpha}}|_{\mathbf{w}} := \sum_{i=1}^n w_i \alpha_i = d$.*
*(b) A nonzero polynomial $f \in \mathbf{C}[x_1, x_2, ..., x_n]$ is **quasihomogeneous of type** $(d; \mathbf{w})$ if all monomials of f are quasihomogeneous of type $(d; \mathbf{w})$.*
*(c) The **weighted degree** is defined as $\deg_{\mathbf{w}}(f) := \max\{|\mathbf{x}^{\boldsymbol{\alpha}}|_{\mathbf{w}} : \mathbf{x}^{\boldsymbol{\alpha}}$ a monomial of $f\}$ where $0 \neq f \in \mathbf{C}[x_1, x_2, ..., x_n]$. We define $\deg_{\mathbf{w}}(0) := -1$.*
*(d) For any given type $(d; \mathbf{w})$ of quasihomogenity we call $\overline{w}_i := \frac{w_i}{d}$ the **normalized weights**.*

Quasihomogeneous polynomials can be characterized by the following group action. Let f be quasihomogeneous of type $(d; \mathbf{w})$ and define

$$\lambda \bullet f(x_1, x_2, ..., x_n) := f(\lambda^{w_1} \cdot x_1, \lambda^{w_2} \cdot x_2, ..., \lambda^{w_n} \cdot x_n)$$

for $\lambda \in \mathbf{C}$.. Then

$$\lambda \bullet f(x_1, x_2, ..., x_n) = \lambda^d \cdot f(x_1, x_2, ..., x_n).$$

This action is called a good $\mathbf{C}^*-$action. One can show that f is quasihomogeneous iff f admits a good $\mathbf{C}^*-$action. If λ is a $d-$th root of unity then $\lambda \bullet f(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n)$.

The object of our study are power series with fixed quasihomogeneous principal part and an isolated singularity at 0.

Definition 22 *Let $0 \neq f \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ be a power series. We define $ord_{\mathbf{w}}(f) = \min\{|\mathbf{x}^{\boldsymbol{\alpha}}|_{\mathbf{w}} : \mathbf{x}^{\boldsymbol{\alpha}}$ a monomial of $f\}$. We define $ord_{\mathbf{w}}(0) := -1$. The power series f is called **semiquasihomogeneous of type** $(d; \mathbf{w})$ **with principal part** f_0 iff f_0 is a quasihomogeneous polynomial of type $(\mathbf{w}, d)$ with an isolated singularity at 0, $f = f_0$ or $ord_{\mathbf{w}}(f - f_0) > d$.*

Proposition 23 *For any semiquasihomogeneous power series f with principal part f_0 we have $\mu(f) = \mu(f_0) < \infty$.*

Proof. By definition, $\mu(f_0) < \infty$. For the proof of the equality we refer, e.g., to [AVG 1985], Ch. 12.2. ■

A semiquasihomogeneous power series f with principal part f_0 defines an isolated hypersurface singularity $(\mathcal{V}(f), 0)$ with principal part f_0.

Note that if f is semiquasihomogeneous with principal part f_0 then right equivalence need not preserve the principal part, cf. Example 2.1.43(b).

Let $f \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ be semiquasihomogeneous with principal part f_0. The power series f can be considered as a function germ $f : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}, 0)$ which can be deformed as follows. An **unfolding** of f over a germ (T, t) is a map $F : (\mathbf{C}^n, 0) \times (T, t) \longrightarrow (\mathbf{C}, 0)$ s.t. the following diagram commutes,

$$\begin{array}{ccc}
(\mathbf{C}^n, 0) & \hookrightarrow & (\mathbf{C}^n, 0) \times (T, t) \\
\downarrow f & & \downarrow \phi \\
(\mathbf{C}, 0) & \hookrightarrow & (\mathbf{C}, 0) \times (T, t) \\
\downarrow & & \downarrow \\
0 & \hookrightarrow & (T, t)
\end{array}$$

where $\phi(x, t) = (F(x, t), t)$, and $F(x, t) = f(x) + g(x, t)$ for some $g \in \mathbf{C}\{\mathbf{x}, \mathbf{t}\}$ with $g(x, 0) = 0$. In other words, $F_{|(\mathbf{C}^n,0)\times(\{0\},t)} = f$. Two unfoldings ϕ, ϕ' of f are **right equivalent** if there exists an isomorphism $\Psi : (\mathbf{C}^n, 0) \times (T, 0) \rightarrow (\mathbf{C}^n, 0) \times (T, 0)\,, \Psi(x, t) = (\psi(x, t), t)$ s.t. $\phi \circ \Psi = \phi'$, i.e., $F(\psi(x, t), t) = F'(x, t)$.

Let $\varphi : (S, s) \rightarrow (T, t)$ be a morphism of space germs and F be an unfolding of f over (T, t). The morphism φ induces an unfolding $\varphi^* F$ of f over (S, s) via $(\varphi^* F)(x, s) = F(x, \varphi(s))$.

An unfolding F of f over (T, t) is a **semiuniversal unfolding** of f if each unfolding G of f over (S, s) can be induced from F by a morphism $\varphi : (S, s) \rightarrow (T, t)$ s.t. the differential $d\varphi$ is unique.

1.3 Fine and coarse moduli spaces

For the definition of moduli spaces we need the notion of a moduli problem and a corresponding moduli functor. A moduli problem typically consists of two things, a specific class of objects and the notion of families of these objects over base spaces of a given category together with an equivalence relation. One may consider global objects like projective varieties or vector bundles, or local objects like singularities and modules. For the global case we refer, e.g., to [MFK 1994], and for the local case we refer to [GP 1991].

Suppose that we have given the notion of a family of objects over a base $B \in \mathcal{C}$, where $\mathcal{C}$ is a category of (base) spaces, together with an equivalence relation of families. Let $\mathcal{F} : \mathcal{C} \rightarrow (sets)$ be a contravariant functor assigning to each base space B the set of isomorphism classes of families with base B. The functor $\mathcal{F}$ is called a **moduli functor** to the moduli problem.

Definition 24 *Let $\mathcal{M} \in \mathrm{Obj}(\mathcal{C})$ and $\Psi : \mathcal{F} \to \mathrm{Mor}(\_, \mathcal{M})$ be a natural transformation. The pair $(\mathcal{M}, \Psi)$ is a **fine moduli space** for $\mathcal{F}$ iff $\Psi : \mathcal{F} \to \mathrm{Mor}(\_, \mathcal{M})$ is an isomorphism.*

Let $\mathcal{M} \in Obj(\mathcal{C})$ and $\Psi : \mathcal{F} \to Mor(\_, \mathcal{M})$ be a natural transformation. Assume that base change is done by means of fibre products and let $X \stackrel{\phi}{\to} B$ be a family over $B \in Obj(\mathcal{C})$ with corresponding morphism $\phi_B := \Psi(\phi) \in Mor(B, \mathcal{M})$. Note that $\mathcal{F}(\{pt\}) \simeq Mor(\{pt\}, \mathcal{M}) \simeq \mathcal{M}$, so, intuitively, closed points of $\mathcal{M}$ represent the isomorphism classes of the objects under consideration, and $\phi_B(b)$ is the moduli point of the fibre $\phi^{-1}(b)$. A fine moduli space implies the existence of a universal family $\mathbf{1} : \mathcal{X} \to \mathcal{M}$ such that each family $X \stackrel{\phi}{\to} B$ is obtained from $\mathcal{X} \to \mathcal{M}$ by a unique base change

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \phi \downarrow & & \downarrow 1 \\ B & \stackrel{\phi_B}{\to} & \mathcal{M}. \end{array}$$

The universal family $\mathbf{1} : \mathcal{X} \to \mathcal{M}$ is the family corresponding to $id : \mathcal{M} \to \mathcal{M}$, i.e., $\Psi(\mathbf{1}) = id$. Conversely, for any morphism $\phi_B : B \to \mathcal{M}$ we obtain via base change a family $D := B \times_{\mathcal{M}} \mathcal{X} \to B$. The families $X \stackrel{\phi}{\to} B$ and $D \to B$ are isomorphic since each of them corresponds to ϕ_B.

Unfortunately, for a reasonable category $\mathcal{C}$, there are only a few problems where a fine moduli space exists. There are two possibilities to remedy this problem. The first one is an enlargement of the category $\mathcal{C}$ (e.g., from schemes to algebraic stacks or algebraic spaces). Or, secondly, one weakens the requirements for the pair $(\mathcal{M}, \Psi)$.

Definition 25 *The pair $(\mathcal{M}, \psi)$ is a **coarse moduli space** for $\mathcal{F}$ iff*
(1) $\psi(\{pt\})$ is bijective for the reduced point pt,
(2) For all other pairs $(\mathcal{M}', \psi')$ where $\psi' : \mathcal{F} \to Mor(\_, \mathcal{M}')$ is a natural transformation, there exists a unique natural transformation $\eta : Mor(\_, \mathcal{M}) \to Mor(\_, \mathcal{M}')$ making the diagram

$$\begin{array}{ccc} & F & \\ \psi \swarrow & & \searrow \psi' \\ Mor(\_, \mathcal{M}) & \stackrel{\eta}{\dashrightarrow} & Mor(\_, \mathcal{M}) \end{array}$$

commute.

As for fine moduli spaces the closed points of a coarse moduli space $(\mathcal{M}, \psi)$ represent the isomorphism classes of the objects under consideration.

Proposition 26 *Let $\mathcal{F} : \mathcal{C} \to (sets)$ be a moduli functor.*
(a) A fine moduli space $(\mathcal{M}, \psi)$ for $\mathcal{F}$ is also a coarse moduli space for $\mathcal{F}$.
(b) A coarse/fine moduli space $(\mathcal{M}, \psi)$ for $\mathcal{F} : \mathcal{C} \to (sets)$ is unique up to unique isomorphism.

Proof. (a) For $(\mathcal{M}', \psi')$, where $\psi' : \mathcal{F} \to Mor(\_, \mathcal{M}')$ is a natural transformation, define η by $\eta_B(\phi) := \psi'_B(\psi_B^{-1}(\phi))$ for $B \in Obj(\mathcal{C})$. Since ψ_B is an isomorphism η is the unique natural transformation making the required diagram commute.
(b) Suppose that $(\mathcal{M}', \psi')$ is another coarse moduli space for $\mathcal{F}$ and let $\eta : Mor(\_, \mathcal{M}) \to Mor(\_, \mathcal{M}')$ and $\eta' : Mor(\_, \mathcal{M}') \to Mor(\_, \mathcal{M})$ be the natural transformation from the definition. By uniqueness of η, η' we obtain $id_{Mor(\_, \mathcal{M})} = \eta' \circ \eta$ and $id_{Mor(\_, \mathcal{M}')} = \eta \circ \eta'$ which shows that η is an isomorphism. ■

1.4 Quotients and linearizations of algebraic group-actions

We discuss the notion of a categorical and geometric quotient for algebraic/ complex reductive Lie groups acting algebraically/holomorphically on an affine variety/complex Stein space. In both categories we provide a construction of a categorical quotient and a criterium for the categorical quotient being a geometric quotient. In the affine category we present a constructive proof of the linearization of a given group-action, which will be used in the computation of the moduli space.

For more general results in the complex-analytic category we refer to [Snow 1982], in the case of reductive algebraic groups we refer to [Kraft 1985] or [MFK 1994], and for results on unipotent algebraic groups we refer to [GP 1992].

In the sequel all algebraic groups and affine varieties are defined over an algebraically closed field of characteristic 0.

1.4.1 Quotients

Let G be an algebraic group, X be an affine variety and

$$\alpha : G \times X \longrightarrow X$$

be an algebraic group action of G on X, denoted by "$\bullet_\alpha$".

Definition 27 *The subring*

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) : \forall \sigma \in G \; f(\sigma \bullet_\alpha x) = f(x)\}$$

of $\mathcal{O}(X)$ is called the ***invariant ring*** *of G* ***on*** *X. If $X = \mathbf{K}^n$ then $\mathcal{O}(X)^G = \mathbf{K}[x_1, x_2, ..., x_n]^G$ is called the* ***invariant ring*** *of G.*

The ring of invariant functions plays a crucial role in the construction of a categorical quotient for the algebraic and complex-analytic category. Certainly, it depends on the category of X.

Definition 28 *Let Y be an affine variety and $\phi : X \to Y$ be a morphism.*
(a) The pair (Y, ϕ) is a ***categorical quotient*** *of X by G iff*

$$\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ \pi_2 \downarrow & & \phi \downarrow \\ X & \xrightarrow{\phi} & Y \end{array} \qquad (*)$$

commutes and for any affine variety Z and morphism $\psi : X \to Z$ which satisfies $(*)$ there exists a unique morphism $\eta : Y \to Z$ s.t. $\eta \circ \phi = \psi$. We sometimes denote Y by X/G.
(b) The pair (Y, ϕ) is a ***geometric quotient*** *of X by G iff (Y, ϕ) is a categorical quotient of X by G and the fibres of ϕ are precisely the orbits of G.*

Proposition 29 *The categorical quotient (Y, ϕ) of X by G is unique up to unique isomorphism.*

Proof. Suppose $(Y, \phi), (Y', \phi')$ are categorical quotients of X by G and let $\eta : Y \to Y'$ and $\eta' : Y' \to Y$ be the corresponding maps. Now $\eta \circ \phi = \phi'$ and $\phi = \eta' \circ \phi'$ imply $\eta \circ \eta' \circ \phi' = \phi'$ and $\phi = \eta' \circ \eta \circ \phi$, so η is an isomorphism. ■

In the sequel we refer to the pair (Y, ϕ) as the categorical quotient.

Remark 2 *(a) We require that the categorical quotient Y of X by G lies in the same category as X, which prevents the existence of quotients for several group actions. If G is not reductive then the invariant ring $\mathcal{O}(X)^G$ may not be finitely generated, but the categorical quotient may exist in the larger category of schemes.*
(b)Each fibre $\phi^{-1}(y), y \in Y$, of a categorical quotient (Y, ϕ) contains precisely one closed orbit of the action of G on X. The quotient $Y = X/G$ parameterizes the closed orbits of G, cf. Ch. II.3.2 of [Kraft 1985].
(c) If $G \subset GL_n(\mathbf{K})$ acts linearly on the affine variety X and G is linearly reductive the ***algebraic quotient*** *of X by G, defined by spec$(\mathcal{O}(X)^G)$, coincides with the categorical quotient of X by G by Theorem 1.4.37(b).*

Example 30 *The linear action of $GL_n(\mathbf{C})$ on $\mathbf{C}^n$ partitions $\mathbf{C}^n$ in the orbits $\mathbf{C}^n \backslash \{0\}$ and $\{0\}$. It follows from the fact $\mathcal{O}(\mathbf{C})^{GL_n(\mathbf{C})} = \mathbf{C}$ and Theorem 1.4.37 that the categorical quotient is the reduced space $spec(\mathbf{C}) = \{pt\}$ which can't be a geometric quotient.*

Existence of quotients: Complex spaces

In the category of complex spaces the notion of a categorical and a geometric quotient are defined as in the category of affine varieties. As in the affine category a quotient is unique up to unique isomorphism. We provide a construction of the categorical quotient in the complex category where we follow [Snow 1982].

In the sequel let X be a so called Stein space and G be a reductive complex Lie group which acts holomorphically on X. For our purposes it suffices to note that $\mathbf{C}^n$ is a Stein space and a reductive algebraic group over $\mathbf{C}$ is a reductive complex Lie group. We refer, e.g., to [GF 1974], Ch. V.2 for the definition of Stein spaces and to [FH 1991] for the definition of reductive complex Lie groups. The construction of the categorical quotient is more difficult as in the algebraic category, since there is no equivariant linear embedding available.

Instead of looking at the orbit space of the action of G on X, which even may not be Hausdorff, we consider the quotient of X by the analytic equivalence relation $\sim_G$ on X, given by $x \sim_G y$ iff $f(x) = f(y)$ for all $f \in \mathcal{O}(X)^G$. Let $\pi : X \to X/_{\sim_G}$ be the projection. Note that $X/_{\sim_G}$ is Hausdorff and π is naturally equivariant, i.e., $\pi \circ \sigma = \pi$ for all $\sigma \in G$. On the topological space $X/_{\sim_G}$ we construct a sheaf $\mathcal{O}_{X/_{\sim_G}}$ of local $\mathbf{C}-$algebras as follows. For $U \subset X/_{\sim_G}$ open we define $\Gamma(U, \mathcal{O}_{X/_{\sim_G}})$ to be the algebra of complex valued continuous functions on $X/_{\sim_G}$ which lift via π to holomorphic functions on $\pi^{-1}(U)$. Note that $\pi^* \mathcal{O}_{X/_{\sim_G}}(U) = \mathcal{O}_X(\pi^{-1}(U))^G$. We obtain a Hausdorff $\mathbf{C}-$ringed space $(X/_{\sim_G}, \mathcal{O}_{X/G})$ which is a categorical quotient. Recall that the categorical quotient is denoted by X/G.

Theorem 31 *Let X be a Stein space and G be a reductive complex Lie group acting holomorphically on X.*
(a) The quotient space $X/_{\sim_G}$ with the projection π is the categorical quotient of X by G.
(b) The categorical quotient X/G is isomorphic to a complex Stein space s.t. $\pi : X \to X/G$ is holomorphic.

Proof. We refer to [Snow 1982]. ■

In general, the categorical quotient is a much smaller object than the orbit space. If they are equal then X/G is a geometric quotient. We conclude with a criterium for X/G being a geometric quotient.

Proposition 32 *$X/_{\sim_G}$ is a geometric quotient iff all orbits of G are closed.*

Proof. We refer to [Snow 1982]. ■

1.4.2 Linearization of an algebraic group-action

Let X be an affine variety and G be an algebraic group over an algebraically closed field $\mathbf{K}$ of characteristic 0. Suppose that G acts algebraically on X via

$$\alpha : G \times X \longrightarrow X.$$

The linearization of α is necessary for the computation of equations of the categorical quotient X/G since the existing algorithms require a linear action, cf. Chapter 3.6. On the coordinate ring of X we obtain the morphism

$$\lambda_\sigma(f)(x) := f(\alpha(\sigma^{-1}, x)) = \alpha^*(f)(\sigma^{-1}, x)$$

for $\sigma \in G, f \in \mathbf{K}[X]$ and $x \in X$, called **left-translation**.

Definition 33 *A **linearization** or **equivariant embedding** of α consists of a closed morphism $\rho : G \to GL_n(\mathbf{K})$ and a closed embedding $\phi : X \to \mathbf{K}^n$ s.t.*

$$\phi(\sigma \bullet_\alpha t) = \rho(\sigma) \cdot \phi(t).$$

Such an equivariant embedding always exists. The following three Theorems appear, e.g., in [Borel 1991].

Theorem 34 *Let G be an algebraic group acting algebraically on a variety X. Any finite dimensional vector subspace V of $K[X]$ can be extended to a finite dimensional vector subspace W of $K[X]$ s.t. $G \bullet_\alpha W \subset W$. Moreover a necessary and sufficient condition that W is invariant under left translation is*

$$\alpha^* W \subseteq \mathbf{K}[G] \otimes W.$$

Proof. W.l.o.g. assume that V is generated by a single element f. Choose a decomposition

$$\alpha^*(f) = \sum_{i=1}^{r} g_i \otimes h_i \in \mathbf{K}[G] \otimes \mathbf{K}[X]$$

s.t. r is minimal. Applying left translation to f yields

$$\lambda_\sigma f(x) = \alpha^*(f)(\sigma^{-1}, x) = \sum_{i=1}^{r} g_i(\sigma^{-1})h_i(x)$$

for all $\sigma \in G$ and $x \in X$. Hence

$$\lambda_\sigma(f) = \sum_{i=1}^{r} g_i(\sigma^{-1})h_i$$

so $h_1, h_2, ..., h_r$ span a $\mathbf{K}-$vectorspace basis of $W \supseteq V$ s.t. $\lambda_\sigma(f) \in W$. To see that W is $G-$invariant take $\sigma, \tau \in G$ and choose $c_1, c_2, ..., c_r \in \mathbf{K}$ s.t. $\lambda_{\sigma^{-1}\tau}(f) = \sum_{i=1}^{r} c_i h_i$. We have $\lambda_\tau(f) = \lambda_{\sigma\sigma^{-1}\tau}(f) = \lambda_\sigma(\sum_{i=1}^{r} c_i h_i) = \sum_{i=1}^{r} c_i(\lambda_\sigma(h_i))$, so $\lambda_\tau(f)$ is contained in the $\mathbf{K}-$vectorspace W_σ spanned by $\lambda_\sigma(h_1), \lambda_\sigma(h_2), ..., \lambda_\sigma(h_r)$. Since r is minimal we have $W \cap W_\sigma = W$ and therefore $\lambda_\sigma(h_i) \in W$. If $\dim_{\mathbf{K}}(V) > 1$ then choose for each basis element a minimal $G-$invariant vectorspace as above. The union of these vectorspaces is then the desired $G-$invariant subspace. Now let $W \subset \mathbf{K}[X]$ be a finite-dimensional subspace and $\{f_i\} \cup \{h_j\}$ be a basis for $\mathbf{K}[X]$ s.t. $\{f_i\}$ spans W. For $f \in W$ and $\sigma \in G$ we have $\lambda_\sigma(f) = \sum r_i(\sigma^{-1})f_i + \sum s_j(\sigma^{-1})h_j$, where $\alpha^* f = \sum r_i \otimes f_i + \sum s_j \otimes h_j$, so $\sigma \cdot f \in W$ iff $s_j(\sigma^{-1}) = 0$ for all $\sigma \in G$. Varying $\sigma \in G$ and $f \in W$ we see that $\lambda_\sigma(W) \subseteq W$ iff $\alpha^* W \subseteq \mathbf{K}[G] \otimes W$. ■

Theorem 35 *Any algebraic group G over $\mathbf{K}$ is isomorphic to a closed subgroup of $GL_n(\mathbf{K})$ for some $n \in \mathbf{N}$.*

Proof. Let $\mathbf{K}[G] = \mathbf{K}[f_1, f_2, ..., f_n]$ s.t. $f_1, f_2, ..., f_n$ are the basis of a $G-$invariant subspace V of $\mathbf{K}[G]$. Now $\alpha^* V \subset \mathbf{K}[G] \otimes V$ by Theorem 1.4.34 and there exist uniquely defined $m_{ij} \in \mathbf{K}[G]$ s.t.

$$\alpha^* f_i = \sum_{j=1}^{m} m_{ij} \otimes f_j.$$

Suppose that there exist $m'_{ij} \in \mathbf{K}[G]$ s.t. $\alpha^* f_i = \sum_{j=1}^{m} m'_{ij} \otimes f_j$. Then $0 = (\alpha^* f_i - \alpha^* f_i)(\sigma, \sigma') = \sum_{j=1}^{m} \left(m_{ij}(\sigma^{-1}) - m'_{ij}(\sigma^{-1})\right) f_j(\sigma')$, i.e., we have $0 = \sum_{j=1}^{m} \left(m_{ij}(\sigma^{-1}) - m'_{ij}(\sigma^{-1})\right) f_j$. Since $f_1, f_2, ..., f_n$ form a basis it follows that $m_{ij} = m'_{ij}$. We claim that

$$\begin{aligned} \rho \quad &: \quad G \to GL_n(\mathbf{K}) \\ \sigma \quad &\mapsto \quad \left(m_{ij}(\sigma^{-1})\right) \end{aligned}$$

is the closed embedding of the algebraic group. Since $f_i(\sigma) = f_i(e\sigma) = \sum_{j=1}^m m_{ij}(e)f_j(\sigma)$ we have $f_i = \sum_{j=1}^m m_{ij}(e)f_j$, i.e. $f_i \in \operatorname{Im}\rho^*$. Hence ρ is a closed embedding by Proposition 1.1.13(a). For $\sigma, \tau \in G$ we have $\lambda_\sigma(\lambda_\tau(f_i)) = \lambda_\tau\left(\sum_{j=1}^m m_{ij}(\sigma^{-1})f_j\right) = \sum_{k=1}^m \sum_{j=1}^m m_{kj}(\tau^{-1})m_{ij}(\sigma^{-1})f_k$ and $\lambda_\sigma \circ \lambda_\tau = \lambda_{\sigma\tau}$. From $\lambda_\sigma(\lambda_\tau(f_i)) = \lambda_{\sigma\tau}(f_i) = \sum_{j=1}^m m_{ij}(\tau^{-1}\sigma^{-1})f_j$ we obtain

$$m_{ij}(\tau^{-1}\sigma^{-1}) = \sum_{j=1}^m m_{kj}(\tau^{-1})m_{ij}(\sigma^{-1})$$

and therefore $\rho(\sigma\tau) = \rho(\sigma) \cdot \rho(\tau)$. Let T_{ij} be coordinates for $\mathbf{K}[GL_n(\mathbf{K})] = \mathbf{K}[T_{11}, ..., T_{nn}, \det^{-1}]$. Then $\rho^*(T_{ij})(\sigma) = T_{ij}(\rho(\sigma)) = m_{ij}(\sigma^{-1})$. ■

Theorem 36 *(Existence of equivariant embeddings) Let G be an algebraic group acting algebraically on a variety X. There exists a finite dimensional vectorspace V, a closed embedding $\phi : X \hookrightarrow V$ and a morphism $\rho : G \to GL(V)$ s.t.*

$$\phi(\sigma \bullet_\alpha x) = \rho(\sigma) \cdot \phi(x)$$

for all $\sigma \in G$ and $x \in X$.

Proof. W.l.o.g. we assume that $f_1, f_2, ..., f_m \in \mathbf{K}[X]$ are algebra generators for $\mathbf{K}[X]$ and span a $G-$invariant $\mathbf{K}-$vectorspace V. By Theorem 1.4.34 we have $\alpha^*V \subset \mathbf{K}[G] \otimes V$ and there exist $m_{ij} \in \mathbf{K}[G]$ s.t. $\alpha^* f_i = \sum_{j=1}^m m_{ij} \otimes f_j$. From the proof of Theorem 1.4.35 it follows that the m_{ij} 's are uniquely determined and the map $\rho : G \to GL_n(\mathbf{K}), \sigma \mapsto (m_{ij}(\sigma^{-1}))$is closed. For each $\sigma \in G$ and $x \in X$ we have

$$f_i(\sigma \bullet_\alpha x) = \sum_{j=1}^m m_{ij}(\sigma^{-1})f_j(x). \tag{$*$}$$

Define $\phi : X \to V = \mathbf{K}^m$ by $y_i := f_i(x)$ for $x \in X$, where $y_1, y_2, ..., y_m$ are coordinates on V. The map ϕ is obviously a morphism and, by definition, we have $\phi^*(y_i) = f_i$, i.e. ϕ^* is surjective. By Proposition 1.1.13(a) ϕ is a closed embedding of X. We can write $(*)$ as

$$\phi(\sigma \bullet_\alpha x) = \rho(\sigma) \cdot \phi(x). \ ■$$

We use these results to construct the categorical quotient for any algebraic action of G on X. Note that for arbitrary algebraic groups the categorical quotient need not exists in the category of affine varieties. Nagata provided a famous counterexample. to Hilbert's 14^{th} problem, namely a group action of an algebraic group where the invariant ring is not finitely generated. Investigations of this kind can be found in the work of Greuel and Pfister (cf. [GP 1992]), where they study actions of unipotent algebraic groups.

Existence of quotients: Algebraic varieties

Let G be an algebraic group, X be an affine variety over an algebraically closed field $\mathbf{K}$ of characteristic 0 and

$$\alpha : G \times X \to X$$

an algebraic action of G on X. In general, the categorical quotient of X by G does not exist, because the invariant ring $\mathcal{O}(X)^G$ of G is not always finitely generated, so $spec(\mathcal{O}(X)^G)$ is not an affine variety. But for several classes of groups, e.g., linearly reductive groups, the categorical quotient exists in the category of affine varieties.

A linear algebraic group $G \subset GL_n(\mathbf{K})$ is called **linearly reductive** if each linear representation of G is completely reducible. If this holds for G then for any affine variety X on which G acts algebraically, the categorical quotient of X by G is an affine variety. For details we refer, e.g., to [Kraft 1985] and [MFK 1994].

In the algebraic category, for computational reasons, we consider only linear algebraic and linearly reductive groups acting linearly on embedded varieties. In this case the categorical quotient is given by $spec(\mathcal{O}(X)^G)$.

Theorem 37 *Let $G \subset GL_n(\mathbf{K})$ be a linear algebraic group which acts linearly on a closed subvariety $X \subset \mathbf{K}^n$.*
(a) If the order of G is finite then G is linearly reductive.
(b) If G is linearly reductive then $spec(\mathcal{O}(X)^G)$ is a categorical quotient of X by G.

Proof. (a) is a standard fact in representation theory, cf., e.g., [FH 1991], Ch. 2.1.
(b)We refer, e.g., to [MFK 1994], Ch. 1, Theorem 1.1. ■

The restriction to linear group actions of linear algebraic groups on embedded varieties does not matter, because any algebraic group action can be linearized and the corresponding categorical quotients are isomorphic.

Theorem 38 *If $\rho : G \to GL_n(\mathbf{K})$ and $\phi : X \to \mathbf{K}^n$ form an equivariant embedding of α, then $\phi(x)/\rho(G)$ is a categorical quotient of X by G.*

Proof. Let $\pi : \phi(X) \to \phi(X)/\rho(G)$ be the map of the categorical quotient (induced by $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$),$\psi : X \to Z$ be a morphism of affine varieties, and $\varphi : X \to \phi(X)/\rho(G), x \mapsto \pi(\phi(x))$. Suppose that $\psi(\sigma \bullet_\alpha x) = \psi(x)$ for all $\sigma \in G$ and $x \in X$. For $\psi' := \psi \circ \left(\phi_{|\phi^{-1}(X)}\right)^{-1} : \phi(X) \to Z$ there exists a unique $\eta : \phi(x)/\rho(G) \to Z$ s.t. $\eta \circ \pi = \psi'$ since $\phi(x)/\rho(G)$ is a categorical quotient. We obtain $\eta \circ \varphi = \eta \circ \pi \circ \phi = \psi$ as desired. ■

The existence of a categorical quotient (Y, π) of X by G follows from Theorem 1.4.36 (linearization) and the two theorems above. Note that the categorical quotient Y inherits several properties, like irreducibility and normality, of X, and the proposition below is only an excerpt. For more details we refer to Ch. 0.2 of [MFK 1994].

Proposition 39 *(a) If X is irreducible or normal then so is Y.*
(b) If X is smooth then Y is Cohen-Macaulay.
(c) If H is a linear algebraic group which acts algebraically on X s.t. the actions of G and H commute then H acts on Y and $\pi : X \to Y$ is $H-$equivariant.

Proof. For (a) and (b) we refer to [MFK 1994], Ch. 0.2, and for (c) to [Kraft 1985], Ch. II.4.3. ■

We conclude with a criterium for the categorical quotient being a geometric quotient, which could be stated in more generality.

Theorem 40 *Let $X \subset \mathbf{C}^n$ be closed subvariety variety and $G \subset GL_n(\mathbf{C})$ be closed linearly reductive subgroup acting linearly on X. If all orbits of G are closed then spec$(\mathcal{O}(X)^G)$ is a geometric quotient.*

Proof. We refer, e.g., to [Kraft 1985], Ch. II, Section 3.2. ■

1.5 How to construct the moduli space

We provide an outline for the construction of coarse moduli spaces for singularities which is due to Greuel and Pfister, cf. [GP 1991]. We apply each step to the case of semiquasihomogeneous hypersurface singularities with fixed principal part w.r.t. right equivalence. In Chapter 2 we provide proofs and details. We follow closely [GP 1991] where this scheme for the construction of a moduli space for a specific class of singularities is described.

The Method

Suppose that we have fixed a class of objects. According to Mumford (cf. [MFK 1994]) one has to fix the functor for which a moduli space is to be constructed.

- Fix an equivalence relation. For subvarieties of a fixed $\mathbb{P}^n$ this might be equality (leading to the Hilbert scheme) or birational equivalence. For hypersurface singularities it could be contact equivalence or right equivalence.

- Define the notion of (flat) families of objects over some category of base spaces, together with the notion of base change.
- The equivalence relation has to be extended to families. In general, there may be several different extensions, which may lead to different concepts of a moduli space.

If the functor is fixed, a general method for constructing a moduli space proposed by Greuel and Pfister (cf. [GP 1991]) is as follows.

1. Fix some rough invariants like the Tjurina number.
2. Construct an algebraic family $X \to T$ with finite dimensional base space T which contains all isomorphism classes to be classified. This family should be locally versal.
3. In general, T will contain many equivalent objects and one tries to interpret the equivalence classes as orbits of an algebraic/Lie group G.
4. (a) Describe a big open subset $T_0 \subset T$ such that T_0/G exists as a complex space or an algebraic variety.
   (b) Better, find a stratification $T = \cup T_i$ of T into locally closed $G-$invariant subvarieties T_i s.t. the closure $\overline{T}_i$ is the union of strata and T_i/G exists.
5. Find fine invariants of the objects to be classified, such that at least the reduction of each stratum id characterized exactly by fixing these invariants.

We apply this principle to the construction of a moduli space for semi-quasihomogeneous hypersurface singularities with fixed principal part w.r.t. right equivalence.

Application

We fix a quasihomogeneous polynomial f_0 with isolated singularity at 0. Our Objects are quasihomogeneous power series with fixed principal part. The functor, denoted by $Unf^{-}_{f_0}$, is constructed as follows:

- The equivalence relation is right-equivalence of power series s.t. the fixed part f_0 is preserved.
- We consider families of unfoldings of f_0 negative weight over complex or algebraic spaces.

- The extension of right equivalence of power series to unfoldings is obvious.

We apply the method:

1. Fortunately, no invariants have to be fixed.
2. The family $F : (\mathbf{C}^n, 0) \times T_- \longrightarrow (\mathbf{C}^n, 0)$, defined in Section 2.4, is the desired family which contains all isomorphism classes of objects to be classified by Theorem 2.2.47.
3. The equivalent objects are the orbits of the group E_{f_0} by Proposition 3.4.53.
4. The quotient space T_-/E_{f_0} is a geometric quotient by Theorem 2.4.61.
5. No invariants have to be fixed.

Remark 3 *If we exchange right-equivalence with contact-equivalence above, then the construction of the moduli space is much more complicated. In this case one has to consider deformations instead of unfoldings and the corresponding group is , i.g., not even reductive. In particular, one has to fix additional invariants in* (1) *and* (5) *like the Tjurina number and the Hilbert function of the Tjurina algebra. In step* (4) *a stratification is required because the topological space T/G is neither an algebraic variety nor a complex space. We refer to [GHP 1997].*

Chapter 2

Theoretical construction of the moduli space

We present the construction of Greuel, Hertling and Pfister (cf. [GHP 1997]) of a coarse moduli space for semiquasihomogeneous power series with fixed part. Let $f \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ be such a power series with principal part $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$. Essentially, it consists of two parts. Firstly, a theorem of Arnold is applied to show that the semiuniversal unfolding of negative weight F (over T_-) of f contains all isomorphism classes (w.r.t. $\sim_r$) of semiquasihomogeneous power series with principal part f_0. Secondly, the stabilizer group $G^{\mathbf{w}}_{f_0}$ of f_0 (and not of the germ defined by f_0) and its action on T_-, given by the group E_{f_0}, is constructed, and it is shown that T_-/E_{f_0} is a coarse moduli space. We follow the presentation in (loc. cit.).

Notation: Let f be a semiquasihomogeneous power series of type $(d; \mathbf{w})$ with principal part f_0, let B be s.t. $\{\mathbf{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in B\}$ is a monomial base of the Milnor algebra M_{f_0}, and set $B_- := \{\boldsymbol{\alpha} \in B : |\boldsymbol{\alpha}|_{\mathbf{w}} > d\}$. The monomials $\mathbf{x}^{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in B_-$ are called **upper monomials** of f_0. Let $k = |B_-|$ and set $T_- := \mathbf{C}^k$.

We need the notion of quasihomogeneous maps, Arnold's Theorem on semiquasihomogeneous power series, and the corresponding group actions on T_-.

2.1 The order of quasihomogeneous maps

Let $R = \mathbf{C}\{x_1, x_2, ..., x_n\}$ be the ring of convergent power series and let $Aut(R)$ denote the group of local $\mathbf{C}-$algebra automorphisms of R. We define the weighted order and the notion of quasihomogenity for morphisms $\varphi \in Aut(R)$. We show that the subgroup of $Aut(R)$ of morphisms with weighted order > 0 forms a normal divisor in the subgroup of morphisms with weighted order ≥ 0 which turns out to be crucial in the construction of the induced group action on T_-. In addition we present a theorem of Greuel, Hertling, and Pfister which states that an isomorphism between two semiquasihomogeneous power series of the same type has weighted order ≥ 0.

Let $\mathbf{w} = (w_1, w_2, ..., w_n) \in \mathbf{N}^n$ be weights and let R_d be the ideal generated by all power series of weighted order greater or equal to d.

Definition 41 *(a)Let $id \neq \varphi : R \to R$ be a ring homomorphism. The* ***weighted order*** *of φ,denoted by $ord_{\mathbf{w}}(\varphi)$, is the maximal integer d s.t. $ord_{\mathbf{w}}(\varphi(x_i) - x_i) \geq w_i + d$ for $1 \leq i \leq n$.*
(b) For $d \geq 0$ we define $Aut_{\geq d}(R) := \{\varphi \in Aut(R) : ord_{\mathbf{w}}(\varphi) \geq d\} \cup \{id\}$ and $Aut_{>d}(R) := Aut_{\geq d+1}(R)$.

Proposition 42 *(a) If $\varphi \in Aut_{\geq d}(R)$ where $d \geq 0$, then for all $\lambda \in \mathbf{N}$ we have $\varphi(R_\lambda) \subseteq R_\lambda$ and $\varphi^{-1}(R_\lambda) \subseteq R_\lambda$.*
(b) For $d \geq 0$ $Aut_{\geq d}(R) \subset Aut(R)$ is a subgroup.
(c) If $q > p \geq 0$ then $Aut_{\geq q}(\mathbf{C}\{\mathbf{x}\}) \subset Aut_{\geq p}(\mathbf{C}\{\mathbf{x}\})$ is a normal subgroup.

Proof. (a) By linearity it suffices to show $\varphi(\mathbf{x}^{\boldsymbol{\alpha}}) \in R_\lambda$ for $|\boldsymbol{\alpha}|_w = \lambda$. Now $ord_{\mathbf{w}}(\varphi(x_i) - x_i) \geq w_i + d$ implies $ord_{\mathbf{w}}(\varphi(x_i)) \geq w_i$, hence $ord_{\mathbf{w}}(\varphi(\mathbf{x}^{\boldsymbol{\alpha}})) = \sum_{i=1}^n ord_{\mathbf{w}}(\varphi(x_i^{\alpha_i})) \geq \sum_{i=1}^n w_i\alpha_i = |\boldsymbol{\alpha}|_w = \lambda$. Now assume that there exists $f \in R_\lambda$ s.t. $\varphi^{-1}(f) \in R \backslash R_\lambda$. Since $\dim_{\mathbf{C}} R/R_\lambda < \infty$ there exits $g \in R\backslash R_\lambda$ s.t. $\varphi^{-1}(g) \in R_\lambda$, but $g = \varphi \circ \varphi^{-1}(g) \subset R_\lambda$, a contradiction.
(b) Let $\varphi, \psi \in Aut_{\geq d}(\mathbf{C}\{\mathbf{x}\})$. We have $ord_w((\varphi \circ \psi)(x_i) - x_i) = ord(\varphi \circ (\psi - id)(x_i) + (\varphi - id)(x_i))$. By assumption, $ord((\psi - id)(x_i)) \geq w_i + d$ and $ord((\varphi - id)(x_i)) \geq w_i + d$, hence $\varphi \circ \psi \in Aut_{\geq d}(\mathbf{C}\{\mathbf{x}\})$ by (a). For φ^{-1} we obtain $ord_{\mathbf{w}}(\varphi^{-1}(x_i) - x_i) = ord_{\mathbf{w}}(\varphi^{-1}(x_i - \varphi(x_i))) \geq w_i + d$ again by applying (a).
(c) Note that for $q > 0$ the components of $\varphi \in Aut_{\geq q}(R)$ satisfy $\varphi(x_i) = x_i + r_{\varphi,i}(\mathbf{x})$ where $ord_{\mathbf{w}}(r_{\varphi,i}(\mathbf{x})) \geq w_i + q$. Hence $ord_{\mathbf{w}}(\varphi) = q$ iff $(\varphi - id)R_\lambda \subset R_{\lambda+q}$. For $\psi \in Aut_{\geq p}(R)$ we obtain $(\psi \circ \varphi \circ \psi^{-1} - id)R_\lambda = \psi \circ (\varphi - id)\psi^{-1}(R_\lambda) \subset \psi(R_{\lambda+q}) \subset R_{\lambda+q}$ by applying (a). ■

The group $Aut_{\geq 0}(R)$ can be seen as the quasihomogeneous analogue to $GL_n(\mathbf{C})$;

Example 43 *(a) Let $n = 2$ and $\mathbf{w} = (1, 2)$. Consider the homomorphisms φ, φ' and φ'' with*

$$\varphi(x_1, x_2) = (x_1 + x_2, 2x_2),$$
$$\varphi'(x_1, x_2) = (x_1 + x_2, x_1^4 + x_2),$$
$$\varphi''(x_1, x_2) = (x_2, x_1).$$

Then φ *has order* 0, φ' *has order* 1, *and* φ'' *has order* -1.
(b) The homomorphisms of order 0 *do not necessarily preserve quasihomogenity. From (a) we see that* φ *is of order* 0 *and* $x_1^2 + x_2$ *is quasihomogeneous but* $\varphi(x_1^2 + x_2) = x_1^2 + 2x_1x_2 + x_2^2 + 2x_2$ *is not quasihomogeneous..*

If f is semiquasihomogeneous of type $(d; \mathbf{w})$ and φ is an automorphism of $\mathbf{C}\{x_1, x_2, ..., x_n\}$ of order 0 then $\varphi(f)$ is semiquasihomogeneous of type $(d; \mathbf{w})$. The converse, due to Greuel, Hertling and Pfister (cf. [GHP 1997]), is by no means trivial and plays an important role for the construction of the moduli space.

Theorem 44 *(Greuel-Hertling-Pfister 1997) Let* f *and* g *be semiquasihomogeneous power series of type* $(d; \mathbf{w})$ *and let* $\varphi \in Aut(\mathbf{C}\{x_1, x_2, ..., x_n\})$. *Then*

$$f = \varphi(g) \Longrightarrow ord_{\mathbf{w}}(\varphi) \geq 0.$$

Proof. The proof uses some facts coming from the Gauss-Manin connection, which we do not introduce. Therefore we refer to [GHP 1997], Theorem 2.1. ■

Hence it suffices to consider only automorphisms $\varphi \in Aut(\mathbf{C}\{x_1, x_2, ..., x_n\})$ which satisfy $ord_{\mathbf{w}}(\varphi) \geq 0$.

2.2 Arnold's Theorem on semiquasihomogeneous power series

We solve the problem of step (2) in the construction scheme by using a result of V.I. Arnold, cf. [Arnold 1974] (or [AVG 1985], Ch. 12.6), which says that any semiquasihomogeneous power series with principal part f_0 is right equivalent to $f_0 + \sum_{\boldsymbol{\alpha} \in B_-} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ for some complex numbers $c_{\boldsymbol{\alpha}}$. Hence the germ of the family

$$F(\mathbf{x}, \mathbf{t}) := F_{\mathbf{t}}(\mathbf{x}) := f_0(\mathbf{x}) + \sum_{\boldsymbol{\alpha} \in B_-} t_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$$

over (T_-, t_0) is a semiuniversal unfolding of F_{t_0} for arbitrary $t_0 \in T_-$, and not only for those t_0 contained in a small neighborhood of 0. The proof provides a construction of a morphism of weighted order > 0 which maps a given semiquasihomogeneous power series (with terms of arbitrary high degree) with principal part f_0 to some $F(\mathbf{x}, \mathbf{c}), \mathbf{c} \in T_-$. This construction is essential for the computation of the induced action of the stabilizer $G_{f_0}^{\mathbf{w}}$ of f_0 on T_-. We follow the presentation in [AVG 1985].

In the sequel let $f_0 \in \mathbf{K}[x_1, x_2, ..., x_n]$ be a quasihomogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at 0. The following Lemmata will be needed to prove Arnold's Theorem.

Lemma 45 *Let f be a power series of weighted order d and ϕ be an automorphism of the form $\phi(\mathbf{x}) = \mathbf{x} - \mathbf{v}$ of order $\delta > 0, \mathbf{v} = (v_1, v_2, ..., v_n)$. Then*

$$f(\varphi(\mathbf{x})) = f(\mathbf{x}) - \sum \frac{\partial f}{\partial x_i} v_i(\mathbf{x}) + r(\mathbf{x})$$

for some r of order $> d + \delta$.

Proof. By linearity we can assume that $f = \mathbf{x}^{\boldsymbol{\alpha}}$. Let $\mathbf{m} = (m_1, m_2, ..., m_n)$ with $m_i \geq 0$ and $|\mathbf{m}| > 1$. A monomial in the term of the Taylor series, which corresponds to $\prod_{i=1}^{n} \frac{\partial^{m_i}}{\partial^{m_i} x_i}$, has exponent vector $\mathbf{p} = \boldsymbol{\alpha} - \mathbf{m} + \sum_{i=1}^{n} \mathbf{l}^{(i)}$ where $\mathbf{l}^{(i)} = \sum_{j=1}^{m_i} \mathbf{l}^{(i,j)}$ and $\mathbf{l}^{(i,j)}$ is one of the exponent vectors $\mathbf{l}$ of the decomposition $v_i = \sum v_{i,\mathbf{l}} x^{\mathbf{l}}$. Let $\mathbf{1}_i$ be the exponent vector with 1 at the $i-$th position and 0 at all other positions. We can write $\mathbf{p}$ as

$$\mathbf{p} = \boldsymbol{\alpha} + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left(\mathbf{l}^{(i,j)} - \mathbf{1}_i\right).$$

By assumption, $\left|\mathbf{l}^{(i,j)} - \mathbf{1}_i\right|_{\mathbf{w}} \geq \delta > 0$ and $|\boldsymbol{\alpha}|_{\mathbf{w}} \geq d$, which implies $|\mathbf{p}|_{\mathbf{w}} \geq d + |\mathbf{m}| \delta > d + \delta$. ∎

Lemma 46 *Any semiquasihomogeneous power series $f_0 + f_1$ with fixed part f_0 and $ord_{\mathbf{w}}(f_1) > d$ can we transformed by a formal diffeomorphism of order > 0 to $f_0 + f_1'$ s.t. the terms of degree less than d' of f_1 and f_1' coincide,and the term of f_1' of degree d' reduce to $\sum_{\boldsymbol{\alpha} \in B_{\_}, |\alpha|_{\mathbf{w}} = d'} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ for some $c_{\boldsymbol{\alpha}} \in \mathbf{C}$.*

Proof. Let g denote the sum of all terms of degree d' in f_1 and h denote the terms of f_1 of degree $< d'$. We can write g as

$$g = \sum_{i=1}^{n} \frac{\partial f_0}{\partial x_i} v_i + \sum_{\boldsymbol{\alpha} \in B_{\_}, |\boldsymbol{\alpha}|_{\mathbf{w}} = d'} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$$

for some $c_{\boldsymbol{\alpha}} \in \mathbf{C}$ since the $\mathbf{x}^{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in B_{\_}, |\boldsymbol{\alpha}|_{\mathbf{w}} = d'$ are basis monomials. Since g is quasihomogeneous, $v_1, v_2, ..., v_n$ can be chosen to be quasihomogeneous of degree $d' - d + w_1, d' - d + w_2, , ..., d' - d + w_n$ respectively. Consider the morphism ϕ defined by $\phi(x_i) = x_i - v_i$. We have $ord_w(\phi(x_i) - x_i) =$

$ord_w(v_i) = w_i + d' - d$, so the order of ϕ equals $d' - d > 0$. By applying Lemma 2.2.45 to $f = f_0 + f_1$ we obtain

$$\begin{aligned} f(\varphi(\mathbf{x})) &= f_0(\mathbf{x}) - \sum_{i=1}^{n} \frac{\partial f_0}{\partial x_i} v_i(\mathbf{x}) + r_0(\mathbf{x}) + f_1(\mathbf{x}) - \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i} v_i'(\mathbf{x}) + r_1(\mathbf{x}) \\ &= f_0(\mathbf{x}) + h(\mathbf{x}) + \sum_{\boldsymbol{\alpha} \in B_-, |\boldsymbol{\alpha}|_{\mathbf{w}} = d'} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \\ &\quad + \left(f_1 - h + - \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i} v_i + r_0 + r_1 \right)(\mathbf{x}) \end{aligned}$$

where h_1 is the sum of all terms m from $f_0 + f_1$ with $d < \deg_{\mathbf{w}}(m) < d'$. Note that the order of the terms in brackets is greater than d'. ■

Theorem 47 *(Arnold 1974). Let f be a semiquasihomogeneous power series with fixed principal part f_0. Then*

$$f \sim_r f_0 + \sum_{\boldsymbol{\alpha} \in B_-} c_{\boldsymbol{\alpha}} \cdot \mathbf{x}^{\boldsymbol{\alpha}}.$$

for some $c_{\boldsymbol{\alpha}} \in \mathbf{C}$.

Proof. Let $d' > d$ be the minimal degree of a term in f not appearing in f_0 and set $d'' := \max\{|\boldsymbol{\alpha}|_w : \boldsymbol{\alpha} \in B\_\}$. We use Lemma 2.2.46 to obtain a diffeomorphism ϕ_1 (of weighted degree $d' - d$) s.t. $\phi_1(f) = f_0 + \sum_{\boldsymbol{\alpha} \in B_-, |\boldsymbol{\alpha}|_{\mathbf{w}} < d'+1} c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} + f_1$ where $ord_w(f_1) > d'$. We successively apply Lemma 2.2.46 and obtain in the $k-$th step

$$\phi_k \circ \phi_{k-1} \circ ... \circ \phi_1(f) = f_0 + \sum_{\boldsymbol{\alpha} \in B_-, |\boldsymbol{\alpha}|_{\mathbf{w}} < d'+k} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} + f_k$$

where $ord_w(f_k) > d' + k$. Note that ϕ_k does not change the coefficients $c_{\boldsymbol{\alpha}}$ where $|\boldsymbol{\alpha}|_{\mathbf{w}} < d' + k$. If $d' + k > d''$ then no new terms of the form $c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}$,$\boldsymbol{\alpha} \in B_-$, are introduced. By Theorem 1.1.16 $\mu(f) < \infty$ since f has an isolated singularity at the origin. Hence there exists $N > 0$ s.t. $ord(f_N) > \mu(f_0)$ (not the weighted order) and Theorem 1.1.20 implies that $f \sim_r f_0 + \sum_{\boldsymbol{\alpha} \in B_-} c_{\boldsymbol{\alpha}} \cdot \mathbf{x}^{\boldsymbol{\alpha}}$ via some $\psi \in Aut(\mathbf{C}\{\mathbf{x}\})$ which has $ord_{\mathbf{w}}(\psi) \geq 0$ by Theorem 2.1.44. ■

Proposition 48 *For $\phi := \phi_N \circ ... \circ \phi_1$ and ψ from the previous proof we have*

$$ord_{\mathbf{w}}(\psi \circ \phi) > 0.$$

Proof. If $ord_{\mathbf{w}}(\psi) > 0$ there is nothing to prove. Suppose $ord_{\mathbf{w}}(\psi) = ord_{\mathbf{w}}(\psi \circ \phi) = 0$. Since $Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ is a normal subgroup there exists $\psi' \in Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})$ with $\psi \circ \phi = \phi \circ \psi'$. Note that $ord_{\mathbf{w}}(\phi) > 0$ and $ord_{\mathbf{w}}(\psi^{-1}) = ord_{\mathbf{w}}(\phi \circ \psi') = 0$. Hence $\psi \circ \phi \circ \psi^{(-1)} = \phi \circ \psi' \circ \psi^{-1}$, but $ord_{\mathbf{w}}(\psi \circ \phi \circ \psi^{(-1)}) > 0 = ord_{\mathbf{w}}(\phi \circ \psi' \circ \psi^{-1})$, a contradiction. ■

Remark 4 *In the proof of Arnold's Theorem it suffices to compute the morphisms ϕ_k for $1 \leq k \leq d'' - d' + 1$ because*

$$\phi_{d''-d'+1} \circ ... \circ \phi_1(f) = f_0 + \sum_{\boldsymbol{\alpha} \in B_-, |\boldsymbol{\alpha}|_{\mathbf{w}}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} + f_{d''-d'+1}$$

and $ord_{\mathbf{w}}(f_{d''-d'+1}) > d''$. The application of ϕ_k for $k > d'' - d' + 1$ increases the weighted order f_{k-1} and does not change $f_0 + \sum_{\boldsymbol{\alpha} \in B_-, |\boldsymbol{\alpha}|_{\mathbf{w}}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$.

2.3 The action of the stabilizer of f_0 on T_-

Let $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ be a quasihomogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at 0. We investigate the stabilizer group $G^{\mathbf{w}}_{f_0}$ of f_0 and its induced action on T_- which play a crucial role in the construction of the moduli space. Namely, the action of $G^{\mathbf{w}}_{f_0}$ on T_- is induced as follows. For $t_0 \in T_-$ and $\varphi \in G^{\mathbf{w}}_{f_0}$ there may not exist a t'_0 s.t. $\varphi(F_{t_0}) = F_{t'_0}$, but there exist $\psi \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ and $t'_0 \in T_-$ s.t. $\psi \circ \varphi(F_{t_0}) = F_{t'_0}$ by Arnold's Theorem. We prove that t'_0 is independent of ψ so φ defines a map $\Theta(\varphi) : T_- \to T_-$. It turns out that the map $\Theta(\varphi)$ is an automorphism of T_- defined by quasihomogeneous polynomials, hence $G^{\mathbf{w}}_{f_0}$ acts algebraically on T_- via $\varphi \bullet t := \Theta(\varphi)(t)$. Note that this action need not be linear. In addition it is shown that the induced group $\Theta(G^{\mathbf{w}}_{f_0}) =: E_{f_0} \subset Aut(T_-)$ is finite. We follow the presentation of [GHP 1997].

Let $m_1, m_2, ..., m_k$ denote the upper monomials of f_0, i.e., $F(\mathbf{x}, \mathbf{t}) = f_0(\mathbf{x}) + \sum_{i=1}^{k} t_i m_i$.

Definition 49 *(a) $\varphi \in Aut(\mathbf{C}\{\mathbf{x}\})$ is called **quasihomogeneous** (w.r.t. $\mathbf{w}$) iff φ maps quasihomogeneous elements to quasihomogeneous elements of the same degree.*
(b) $G^{\mathbf{w}} := \{\varphi \in Aut(\mathbf{C}\{\mathbf{x}\}) : \varphi$ is quasihomogeneous w.r.t. $\mathbf{w}\}$.

If φ is quasihomogeneous then $\varphi(x_i)$ is quasihomogeneous of degree w_i, i.e., the components φ_i are quasihomogeneous polynomials. Consider the natural inclusion $i : G^{\mathbf{w}} \hookrightarrow Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})$ and the projection $\pi : Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\}) \to Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})/Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$.

Proposition 50 *The set $G^{\mathbf{w}}$ is a group, in particular,*

$$Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})/Aut_{>0}(\mathbf{C}\{\mathbf{x}\}) \cong G^{\mathbf{w}}.$$

Proof. $Ker(\pi \circ i) = \{\varphi \in G^{\mathbf{w}} : i(\varphi) \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})\} = G^{\mathbf{w}} \cap Aut_{\geq 1}(\mathbf{C}\{\mathbf{x}\}) = \{id\}$. To show surjectivity we construct the inverse to $\pi \circ i$. Let $\varphi \in Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})$ be a representative of $[\varphi] \in Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})/Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$. If $\varphi \in G^{\mathbf{w}}$.then set $\phi(\varphi) := \varphi$. Assume $\varphi \notin G^{\mathbf{w}}$. We have $\varphi(x_i) \in A_{w_i}$ and $\varphi(x_i) = y_i + z_i$, where y_i is quasihomogeneous of degree w_i and $ord(z_i) > w_i$. We define $\phi(\varphi)(x_i) := y_i$ and denote $\phi(\varphi)$ by $\widetilde{\varphi}$. For the determinants of the Jacobians of φ and $\widetilde{\varphi}$, evaluated at 0 we obtain

$$0 \neq \det\left(\frac{\partial(y_i + z_i)}{\partial x_j}\right) = \det\left(\frac{\partial y_i}{\partial x_j}\right) + R(z).$$

The term R contains the derivatives of z is equal to zero at the origin. Therefore both determinants coincide and $\widetilde{\varphi}$ is invertible. It remains to show that $(\pi \circ i \circ \phi)(\varphi) = [\varphi]$. We claim that $ord(\widetilde{\varphi}^{(-1)} \circ \varphi) > 0$,.i.e. $\widetilde{\varphi}^{(-1)} \circ \varphi \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ which implies $[\tilde{\varphi}] = [\varphi]$. We have

$$\begin{aligned}
ord_{\mathbf{w}}(\widetilde{\varphi}^{(-1)} \circ \varphi(x_i) - x_i) &= ord_{\mathbf{w}}(\widetilde{\varphi}^{(-1)}(y_i + z_i) - x_i) \\
&= ord_{\mathbf{w}}(\widetilde{\varphi}^{(-1)}(y_i) + \widetilde{\varphi}^{(-1)}(z_i)) - x_i) \\
&= ord_{\mathbf{w}}(x_i + \widetilde{\varphi}^{(-1)}(z_i)) - x_i) \\
&= ord_{\mathbf{w}}(\widetilde{\varphi}^{(-1)}(z_i)) > w_i. \blacksquare
\end{aligned}$$

Proposition 51 *For any semiquasihomogeneous power series f with principal part f_0 there exists an automorphism $\psi \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ and a $t_0 \in T_-$ s.t. $\psi(f) = F_{t_0}$. If $\psi' \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ and $\psi'(F_{t_0}) = F_{t_0'}$ for some $t_0' \in T_-$ then $t_0 = t_0'$.*

Proof. The existence of ψ and t_0 follows from Arnold's Theorem (Theorem 2.2.47). Assume that there exists $\psi' \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\}), t_0' \in T_-$ s.t. $\psi'(F_{t_0}) = F_{t_0'}$ and $t_0 \neq t_0'$. Consider the automorphisms $\psi_s(x_i) := x_i + s \cdot (\psi'(x_i) - x_i)$. Since $\psi_0 = id, \psi_1 = \psi'$, and $ord_{\mathbf{w}}(\psi_s(x_i) - x_i) = ord_{\mathbf{w}}(x_i + s \cdot (\psi'(x_i) - x_i) - x_i) = ord_{\mathbf{w}}(s \cdot (\psi'(x_i) - x_i))$ we have $ord_{\mathbf{w}}(\psi_s) = ord_{\mathbf{w}}(\psi') > 0$ for $s \neq 0$. The family $G_s := \psi_s(F_{t_0})$ is semiquasihomogeneous with principal part f_0 and connects F_{t_0} and $F_{t_0'}$. We claim that this family is right equivalent to a trivial subfamily of F which contradicts Proposition 2.4.59(b). Arnold's Theorem implies the existence of an automorphism χ_s for each s and of a holomorphic map $\sigma : \mathbf{C} \to T_-$ s.t. $\chi_s(G_s) = F_{\sigma(s)}$, i.e., the family G is right equivalent to the family $F_{\sigma(s)}$. Substituting $\psi_s(F_{t_0})$ for G_s we obtain $\chi_s \circ \psi_s(F_{t_0}) = F_{\sigma(s)}$ which implies

$$F_{t_0} = (\chi_s \circ \psi_s)^{-1}(F_{\sigma(s)}).$$

Hence the family G is trivial. ■

Definition 52 *The stabilizer group of f_0 is $G^{\mathbf{w}}_{f_0} := \{\varphi \in G^{\mathbf{w}} : \varphi(f_0) = f_0\}$.*

Proposition 53 *(a) For any $\varphi \in G^{\mathbf{w}}_{f_0}$ and $t \in T_-$ there exists a unique $s = \Theta(\varphi)(t)$ and an automorphism $\psi \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ s.t. $\psi \circ \varphi(F_t) = F_s$.*
(b) The function $\Theta : G^{\mathbf{w}}_{f_0} \to Aut(T_-)$ is a group homomorphism.
(c) The components of $\Theta(\varphi)(t_i)$ are quasihomogeneous polynomials of degree $\deg_{\mathbf{w}}(m_i) - d$. In particular, the group $\Theta(G^{\mathbf{w}}_{f_0})$ acts algebraically on T_- and this action commutes with a $\mathbf{C}^*-$action on T_-.

Proof. (a) Existence of $s \in T_-$ follows from Theorem 2.2.47, uniqueness from Proposition 2.2.48 and Proposition 2.3.51.
(b) Let $\varphi \in G^{\mathbf{w}}_{f_0}$ and $t \in T_-$. If $\Theta(\varphi)(t) = \Theta(\varphi)(t')$ then $\psi \circ \varphi(F_t) = \psi' \circ \varphi(F_{t'})$ for some $\psi, \psi' \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$. Since $Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ is a normal subgroup it follows from $\varphi^{-1} \circ \psi^{-1}\psi' \circ \varphi(F_{t'}) = F_t$ and Proposition 2.3.51 that $t = t'$, which proves the bijectivity of $\Theta(\varphi)$. Let $\varphi' \in G^{\mathbf{w}}_{f_0}, \psi, \psi', \widetilde{\psi} \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ s.t.

$$\begin{aligned} \psi' \circ \varphi' \circ \psi \circ \varphi(F_t) &= F_s, \\ \widetilde{\psi} \circ \varphi' \circ \varphi(F_t) &= F_{s'} \end{aligned}$$

for some $s, s' \in T_-$. Since $Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$ is a normal subgroup of $Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})$ there exists ψ'' s.t. $\varphi' \circ \psi = \psi'' \circ \varphi'$. Note that $ord_{\mathbf{w}}(\psi' \circ \psi'') > 0$. Hence

$$F_s = \psi' \circ \psi'' \circ \varphi' \circ \varphi(F_t) = \widetilde{\psi} \circ \varphi' \circ \varphi(F_t) = F_{s'}$$

by the uniqueness of s as stated in Proposition 2.3.51. The above equalities can be rewritten as

$$\Theta(\varphi' \circ \varphi)(t) = \Theta(\varphi') \circ \Theta(\varphi)(t)$$

for all $t \in T_-$, so Θ is a group homomorphism.
(c) Let $t_1, t_2, ..., t_k$ be coordinates on T_-, extend $\mathbf{w}$ by k zeros, set $\mathbf{w}' = (w_1, w_2, ..., w_n, d - \deg_{\mathbf{w}}(m_1), ..., d - \deg_{\mathbf{w}}(m_k))$ and extend φ by $\varphi$by $\varphi(t_i) = t_i$. Note that F is quasihomogeneous of type $(d, \mathbf{w}')$ and φ is quasihomogeneous... Hence

$$\varphi(F) = \varphi(f_0 + \sum_{i=1}^{k} t_i m_i) = f_0 + \sum_{i=1}^{k} t_i \varphi(m_i)$$

is quasihomogeneous of type $(d,\mathbf{w}')$. Consider the map ϕ_r constructed in the $r-$th step of the iteration in the proof of Theorem 2.2.47 from the decomposition $g(\mathbf{x})=\sum_{i=1}^n \frac{\partial f_0}{\partial x_i}v_i+\sum_{i=1,\deg_{\mathbf{w}}(m_i)=d'}^k c_i m_i$ of the terms of degree $d'>d$ of $f_0+\sum_{i=1}^k\varphi(m_i)$ (w.r.t. $\mathbf{w}$). If m is a monomial s.t. $\deg_{\mathbf{w}}(m)=d'$ and $\beta(\mathbf{t})m$ is contained in $\varphi(F)$ then $m=\sum_{i=1}^n\frac{\partial f_0}{\partial x_i}\beta(\mathbf{t})v_{i,m}+\beta(\mathbf{t})m_i$ is a decomposition of m (one of the two summands vanishes). In this way we obtain a decomposition

$$g(\mathbf{x},\mathbf{t})=\sum_{i=1}^n\frac{\partial f_0}{\partial x_i}(\mathbf{x})\cdot v_i(\mathbf{x},\mathbf{t})+\sum_{i=1,\deg_{\mathbf{w}}(m_i)=d'}c_i(\mathbf{t})\cdot m_i$$

of the terms of degree d' of F (w.r.t. $\mathbf{w}$) which is quasihomogeneous of type $(d,\mathbf{w}')$, in particular, $\deg_{\mathbf{w}'}(v_i)=w_i$ and $\deg_{\mathbf{w}'}(c_i(\mathbf{t}))=d-\deg(m_i)$. Since $\phi_r(x_i)-x_i=v_i$ we have $\deg_{\mathbf{w}'}(\phi_r)=\deg_{\mathbf{w}'}(v_i)=w_i$ and ϕ_r is quasihomogeneous w.r.t. to $\mathbf{w}'$. We obtain

$$\phi_r\circ\phi_{r-1}\circ\ldots\circ\phi_1(\varphi(F))=f_0+\sum_{i=1,\deg_{\mathbf{w}}(m_i)\le d'}c_i(\mathbf{t})\cdot m_i+R(\mathbf{x},\mathbf{t})$$

where $ord_{\mathbf{w}}(R(\mathbf{x},\mathbf{t}))>d'$. The image $\Theta(\varphi)(t_i)$ is given by the quasihomogeneous polynomial $c_i(\mathbf{t})$. $\blacksquare$

Definition 54 *The group $E_{f_0}:=\Theta(G_{f_0}^{\mathbf{w}})$ is the induced subgroup of $Aut(T_-)$ of the action of $G_{f_0}^{\mathbf{w}}$ on T_-.*

Remark 5 *(a) Unfortunately (for computational purposes) the map ψ of the previous Proposition cannot be omitted. Consider the homogeneous polynomial $f_0=x_1^8+14x_1^4x_2^4+x_2^8$. The permutation $\pi=(12)$ which permutes the variables, is contained in $G_{f_0}^{\mathbf{w}}$. The semiuniversal unfolding of negative weight with principal part f_0 is given by*

$$\begin{aligned}F_{\mathbf{t}}=f_0+t_1x_1x_2^8+t_2x_1^2x_2^7+t_3x_1^3x_2^6+t_4x_2^9+t_5x_1x_2^9+t_6x_1^2x_2^8\\+t_7x_2^{10}+t_8x_1x_2^{10}+t_9x_1^2x_2^9+t_{10}x_1^2x_2^{10}.\end{aligned}$$

An application of π to $F_{\mathbf{t}}$ yields

$$\begin{aligned}\pi(F_{\mathbf{t}})=f_0+t_1x_1^8x_2+t_2x_1^7x_2^2+t_3x_1^6x_2^3+t_4x_1^9+t_5x_1^9x_2+\\t_6x_1^8x_2^2++t_7x_1^{10}+t_8x_1^{10}x_2+t_9x_1^9x_2^2+t_{10}x_1^{10}x_2^2.\end{aligned}$$

But none of the terms in $\pi(F_{\mathbf{t}})-f_0$ is contained in the upper base. As an example, consider $f=f_0+t_4x_2^9$ (the restriction of F to the subspace $t_i=0$ for $i\neq 4$). We apply ψ from Proposition 2.3.51 to $\pi(f)=f_0+t_4x_1^9$ and

obtain $\psi \circ \pi(f) = f_0 + t_4 x_1 x_2^8 - \frac{1}{8} t_4^2 x_1^2 x_2^8$.
(b) It follows from (a) that, in general, E_{f_0} *does not act linearly on* T_-*, even if* $G_{f_0}^{\mathbf{w}} \subset GL_n(\mathbf{C})$*. From a theoretical point of view, this is no restriction since the action is algebraic, so the quotient* T_-/E_{f_0} *still exists in the category of affine varieties. In order to compute the equations of* T_-/E_{f_0} *one has to linearize the action of* E_{f_0} *on* T_- *which is the main topic in Section 3.6.*
(c) Since the action of $G_{f_0}^{\mathbf{w}}$ *on* T_- *is given by quasihomogeneous polynomials the quotient* $T_-/G_{f_0}^{\mathbf{w}}$ *admits a* $\mathbf{C}^*-$*action. It turns out that this action is preserved by the linearization, i.e., the linear action of* E'_{f_0} *commutes with a* $\mathbf{C}^*-$*action, too.*

The construction of the quotient T_-/E_{f_0} is considerably simplified by the fact that E_{f_0} is finite, which is proved below. Together with Theorem 1.4.40(b) this implies that T_-/E_{f_0} is even a geometric quotient. In the sequel we denote by

$$V := \bigoplus_{|\boldsymbol{\alpha}|_{\mathbf{w}}=d} \mathbf{C} \cdot \mathbf{x}^{\boldsymbol{\alpha}}$$

the $\mathbf{C}-$vectorspace of quasihomogeneous polynomials of weighted degree d.

Lemma 55 *The tangent space* $T_{f_0} G^w(f_0)$ *of the orbit* $G_w(f_0)$ *at* f_0 *is given by* $j(f_0) \cap V$*.*

Proof. The tangent space $T_{f_0} G^{\mathbf{w}}(f_0)$ is the image of the tangent space $T_{id} G^{\mathbf{w}}$ under the orbit map $\omega_{f_0} : \sigma \mapsto \sigma \cdot f_0$. Consider small curve germs

$$\sigma_t(x) := x + \varepsilon(x,t) : (\mathbf{C}^n, 0) \times (\mathbf{C}, 0) \to (\mathbf{C}^n, 0),$$

where $\varepsilon(x,t) = \varepsilon^{(1)}(x)t + \varepsilon^{(2)}(x)t^2 + ...$, in particular, $\sigma_0 = id$. The image of the tangent map are all vectors of the form

$$\frac{d}{dt}(f_0(x + \varepsilon(x,t)) = \sum_{i=1}^{n} \frac{\partial f_0}{\partial x_i}(x) \varepsilon_i^{(1)}(x) \in m_x \cdot j(f_0)$$

which are all elements of $j(f_0) \cap V$. ■.

Theorem 56 *Let* $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ *be a quasihomogeneous polynomial of type* $(d, \mathbf{w})$ *with an isolated singularity at* 0*.*
(a) The group $G_{f_0}^{\mathbf{w}}$ *is finite if* $\overline{w}_1 < \frac{1}{2}, ..., \overline{w}_{n-1} < \frac{1}{2}, \overline{w}_n \leq \frac{1}{2}$*.*
(b) The group E_{f_0} *is finite.*

Proof. (a) The dimension of the algebraic group $G^{\mathbf{w}}$ is

$$\dim G^{\mathbf{w}} = \sum_{i=1}^{n} \#(\text{monomials od degree } w_i).$$

and $G^{\mathbf{w}}$ acts on V. We define

$$j_i(f_0) := \left\langle \frac{\partial f_0}{\partial x_1}, \frac{\partial f_0}{\partial x_2}, ..., \frac{\partial f_0}{\partial x_{i-1}}, \frac{\partial f_0}{\partial x_{i+1}}, ..., \frac{\partial f_0}{\partial x_n} \right\rangle.$$

By Lemma 2.3.55 we have for the tangent space $T_{f_0}G^{w}(f_0) \subset T_{f_0}V$ in f_0 that

$$T_{f_0}G^{\mathbf{w}}(f_0) \cong j(f_0) \cap V.$$

We claim that $\left\{ \mathbf{x}^{\boldsymbol{\alpha}} \frac{\partial f_0}{x_i} : \deg_{\mathbf{w}}(\mathbf{x}^{\boldsymbol{\alpha}}) = w_i \right\}$ is a basis of $T_{f_0}G^{\mathbf{w}}(f_0)$. For any relation with

$$0 = \sum_{i=1}^{n} \sum_{|\boldsymbol{\alpha}|_{\mathbf{w}} = w_i} a_{i,\boldsymbol{\alpha}} \cdot \mathbf{x}^{\boldsymbol{\alpha}} \frac{\partial f_0}{\partial x_i} = \sum_{i=1}^{n} b_i \frac{\partial f_0}{\partial x_i}$$

where $a_{i,\boldsymbol{\alpha}} \in \mathbf{C}$ and $b_i = \sum_{\deg_{\mathbf{w}}(\mathbf{x}^{\boldsymbol{\alpha}}) = w_i} a_{i,\boldsymbol{\alpha}} \cdot \mathbf{x}^{\boldsymbol{\alpha}}$, we have, provided that $b_i \neq 0$, $\deg_{\mathbf{w}}(b_i) = w_i < d - w_j = \deg_{w}(\frac{\partial f_0}{\partial x_j})$ for $i \neq j$. Therefore $b_i \notin j_i(f_0)$ or $b_i = 0$. Since f_0 has an isolated singularity at 0 and $\mathbf{C}\{\mathbf{x}\}$ is local, any permutation of the sequence $\frac{\partial f_0}{\partial x_1}, \frac{\partial f_0}{\partial x_2}, ..., \frac{\partial f_0}{\partial x_n}$ is regular and $\frac{\partial f_0}{\partial x_i}$ is not a zerodivisor in $\mathbf{C}\{\mathbf{x}\}/j_i(f_0)$. This implies $b_i = 0$ for $1 \leq i \leq n$,

$$j(f_0) \cap V = \bigoplus_{i=1}^{n} \bigoplus_{|\boldsymbol{\alpha}|_{\mathbf{w}} = w_i} \mathbf{C} \cdot \mathbf{x}^{\boldsymbol{\alpha}} \frac{\partial f_0}{\partial x_i},$$

and

$$\dim G^{\mathbf{w}}_{f_0} = \dim G^{w} - \dim_{\mathbf{C}} (j(f_0) \cap V) = 0.$$

(b) W.l.o.g. $\overline{w}_1 < \frac{1}{2}, \overline{w}_2 < \frac{1}{2}, ..., , \overline{w}_r < \frac{1}{2}, \overline{w}_{r+1} \leq \frac{1}{2}, ..., \overline{w}_n \leq \frac{1}{2}$. By the generalized Morse Lemma and by Theorem 2.1.44 there exists $\varphi \in G^{\mathbf{w}}$ s.t.

$$\varphi(f_0) = g_0(x_1, x_2, ..., x_r) + x_{r+1}^2 + ... + x_n^2.$$

We define $h_0 := x_{r+1}^2 + ... + x_n^2$ and denote by $T_-^{f_0}, T_-^{g_0+h_0}, T_-^{g_0}$ the corresponding base spaces of the semiuniversal unfoldings of negative weight with fixed principal part $f_0, g_0 + h_0$ and g_0 respectively. We obtain $\varphi^{-1} G^{\mathbf{w}}_{f_0} \varphi = G^{\mathbf{w}}_{g_0+h_0}$ and isomorphisms among the Milnor algebras $M_{f_0} \cong M_{g_0+h_0} \cong M_{g_0}$, which implies

$$T_-^{f_0} = T_-^{g_0+h_0} = T_-^{g_0}.$$

An element $\sigma \in G^{\mathbf{w}}_{g_0+h_0}$ satisfies $\sigma(x_i) = p_i(x_1, x_2, ..., x_r)$ for $1 \leq i \leq r$ since $w_i < w_{r+j}$ for $1 \leq j \leq n-r-1$. Since σ is an isomorphism it follows that $\sigma'(x_i) := p_i, 1 \leq i \leq r$, is contained in $G^{\mathbf{w}}_{g_0}$. Conversely, any element $\sigma' \in G^{\mathbf{w}}_{g_0}$ can be lifted to an element $\sigma \in G^{\mathbf{w}}_{g_0+h_0}$ by $\sigma(x_i) := \sigma'(x_i)$ for $i \leq r$ and $\sigma(x_i) := x_i$ otherwise, hence we obtain a surjective map $G^{\mathbf{w}}_{g_0+h_0} \to G^{\mathbf{w}}_{g_0}$. Let $\sigma \in G^{\mathbf{w}}_{g_0+h_0}$ and $\sigma' \in G^{\mathbf{w}}_{g_0}$ be its image and note that none of the upper monomials $m_1, m_2, ..., m_k$ of g_0+h_0 contain $x_{r+1}, x_{r+2}, ..., x_n$. Therefore $\sigma(F) = \sigma'(F)$ which implies $\Theta(\sigma) = \Theta(\sigma')$. Hence $E_{g_0+h_0} = E_{g_0} = \Theta(G^{\mathbf{w}}_{g_0})$ is finite since $G^{\mathbf{w}}_{g_0}$ is finite by (a). ■

Remark 6 *Under the above hypotheses it is tempting, but wrong, to conclude that* $G^{\mathbf{w}}_{f_0} \subset GL_n(\mathbf{C})$ *or that* $\oplus_{i=1}^n \mathbf{C}\frac{\partial f_0}{\partial x_i}$ *is a* $G^{\mathbf{w}}_{f_0}$*−module. Consider the quasihomogeneous polynomial* $f_0 = x^2 - xy^2 + y^4$ *of type* $(4; 2, 1)$ *with an isolated singularity at* 0 *(*$\mu_{f_0}(0) = 3$*). The the stabilizer group* $G^{\mathbf{w}}_{f_0}$ *has order* 8 *and is generated by the two automorphisms* σ, τ*, where*

$$\sigma(x) = -x + y^2,$$
$$\sigma(y) = y,$$

and $\tau(x) = -x, \tau(y) = iy$*. It is obvious that* $G^{\mathbf{w}}_{f_0} \not\subseteq GL_2(\mathbf{C})$*. Moreover,* $\mathbf{C}\frac{\partial f_0}{\partial x} \oplus \mathbf{C}\frac{\partial f_0}{\partial y}$ *is not a* $G^{\mathbf{w}}_{f_0}$*−module since* $\sigma^{-1} \bullet \frac{\partial f_0}{\partial y} = 2y\frac{\partial f_0}{\partial x} + \frac{\partial f_0}{\partial y}$*.*

2.4 The moduli space

For the construction of a moduli space for semiquasihomogeneous hypersurface singularities with fixed principal part f_0 w.r.t. right equivalence we have to construct a moduli functor as described in Section 1.5, denoted by $Unf^-_{f_0}$. Note that In order to do so we have to extend the notion of unfoldings of negative weight and right equivalence of unfoldings to families of unfoldings of negative weight and right equivalence of such families and we mention that the family $F : \mathbf{C}^n \times T_- \to \mathbf{C}$ is semiuniversal. Initially, the moduli space $\mathcal{M}$ is a complex space (more precisely, a Stein space), but it turns out that $\mathcal{M}$ also carries the structure of an affine variety. In addition, $\mathcal{M} \neq \{0\}$ implies that $\mathcal{M}$ is not a fine moduli space.

Throughout this section let $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ be a quasihomogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at 0. Let $\mathcal{S}$ denote one of the categories complex/algebraic spaces, or complex germs. Note that $T_- = \mathbf{C}^k$ where k is the number of upper monomials of a monomial basis of the Milnor algebra M_{f_0}. We follow the construction of [GHP 1997].

2.4.1 The functor $Unf_{f_0}^-$

We have to extend the notion of an unfoldings of negative weight with fixed principal part f_0 to families of unfoldings of negative weight.

Definition 57 *Let $S \in \mathcal{S}$ be a base space.*
(a) A ***family of unfoldings over*** *S is a morphism $\phi : (\mathbf{C}^n, 0) \times S \to (\mathbf{C}, 0) \times S$ s.t.*

$$\begin{array}{ccc} (\mathbf{C}^n, 0) \times S & \stackrel{\phi}{\longrightarrow} & (\mathbf{C}, 0) \times S \\ \searrow & & \swarrow \\ & S & \end{array}$$

commutes.
(b) A ***morphism*** *of two families ϕ, ϕ' of unfoldings over S is a morphism $\Psi : (\mathbf{C}^n, 0) \times S \to (\mathbf{C}^n, 0) \times S, \Psi(x, s) = (\psi(x, s), s)$ s.t. $\phi \circ \Psi = \phi'$.*
(c) Two families ϕ, ϕ' of unfoldings over S are ***right equivalent****, denoted by $\phi \sim_r \phi'$, iff there exists a morphism Ψ of ϕ and ϕ' s.t. for all $s \in S, \psi_s \in Aut((\mathbf{C}^n, 0))$.*

Let ϕ, ϕ' be two right equivalent families (via Ψ) of unfoldings over $S \in \mathcal{S}$. For each $s \in S$, the germ $\phi : (\mathbf{C}^n, 0) \times (S, s) \to (\mathbf{C}, 0) \times (S, s), \phi(x, s) = (G_s(x), s)$, is an unfolding of G_s over (S, s). The right equivalence can be written as $G_s(\psi(x, s)) = G_s(\psi_s(x)) = G'_s(x)$.

Definition 58 *A morphism*

$$\begin{aligned} \phi : (\mathbf{C}^n, 0) \times S &\longrightarrow (\mathbf{C}, 0) \times S, \\ (x, s) &\longmapsto (G(x, s), s) = (G_s(x), s) \end{aligned}$$

is called a ***family of unfoldings of negative weight with principal part*** *f_0, also called a* ***family of f_0^-−unfoldings****, iff ϕ is a family of unfoldings over S and the following condition holds. For any $s \in S, G_s : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ is right equivalent to a semiquasihomogeneous power series with principal part f_0 and the germ of G at $s, G : (\mathbf{C}^n, 0) \times (S, s) \to (\mathbf{C}, 0) \times (S, s)$ is an unfolding of G_s of negative weight.*

Let ϕ be a family of unfoldings of negative weight with principal part f_0 over $S \in \mathcal{S}$, let $T \in \mathcal{S}$ and $\varphi : T \to S$ be a morphism. The induced map

$$\begin{aligned} \varphi^*\phi \quad &: \quad (\mathbf{C}^n, 0) \times T \longrightarrow (\mathbf{C}, 0) \times T, \\ (x, t) \quad &\longmapsto \quad (G(x, \varphi(t)), t) \end{aligned}$$

is a family of unfolding of negative weight with principal part f_0 over T, also denoted by φ^*G.

Let $\{m_1, m_2, ..., m_k\}$ be the set of upper monomials of the Milnor algebra M_{f_0}. We define a family of f_0^-−unfoldings

$$\begin{aligned} F &: \mathbf{C}^n \times T_- \longrightarrow \mathbf{C}, \\ F(x,t) &:= f_0(x) + \sum_{i=1}^{k} t_i m_i \end{aligned}$$

and call it the **semiuniversal family of unfoldings of negative weight of semiquasihomogeneous power series with principal part** f_0. This name is justified by the proposition below.

Proposition 59 *Let f be a semiquasihomogeneous power series with principal part f_0.*
(a) For each $t \in T_-$ the restriction of F to the germ (T_-, t) is a semiuniversal unfolding of F_t of negative weight.
(b) T_- does not contain trivial subfamilies.
(c) $T_- = \{0\}$ iff f_0 is simple or simple elliptic.
(d) If f_0 is neither simple nor simple elliptic then there exist $t, t' \in T_-$ arbitrary close to 0 s.t. $F_t \sim_r F_{t'}$ (that is, the unfolding F is not universal in any neighborhood of 0).

Proof. For the proof of (a) and (b) we refer to [GHP 1997], Remark 1.1 and Lemma 1.2.
(c) due to [Saito 1974].
(d) The group μ_d of $d-$th roots of 1 acts on T_- via $\lambda \bullet t_i = \lambda^{\deg(t_i)} t_i$ where $\deg(t_i) := d - \deg_{\mathbf{w}}(m_i)$. For $\lambda \in \mu_d$ we obtain $F_t(\lambda \bullet x) = F(\lambda \bullet x, t) = f_0(x) + \sum_{i=1}^{k} \lambda^{-\deg(t_i)} t_i m_i = F_{\lambda^{-1} \bullet t}(x)$, i.e., $F_t \sim_r F_{\lambda^{-1} \bullet t}$. ■

The functor for our moduli problem is defined as follows

$$\begin{aligned} Unf_{f_0}^- &: \mathcal{S} \longrightarrow (sets), \\ S &\in Obj(\mathcal{S}) \longmapsto \left\{ [\phi]_{\sim_r} : \phi \text{ a family of } f_0^- - \text{unfoldings over } S \right\}, \\ T \overset{\varphi}{\to} S &\in Mor(T,S) \longmapsto \left([\phi]_{\sim_r} \mapsto [\varphi^* \phi]_{\sim_r} \right). \end{aligned}$$

Proposition 60 *For the reduced point $\{pt\} \in \mathcal{S}$ we obtain*

$$Unf_{f_0}^-(\{pt\}) = \left\{ [f]_{\sim_r} : f \text{ semiquasihomogeneous with principal part } f_0 \right\}.$$

Proof. By definition, a family ϕ of negative weight with principal part f_0 over $\{pt\}$ is a quasihomogeneous power series G with principal part f_0. ■

By Theorem 2.2.47 the quasihomogeneous power series G is right equivalent to F_t for some $t \in T_-$. Hence $Unf^-_{f_0}(\{pt\})$ is the desired set of equivalence classes classifying semiquasihomogeneous hypersurface singularities with principal part f_0 w.r.t. right equivalence. We can solve the classification problem by providing a coarse moduli space for $Unf^-_{f_0}$.

If $\mathcal{S}$ is any of the categories complex/algebraic spaces we denote $Unf^-_{f_0}$ by $Unf^-_{f_0,hol}$ and $Unf^-_{f_0,alg}$ respectively.

2.4.2 The moduli space

We provide a proof that coarse moduli spaces $\mathcal{M}_{hol}$ for $Unf^-_{f_0,hol}$ and $\mathcal{M}_{alg}$ for $Unf^-_{f_0,alg}$ exist and that they are affine varieties.

Theorem 61 *(Greuel-Hertling-Pfister 1997). Let $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0. The geometric quotient*

$$\mathcal{M}_{hol} := T_-/E_{f_0}$$

is a coarse moduli space for the functor $Unf^-_{f_0,hol}$.

Proof. By Theorem 2.3.56.(b) and Proposition 2.3.53.(c) E_{f_0} is finite and acts algebraically on T_-, in particular it acts holomorphically on T_-, hence $\mathcal{M}_{hol}$ is a geometric quotient. by Theorem 1.4.32. The class of $t \in T_-$ is denoted by $[t] \in \mathcal{M}_{hol}$. We have to construct a natural transformation

$$\psi : Unf^-_{f_0,hol} \longrightarrow Mor(\_, T_-/E_{f_0}).$$

Let $f \sim_r f' \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ be semiquasihomogeneous power series of type $(d; \mathbf{w})$ with principal part f_0 s.t. $f \sim_r F_t$ and $f' \sim_r F_{t'}$ for some $t, t' \in T_-$. We obtain a morphism φ s.t. $\varphi(F_t) = F_{t'}$ and $\varphi(f_0) = f_0$. By Proposition 2.3.51 $ord_{\mathbf{w}}(\varphi) = 0$ which implies $\varphi \in E_{f_0}$ and $[t] = [t']$. In this way we obtain a bijection $\psi(pt)$ from the set of right equivalence classes of semiquasihomogeneous power series with principal part f_0 and $\mathcal{M}_{hol}$. For $S \in$ (complex spaces) the map

$$\psi(S) : Unf^-_{f_0,hol}(S) \to Mor(S, T_-/E_{f_0})$$

is constructed as follows. Let $G \in Unf^-_{f_0,hol}(S)$ be a representative and chose a covering $\{U_i\}_{i\in I}$ of S in such a way that Proposition 2.4.59 yields maps $\varphi_i : U_i \to (T_-, t_i)$ and $\varphi_i^* F \sim_r G\mid_{U_i}$. Note that these morphisms are not necessarily unique. Suppose that $U_i \cap U_j =: U_{ij}$ is nonempty and let $s \in U_{ij}$. We obtain $F_{\varphi_i(s)} \sim_r G_s \sim_r F_{\varphi_j(s)}$ which implies $[\varphi_i(s)] = [\varphi_j(s)]$ (as above).

Therefore we can glue $\{\varphi_i\}_{i\in I}$ to a morphism $\varphi : S \to T_-/E_{f_0}$. If $G \sim_r G'$ via ψ then $[\varphi(s)] = [\varphi \circ \psi^{-1}(s)]$ since $F_{\varphi(s)} \sim_r G_s \sim_r G'_{\psi^{-1}(s)} \sim_r F_{\varphi\circ\psi^{-1}(s)}$. Let $\{U'_i\}_{i\in I}$ be a covering of S and let $\{\varphi'_i\}_{i\in I}$ be morphisms as above, i.e. $\widetilde{\varphi}_i^* F \sim_r G \mid_{U'_i}$ and they glue to φ'. We have to show that $\varphi([s]) = \varphi'([s])$. For $s \in U_i \cap U'_j$ we have $F_{\varphi_i(s)} \sim_r G_s \sim_r F_{\widetilde{\varphi}_j(s)}$ which implies $[\varphi_i(s)] = [\varphi_j(s)]$. Hence the map $\psi(S) : [G]_{\sim_r} \mapsto \varphi$ is well defined. For $S' \in$ (complex spaces) and $\phi : S' \to S$ a morphism we claim that

$$\begin{array}{ccc} Unf^-_{f_0,hol}(S) & \to & Mor(S, T_-/E_{f_0}) \\ \downarrow & & \downarrow \\ Unf^-_{f_0,hol}(S') & \to & Mor(S', T_-/E_{f_0}) \end{array}$$

commutes. Let $\varphi' = \psi(S')(\phi^* G)$ and let G and φ be as above. We have to show that $\varphi' = \varphi \circ \phi$. Since $F_{\varphi'(s)} \sim_r G_{s'}$ for $s' \in S'$, we obtain for $s \in S$ and $s' := \phi(s)$

$$F_{\varphi(\phi(s))} \sim_r G_{\phi(s)} = G_{s'} \sim_r F_{\varphi'(s)}$$

which implies the claim. ■

Corollary 62 *(a) The geometric quotient $\mathcal{M}_{alg} := T_-/E_{f_0}$ is a coarse moduli space for the functor $Unf^-_{f_0,alg}$.*
(b) T_-/E_{f_0} is an affine variety.

Proof. (a) As in the Theorem 2.4.61.
(b) Since E_{f_0} is finite and acts algebraically on T_-, all orbits are closed. Consider the linearization of E_{f_0}, given by $\rho : E_{f_0} \to GL_l(\mathbf{C})$ and $\phi : T_- \hookrightarrow \mathbf{C}^l$, and set $E'_{f_0} := \rho(E_{f_0})$. Since E_{f_0} is finite the group E'_{f_0} is linearly reductive by Theorem 1.4.37(a). By Theorem 1.4.38 the categorical quotients T_-/E_{f_0} and T_-/E'_{f_0} are isomorphic, and by Theorem 1.4.37 the categorical quotient of $\phi(T_-)/E'_{f_0}$ is isomorphic to $spec(\mathcal{O}(\phi(T_-))^{E'_{f_0}})$, hence it is an affine variety. ■

Proposition 63 *As an affine variety the moduli space $\mathcal{M}_{hol}$ is reduced, irreducible, has the Cohen-Macaulay property, and admits a $\mathbf{C}^*-$action.*

Proof. Obviously, T_- is smooth and irreducible. By Proposition 2.3.53(c) E_{f_0} admits a $\mathbf{C}^*-$action. The claim follows from Proposition 1.4.39. ■

Remark 7 *A fine moduli space for $Unf^-_{f_0}$ does not exist even if $\mathcal{S}$ is the category of complex germs. Suppose $\mathcal{M}$ is a fine moduli space. Then, by Proposition 1.3.26(a) $\mathcal{M}$ is also a coarse moduli space, so $\mathcal{M} \simeq T\_/E_{f_0}$. There exists a universal unfolding over the germ $(T_-/E_{f_0}, 0)$ which can be* *induced from the semiuniversal unfolding F over the germ $(T_-, 0)$ and vice versa. Since T_- does not contain trivial subfamilies the semiuniversal family F over $(T_-, 0)$ would be universal, which contradicts Proposition 2.4.59(d), cf. [GHP 1997].*

Chapter 3

Algorithmic construction of the moduli space

We provide algorithms for computing equations of the moduli space $\mathcal{M}_{hol}$, abbreviated by $\mathcal{M}$, for the functor $Unf^{-}_{f_0,hol}$ where $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ is a quasihomogeneous polynomial with an isolated singularity at 0. $\mathcal{M}_{hol}$ is the moduli space of quasihomogeneous power series with principal part f_0 w.r.t. $\sim_r$, or, in other words, for semiquasihomogeneous hypersurface singularities with principal part f_0. As we have seen in Section 2.4.2 $\mathcal{M}$ carries the structure of an affine variety, hence $\mathcal{M}$ is isomorphic to a closed subvariety $\mathcal{M}'$ of some $\mathbf{C}^m$. It is the aim of this chapter to compute equations for $\mathcal{M}'$. Note that the moduli space $\mathcal{M}$ is determined only up to unique isomorphism.

In the subsequent sections we provide the main algorithm together with necessary subalgorithms. Each algorithm is applied to the corresponding data coming from the quasihomogeneous polynomial $f_0 = x^2z + x^2y - z^5 + y^5$.

Note: In this chapter we provide a description of the algorithms and not of the implementation. No implementation details or tricks are mentioned, in particular we neglect the use of data types (such as `ideal, poly, ring`) which is essential for any use of `Singular`. Furthermore, the implemented algorithms may return more complex objects than described here. For a description of the interface of the implementation we refer to Chapter 5.

A procedure name in `typewriter` refers to a procedure in Singular or to an algorithm described here.

3.1 An algorithm for computing equations of the moduli space

Let $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0. The coarse moduli space $\mathcal{M}_{hol}$ for the functor $Unf^-_{f_0,hol}$ is isomorphic to the quotient T_-/E'_{f_0} which can be embedded in $\mathbf{C}^m$ by computing the invariant ring of the group E'_{f_0} which is the linearization of the action of E_{f_0} on T_-. We present the main algorithm which computes the desired equations of $\mathcal{M}_{hol}$ as a closed subvariety of $\mathbf{C}^m$. Roughly speaking, the algorithm consists of the computation of the quotient of T_- w.r.t. the action of the finite group E_{f_0} which is induced from the stabilizer group $G^{\mathbf{w}}_{f_0}$ acting on T_-, where the latter action is computed from Arnold's Theorem (cf. Theorem 2.2.47). If E_{f_0} does not act linearly on T_- the action has to be linearized and T_- and E_{f_0} have to be replaced above by $\phi(T_-)$ and E'_{f_0} respectively. More precisely, the computation is divided in the following 6 steps.

1. Compute equations of the stabilizer group $G^{\mathbf{w}}_{f_0}$ of f_0.

2. Compute the induced action α of $G^{\mathbf{w}}_{f_0}$ on T_- by means of Arnold's Theorem on semiquasihomogeneous power series. Note that $\sigma \bullet_\alpha t := \Theta(\sigma)(t)$ where Θ is the map defined in Proposition 2.3.53.

3. If necessary, linearize the action α, i.e., find a representation $\rho : G^{\mathbf{w}}_{f_0} \to GL_l(\mathbf{C})$ and a closed embedding $\phi : T_- \hookrightarrow \mathbf{C}^l$ s.t.
$$\phi(\sigma \bullet_\alpha x) = \rho(\sigma) \cdot \phi(x).$$

4. Compute the ideal of the group $E'_{f_0} := \rho(G^{\mathbf{w}}_{f_0})$. Note that E'_{f_0} acts linearly on $\mathbf{C}^l$ and is the linearization of the action of E_{f_0} on T_-.

5. Compute generators $h_1, h_2, ..., h_m$ of the invariant ring $\mathbf{C}[t_1, t_2, ..., t_l]^{E'_{f_0}}$ of E'_{f_0} by means of Derksen's algorithm. In general, this step requires the application of the Reynolds operator of E'_{f_0}.

6. Compute the ideal I of the image of $\phi(T_-)$ under the morphism
$$\Phi : \mathbf{C}^l \to \mathbf{C}^l/E'_{f_0} \hookrightarrow \mathbf{C}^m$$
given by $\Phi^*(y_i) = h_i$. The coarse moduli space for $Unf^-_{f_0}$ is given by the variety $\mathcal{V}(I)$ defined by I, i.e.,
$$\mathcal{V}(I) \simeq \mathcal{M}_{hol} \simeq Unf^-_{f_0}(\{pt\}).$$

In the algorithm the (possibly nonlinear) action α is represented by the polynomials $p_1(\mathbf{s},\mathbf{t}),...,p_k(\mathbf{s},\mathbf{t})$ where $\mathbf{s}=(s_1,s_2,...,s_r),\mathbf{t}=(t_1,t_2,...,t_k)$. The linearized action is represented by the polynomials $p_1(\mathbf{s},\mathbf{t}),...,p_l(\mathbf{s},\mathbf{t})$ where $l\geq k$ and $\mathbf{t}=(t_1,t_2,...,t_l)$, which also encode the representation ρ. From a theoretical point of view, the computation of E'_{f_0} is not necessary, but this may simplify the computation of the Reynolds operator since $\left|E'_{f_0}\right|\leq\left|G^{\mathbf{w}}_{f_0}\right|$.

Algorithm `ModEqn`(f_0)

In: quasihomogeneous polynomial f_0 with an isolated singularity at 0.
Out: equations $f_1,f_2,...,f_q$ of the moduli space of $Unf^-_{f_0}$.
begin
$\mathbf{m}$:=`UpperMonomials`(f_0); //
k :=`size`$(\mathbf{m})$;
$G^{\mathbf{w}}_{f_0}$:=`StabEqns`(f_0); // variables $s_1,...,s_r$
// variables are $s_1,s_2,...,s_r,t_1,t_2,...,t_k,x_1,x_2,...,x_n$
$F_t:=f_0+\sum_{i=1}^{k}t_i\mathbf{m}[i]$;
$\{p_1(\mathbf{s},\mathbf{t}),...,p_k(\mathbf{s},\mathbf{t})\}$:=`ArnoldAction`$(G^{\mathbf{w}}_{f_0},F_t)$;
$(\{p_1(\mathbf{s},\mathbf{t}),...,p_l(\mathbf{s},\mathbf{t})\},\boldsymbol{\phi})$:=`LinearizeAction`$(G^{\mathbf{w}}_{f_0},\{p_1,...,p_k\})$
// variables are $s_1,s_2,...,s_r,t_1,t_2,...,t_k,t_{k+1},...,t_l$
$\{g_1,...,g_l\}:=\{\phi_1(t_1,...,t_k),\phi_2(t_1,...,t_k),...,\phi_l(t_1,...,t_k)\}$;
E'_{f_0} := `ImageVariety`$(G^{\mathbf{w}}_{f_0},p_1(\mathbf{s},\mathbf{t}),...,p_l(\mathbf{s},\mathbf{t}))$;
// variables are $s_1,s_2,...,s_{r'},t_1,t_2,...,t_k,t_{k+1},...,t_l$;
$\{g'_1,...,g'_{l'}\}$:= `ImageVariety`$(\{0\},\{g_1,...,g_l\})$; // image of parameterization
$\{f_1,...,f_q\}$:=`QuotientEquations`$(E'_{f_0},\{p_1(\mathbf{s},\mathbf{t}),...,p_l(\mathbf{s},\mathbf{t})\},\{g_1,...,g_l\})$;
return$(\{f_1,...,f_q\})$
end;

3.2 Computing the image of an ideal

The computation of the image of an ideal (or variety) is needed two times in the algorithm. Let $I:=\langle f_1,f_2,...,f_r\rangle\subset\mathbf{K}[x_1,x_2,...,x_n]$ and $J\subset\mathbf{K}[y_1,y_2,...,y_m]$ be defining ideals of subvarieties $V\subset\mathbf{K}^n$ and $W\subset\mathbf{K}^m$ respectively. A morphism

$$\phi:V\longrightarrow W$$

is defined on the ring level as a homomorphism

$$\phi^*:\mathbf{K}[y_1,y_2,...,y_m]/J\longrightarrow\mathbf{K}[x_1,x_2,...,x_n]/I$$

where $\phi^*(J) \subset I$. The Zariski-closure of the image of ϕ is a closed subvariety $\overline{\phi(V)} \subset \mathbf{K}^m$. The defining ideal of $\overline{\phi(V)}$ can be computed as follows (cf. Proposition 2.1.3 in [Vasconcelos 1997]).

Proposition 64 *The defining ideal of $\overline{\phi(V)}$ is given by*

$$\mathcal{I}(\overline{\phi(V)}) = \langle y_1 - \phi_1^*(\mathbf{x}), ..., y_m - \phi_m^*(\mathbf{x}), f_1, ..., f_r \rangle \cap \mathbf{K}[y_1, y_2, ..., y_m],$$

where $\overline{\phi(V)}$ denotes the Zariski closure of $\phi(V)$.

Proof. The proof is a slight modification of the proof of Theorem 3.3.1 of [CLS 1992] where the case $I = 0$ is considered. By $\mathbf{x}$ we denote the point $(x_1, x_2, ..., x_n) \in \mathbf{K}^n$. The graph Γ_ϕ of ϕ is given by

$$\Gamma_\phi := \{\mathbf{x}, \phi_1(\mathbf{x}), \phi_2(\mathbf{x})..., \phi_m(\mathbf{x}) : \mathbf{x} \in V\} \subset \mathbf{K}^n \times \mathbf{K}^m$$

and its Zariski-closure has defining ideal

$$\mathcal{I}(\overline{\Gamma_\phi}) = \langle f_1, f_2, ..., f_r, y_1 - \phi_1^*(\mathbf{x}), y_2 - \phi_2^*(\mathbf{x}), ..., y_m - \phi_m^*(\mathbf{x}) \rangle .$$

The claim follows from elimination theory. ■.

Note that the ideal J is irrelevant (as soon as $\phi^*(J) \subset I$) in the practical computation of $\mathcal{I}(\overline{\Gamma_\phi})$. In our applications of the algorithm below we will not make use of J.

Algorithm `ImageVariety`$(\{f_1, f_2, ..., f_r\}, \{\phi_1, \phi_2, ..., \phi_m\})$

In: Generators $f_1, f_2, ..., f_r$ for I, the components of $\phi = (\phi_1, \phi_2..., \phi_m)$.
Out: $(g_1, g_2, ..., g_p)$ equations of $\overline{\phi(V)}$.
Fix an elimination order on $K[x_1, x_2, ..., x_n, y_1, y_2, ..., y_m]$ s.t. $x_i > y_j$
begin
$\{g_1, ..., g_p\} :=$ `eliminate`$(\{f_1, ..., f_r, y_1 - \phi_1, ..., y_m - \phi_m\}, y_1 \cdot ... \cdot y_m)$;
return$(\{g_1, g_2, ..., g_p\})$
end;

3.3 A basis for T_-

Recall that any semiquasihomogeneous power series $f \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ of type $(d; \mathbf{w})$ with principal part f_0 is right equivalent to $f_0 + \sum_{i=1}^k c_i m_i$ where $m_1, m_2, ..., m_k$ are the monomials from a basis of the Milnor algebra of f_0 which satisfy $\deg_{\mathbf{w}}(m_i) > d$. By definition, f_0 has an isolated singularity at 0.

Algorithm `UpperMonomials`(f_0)

In: quasihomogeneous polynomial f_0 with an isolated singularity at 0.
Out: $\{m_1, m_2, ..., m_k\}$ upper monomials.
begin
$\{m'_1, ..., m'_{k'}\}$:=`kbase(std(jacob(`f_0`)))`;
$\{m_1, ..., m_k\} := \{m'_j : 1 \leq j \leq k', \deg_{\mathbf{w}}(m'_j) > d\}$;
return($\{m_1, m_2, ..., m_k\}$)
end;

Example 65 *The polynomial $f_0 = x^2z + x^2y - z^5 + y^5$ is quasihomogeneous of type $(5; 2, 1, 1)$ and has an isolated singularity at 0 ($\mu_{f_0}(0) = 24$). An upper basis of the Milnor algebra $M_{f_0} = \mathbf{C}\{x, y, z\}/\left\langle \frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}, \frac{\partial f_0}{\partial z} \right\rangle$ is given by the monomials y^3z^3, x^2y^3, x^2y^2 of weighted degrees $6, 7$ and 6 respectively. The semiuniversal unfolding of f_0 of negative weight is given by*

$$F = f_0 + t_1y^3z^3 + t_2x^2y^3 + t_3x^2y^2.$$

3.4 The stabilizer group $G^{\mathbf{w}}_{f_0}$

Let $f \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ be a quasihomogeneous power series of type $(d; \mathbf{w})$ with principal part $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$. We are interested in quasihomogeneous automorphisms of $\mathbf{C}\{x_1, x_2, ..., x_n\}$ w.r.t. the weights $w_1, w_2, ..., w_n$ which leave f_0 invariant. Each component φ_i of a quasihomogeneous automorphism is quasihomogeneous of degree w_i, hence φ is of the form

$$\varphi(x_i) = \sum_{j=1, w_j | w_i}^{n} a_{ij} x_j^{\frac{w_i}{w_j}} + \sum_{|\boldsymbol{\alpha}|_w = w_i, \alpha_i = 0} b_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$$

for some $a_{ij}, b_{\boldsymbol{\alpha}} \in \mathbf{C}$ s.t. $\det(\frac{\partial \varphi_i}{\partial x_j}(0)) \neq 0$. We can compute equations defining the variety of $G^{\mathbf{w}}_{f_0}$ as a set by comparing coefficients of f_0 and $f_0 \circ \varphi$. By definition, f_0 has an isolated singularity at 0.

Proposition 66 *Let $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ be a quasihomogeneous with an isolated singularity at 0 and I be the ideal generated by the equations obtained by comparing coefficients of f_0 and $f_0 \circ \varphi$. The affine variety $\mathcal{V}(\sqrt{I})$ is an algebraic group.*

Proof. It suffices to show that each the elements of $\mathcal{V}(\sqrt{I})$ are invertible. Assume that $\varphi \in \mathcal{V}(\sqrt{I})$ and $\det(\frac{\partial \varphi_i}{\partial x_j}(0)) = 0$. There exists a coordinate change s.t. $\frac{\partial\varphi_1}{\partial x_j}(0) = 0$ for $1 \leq j \leq n$, i.e., $\varphi(x_1) = \sum_{|\boldsymbol{\alpha}|_w = w_1, \alpha_1 = 0} b_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$. This implies $\frac{\partial\varphi_1}{\partial x_1} = 0$. Note that From the chain rule we obtain

$$\frac{\partial f_0}{\partial x_1} = \sum_{i=1}^{n} \frac{\partial f_0}{\partial x_i}\frac{\partial \varphi_i}{\partial x_1} = \sum_{i>1}^{n} \frac{\partial f_0}{\partial x_i}\frac{\partial \varphi_i}{\partial x_1}$$

since $\frac{\partial\varphi_1}{\partial x_1} = 0$. Hence $\frac{\partial f_0}{\partial x_1}, \frac{\partial f_0}{\partial x_2}, ..., \frac{\partial f_0}{\partial x_n}$ is not a regular sequence which contradicts Proposition 1.1.17. ■

In the implementation the field $\mathbf{K}$ is a finite extension of (or equal to) $\mathbf{Q}$.

Algorithm `StabEqns`(f_0)

In: quasihomogeneous polynomial $f_0 \in \mathbf{K}[x_1, x_2, ..., x_n]$ with an isolated singularity at 0.
Out: equations for the stabilizer in $\mathbf{K}[s_1, s_2, ..., s_r]$.
begin
set $\varphi(x_i) = \sum_{j=1, w_j | w_i}^{n} a_{ij} x_j^{\frac{w_i}{w_j}} + \sum_{|\boldsymbol{\alpha}|_w = w_i, \alpha_i = 0} b_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$;
let $f_1, f_2, ..., f_k$ be the equations obtained by comparing coefficients in $f_0 = f_0 \circ \varphi$.
$\langle g_1, g_2, ..., g_l \rangle =$ `radical`$(\langle f_1, f_2, ..., f_k \rangle)$;
return$(\langle g_1, g_2, ..., g_l \rangle)$
end;

Remark 8 *In all examples of this thesis the ideal $\langle f_1, f_2, ..., f_k \rangle$ was already a radical, but it seems to be tricky to prove this for all quasihomogeneous polynomials with an isolated singularity at 0. To avoid the computation of the radical one can set* `RADICAL=0` *in the actual implementation (cf. Ch. 5.1).*

Example 67 *(cont. of Example 3.3.65) For $f_0 = x^2z + x^2y - z^5 + y^5$ a quasihomogeneous morphism φ is of the form*

$$\begin{aligned} \varphi(x) &= s_1x + s_2y^2 + s_3yz + s_4z^2, \\ \varphi(y) &= s_5y + s_6z, \\ \varphi(z) &= s_7y + s_8z \end{aligned}$$

for some $s_i \in \mathbf{C}$ s.t. $\det(\frac{\partial\varphi}{\partial x_i}(0)) \neq 0$. By comparing coefficients in

$$f(\varphi(x), \varphi(y), \varphi(z)) = f(x, y, z)$$

we obtain the following set of 12 *equations:*

$$\begin{aligned}2s_1s_3s_5+2s_1s_2s_6+2s_1s_3s_7+2s_1s_2s_8,\, 2s_1s_4s_5+2s_1s_3s_6+2s_1s_4s_7+2s_1s_3s_8,\\ 2s_1s_4s_6+2s_1s_4s_8,\, s_5^5-s_7^5+s_2^2s_5+s_2^2s_7-1,\, s_1^2s_5+s_1^2s_7-1,\, s_1^2s_6+s_1^2s_8-1,\\ 5s_5^4s_6-5s_7^4s_8+2s_2s_3s_5+s_2^2s_6+2s_2s_3s_7+s_2^2s_8,\, 2s_1s_2s_5+2s_1s_2s_7,\\ 10s_5^3s_6^2-10s_7^3s_8^2+s_3^2s_5+2s_2s_4s_5+2s_2s_3s_6+s_3^2s_7+2s_2s_4s_7+2s_2s_3s_8,\\ 10s_5^2s_6^3-10s_7^2s_8^3+2s_3s_4s_5+s_3^2s_6+2s_2s_4s_6+2s_3s_4s_7+s_3^2s_8+2s_2s_4s_8.\end{aligned}$$

Note that these equations generate a radical ideal. A standard basis w.r.t. lexicographical is given by

$$\begin{aligned}s_8^6-s_8,\, s_7s_8,\, s_5-s_6+s_7-s_8,\, s_4,\, s_3+s_4,\, s_2,\\ s_6+s_7^5s_8-5s_7^4s_8^2+10s_7^3s_8^3-10s_7^2s_8^4+5s_7s_8^5-s_7-s_8^6+s_8,\\ s_7^5-5s_7^4s_8+10s_7^3s_8^2-10s_7^2s_8^3+5s_7s_8^4-s_8^5+1,\\ 15s_1^2+16s_6^4-16s_6^3s_8+16s_6^2s_8^2-16s_6s_8^3-s_7^4+6s_7^3s_8-16s_7^2s_8^2+26s_7s_8^3-15s_8^4\end{aligned}$$

which implies $s_2 = s_3 = s_4 = 0$*. We rename the variables as follows.* $s_1 = s_1, s_2 = s_5, s_3 = s_6, s_4 = s_7, s_5 = s_8$*, so* φ *writes as*

$$\begin{aligned}\varphi(x) &= s_1x,\\ \varphi(y) &= s_2y + s_3z,\\ \varphi(z) &= s_4y + s_5z.\end{aligned}$$

The corresponding standard basis is given by

$$\begin{aligned}s_2-s_3+s_4-s_5,\, s_4s_5,\, s_4^5-5s_4^4s_5+10s_4^3s_5^2-10s_4^2s_5^3+5s_4s_5^4-s_5^5+1,\\ s_3+s_4^5s_5-5s_4^4s_5^2+10s_4^3s_5^3-10s_4^2s_5^4+5s_4s_5^5-s_4-s_5^6+s_5,\, s_5^6-s_5,\\ 15s_1^2+16s_3^4-16s_3^3s_5+16s_3^2s_5^2-16s_3s_5^3-s_4^4+6s_4^3s_5-16s_4^2s_5^2+26s_4s_5^3-15s_5^4.\end{aligned}$$

These equations define the group $G_{f_0}^{(2,1,1)} \subset GL_3(\mathbf{C})$ *which has order* 20 *and is generated by the matrices*

$$\begin{pmatrix} \xi^2 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{pmatrix}, \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

where $\xi = \exp(\frac{2\pi i}{5})$*. Note that* $s_1 = m_{11}, s_2 = m_{22}, s_3 = m_{23}, s_4 = m_{32}, s_5 = m_{33}$ *and all other* m_{ij}*'s are* 0*.*

3.5 The action of $G^{\mathbf{w}}_{f_0}$ on T_-

Let $f \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ be a quasihomogeneous power series of type $(d; \mathbf{w})$ with principal part $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$. As we have seen in Chapter 2.2.3 the group $G^{\mathbf{w}}_{f_0}$ acts on T_- via

$$\begin{array}{rcl} G^{\mathbf{w}}_{f_0} \times T_- & \longrightarrow & T_- \\ (\varphi, t) & \longmapsto & \Theta(\varphi)(t) = s \end{array}$$

where we take the $s \in T_-$ s.t. $\psi \circ \varphi(F_t) = F_s$ for some $\psi \in Aut_{>0}(\mathbf{C}\{\mathbf{x}\})$. The homomorphism Θ is well defined by Proposition 2.3.53(a) and the action on T_- is algebraic by Proposition 2.3.53(c). We provide an algorithm for the computation of the action of $G^{\mathbf{w}}_{f_0}$ on T_-. The algorithm is an implementation of the proof of Arnold's Theorem on quasihomogeneous power series (Theorem 2.2.47) and delivers the morphism ϕ of order > 0 s.t.

$$\phi(\varphi(F_t)) = F_s + r(\mathbf{x})$$

where $ord_{\mathbf{w}}(r) > d'' := \max\left\{|\boldsymbol{\alpha}|_{\mathbf{w}} : \boldsymbol{\alpha} \in B_-\right\}$. By Remark 2.2.4 we may assume $ord(r) > \mu(f_0)$ (not the weighted order) and by Theorem 1.1.20 there exist $\psi \in Aut(\mathbf{C}\{\mathbf{x}\})$ s.t. $F_s + r(\mathbf{x}) \sim_r F_s$. The Theorem of Greuel, Hertling, and Pfister (Theorem 2.1.44) and Proposition 2.2.48 imply that $ord_{\mathbf{w}}(\psi \circ \phi) > 0$, hence $s \in T_-$ is uniquely determined by the homomorphism ϕ. It remains to compute the action $t_i \mapsto p_i(\mathbf{s}, \mathbf{t})$ by setting $p_i(\mathbf{s}, \mathbf{t}) :=$ coefficient of m_i in $\phi(F_t)$ where $m_1, m_2, ..., m_k$ are the upper monomials of f_0.

Algorithm `ArnoldAction`$(f_0, G^{\mathbf{w}}_{f_0}, B)$

In: quasihomogeneous polynomial $f_0 \in \mathbf{K}[x_1, x_2, ..., x_n]$ with an isolated singularity at 0.
equations for $G^{\mathbf{w}}_{f_0}$ in $\mathbf{K}[s_1, s_2, ..., s_r]$, upper monomials $B = \{m_1, ..., m_k\}$
Out: $(p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), ..., p_k(\mathbf{s}, \mathbf{t}))$ polynomials defining the action of $G^{\mathbf{w}}_{f_0}$ on $T_- = \mathbf{C}^k$.
basering has parameters $s_1, .., s_r, t_1, ..., t_k$ and variables $x_1, ..., x_n$.
begin
$lb := \min\left(|m|_{\mathbf{w}} : m \in B\right)$;
$ub := \max\left(|m|_{\mathbf{w}} : m \in B\right)$;
$F := f_0 + \sum_{i=1}^{k} t_i m_i$;
for $d := lb$ **to** ub **do**
. $g :=$ sum of all monomials in F of weighted degree d;
. $g_1 :=$ `reduce`$(g,$`jacob`$(f_0))$;
. $\mathbf{v} :=$ `lift`$($`jacob`$(f_0), g - g_1)$; // $g - g_1 = \sum_{i=1}^{n} \frac{\partial f_0}{\partial x_i} v_{1i}$;
. $\varphi(x_i) := x_i - v_{1i}(\mathbf{x})$;

```
.     F := φ(F);
.     reduce F w.r.t. the ideal G^w_{f0},.i.e., consider s1, ..., sr as variables.
end;
for i := 1 to k do
.     pi := coefficient of mi in F;
end;
return({p1, p2, ..., pk});
end.
```

Example 68 *(cont. of Example 3.4.67) Let $f_0 = x^2z + x^2y - z^5 + y^5$. The action of the group $G^{(2,1,1)}_{f_0}$ on $T_- = \mathbf{C}^3$, denoted by α, is given by the following polynomials*

$$\alpha^* t_1 = p_1(\mathbf{s}, t_1, t_2, t_3) = (s_5 - s_4)\, t_1,$$
$$\alpha^* t_2 = p_2(\mathbf{s}, t_1, t_2, t_3) = \left(s_5^2 - s_4^2\right) t_2 + 2s_4^2 t_3^2,$$
$$\alpha^* t_3 = p_3(\mathbf{s}, t_1, t_2, t_3) = (s_4 + s_5)\, t_3$$

where $\mathbf{s} = (s_1, s_2, s_3, s_4, s_5)$. In this case, the action of $G^{(2,1,1)}_{f_0}$ is not linear. Note that p_1, p_2, p_3 are quasihomogeneous w.r.t. t_1, t_2, t_3 where $\deg(t_1) = \deg(t_3) = 1$ and $\deg(t_2) = 2$.

3.6 Linearization of the group-action.

In order to compute equations of a categorical quotient of an affine variety X by an algebraic group G we need generators of the invariant ring of G. The existing algorithm require that the action α of G on X is linear. Note that the quotient X/G is isomorphic to the quotient $\phi(X)/\rho(G)$ where (ρ, ϕ) is a linearization of α by Theorem 1.4.38. Therefore we need a linearization of the action of G on X. How to compute equations of the quotient is explained in Section 3.7 below. In this section we provide an algorithm for the linearization of an algebraic group action of an algebraic group G on $\mathbf{C}^k$. In the sequel we denote the action $\alpha(\sigma, t)$ occasionally by $\sigma \bullet_\alpha t$.

We are interested in the action α of the algebraic group $G^{\mathbf{w}}_{f_0}$ on T_- defined as follows

$$\begin{array}{rcl} \alpha & : & G^{\mathbf{w}}_{f_0} \times T_- \longrightarrow T_-, \\ (\sigma, t) & \longmapsto & \Theta(\sigma)(t) \end{array}$$

where $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ is the principal part of a quasihomogeneous power series. In general, this action is not linear, cf. Example 3.5.68 or Remark

2.3.5(b). Recall that $\Theta : G^{\mathbf{w}}_{f_0} \to E_{f_0}$ is onto and E_{f_0} is finite. It follows from Theorem 1.4.40 that $T_{\_}/E_{f_0}$ is a geometric quotient. In order to compute the corresponding equations we need a linearization of the action of E_{f_0} on $T_{\_}$. Since $G^{\mathbf{w}}_{f_0}/\ker\Theta \simeq E_{f_0}$ we linearize the action of $G^{\mathbf{w}}_{f_0}$ on T_{-}, i.e., we construct closed morphisms $\rho : G^{\mathbf{w}}_{f_0} \to GL_l(\mathbf{C})$ and $\phi : T_{-} \hookrightarrow \mathbf{C}^l$ s.t. $\phi(\sigma \bullet_\alpha t) = \rho(\sigma) \cdot \phi(t)$. Equations for the quotient can be constructed from the invariant ring of $G^{\mathbf{w}}_{f_0}$ w.r.t. to the linearized action provided that $\left|G^{\mathbf{w}}_{f_0}\right| < \infty$ or from the invariant ring of the finite group $E'_{f_0} := \rho(G^{\mathbf{w}}_{f_0})$. The performance of `InvariantRing` depends heavily on the group, in particular if the Reynolds operator has to be computed. We refer to Section 5.4.

Note: If $G^{\mathbf{w}}_{f_0}$ acts linearly on T_{-} then $E_{f_0} = E'_{f_0}$.

Algorithm `LinearizeAction`$(G, \{p_1, p_2, ..., p_k\})$

In: Ideal of G in $\mathbf{K}[s_1, s_2, ..., s_r]$, polynomials $p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), ..., p_k(\mathbf{s}, \mathbf{t})$ defining an action of G on $\mathbf{K}^k$.
Out: $\{p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), ..., p_l(\mathbf{s}, \mathbf{t})\}$, defining a linear action of G on $\mathbf{K}^l$, $\{B_1(\mathbf{t}), B_2(\mathbf{t}), ..., B_l(\mathbf{t})\}$ an equivariant embedding $\mathbf{K}^k \hookrightarrow \mathbf{K}^l$.
begin;
$B := \{t_1, t_2, ..., t_k\}$;
for $i := 1$ **to** k **do**
. $h^{(i)}$:=`MinimalDecomposition`(p_i);
. write $p_i = \sum_{j=1}^{n_i} h^{(i)}_{1j} \otimes h^{(i)}_{2j} \in \mathbf{K}[G] \otimes \mathbf{K}[t_1, t_2, ..., t_k]$ with n_i minimal
. **for** $j := 1$ **to** n_i **do**
. **if** $h^{(i)}_{2j} \notin \langle B \rangle_{\mathbf{K}}$ **then begin**
. $B := B \cup \left\{h^{(i)}_{2j}\right\}$;
. $p_{|B|} := h^{(i)}_{2j}(p_1, p_2, ..., p_k)$;
. **end if**
. **end**;
end;
for $i := 1$ **to** $|B|$ **do**
. $p_i = \sum_{j=1}^{|B|} h^{(i)}_{1j} \otimes B_j$; // decomposition of p_i
. $p'_i := \sum_{j=1}^{|B|} h^{(i)}_{1j} \otimes t_j$;
end;
return$(\left\{p'_1, p'_2, ..., p'_{|B|}\right\}, B)$;
end.

The minimal decomposition of the polynomials p_i can be done as follows. Note that the Singular procedure `coef`$(p, x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_k})$ returns a $2 \times m$ matrix M s.t. $p = \sum_{i=1}^m M_{1i} \cdot M_{2i}$, where M_{1i} contains none of the variables $x_{i_1}, x_{i_2}, ..., x_{i_k}$ while $M_{2i} \in \mathbf{K}[x_{i_1}, x_{i_2}, ..., x_{i_k}]$.

Algorithm `MinimalDecomposition`(p)

In: polynomial $p(s_1, s_2, ..., s_r, t_1, t_2, ..., t_k)$.
Out: $2 \times m$ matrix D s.t. $p = \sum_{i=1}^{m} D_{1i} \cdot D_{2i}$ with m minimal, and $D_{1i} \in \mathbf{K}[s_1, s_2, ..., s_r], D_{2i} \in \mathbf{K}[t_1, t_2, ..., t_k]$.
begin;
$M := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \texttt{coef}(p, t_1 \cdot t_2 \cdot ... \cdot t_k)$; // exchange rows
for all $i < j \leq \texttt{ncols}(M)$ **do**
. **if** $M_{2i} = M_{2j}$ **then** $M_{1i} := M_{1i} + M_{1j}$; delete $j - th$ column;
end;
$M' := \texttt{coef}(p, s_1 \cdot s_2 \cdot ... \cdot s_r)$;
for all $i < j \leq \texttt{ncols}(M')$ **do**
. **if** $M'_{2i} = M'_{2j}$ **then** $M'_{1i} := M'_{1i} + M'_{1j}$; delete $j - th$ column;
end;
$D := M$;
for $i := 1$ **to** $\texttt{ncols}(M)$ **do**
. $N := \texttt{MinimalDecomposition}(p - M_{1i}M_{2i})$;
. **if** $\texttt{ncols}(N) + 1 < \texttt{ncols}(D)$ **then** $D := (M_{1i}, M_{2i})^T + N$;
end;
for $i := 1$ **to** $\texttt{ncols}(M')$ **do**
. $N := \texttt{MinimalDecomposition}(p - M'_{1i}M'_{2i})$;
. **if** $\texttt{ncols}(N) + 1 < \texttt{ncols}(D)$ **then** $D := (M'_{1i}, M'_{2i})^T + N$;
end;
return(D);
end.

Proposition 69 *Let α be an action of G on $\mathbf{C}^k$ which commutes with a $\mathbf{C}^*-$action and let* $\deg(t_i)$ *be s.t. the defining polynomials of α are quasihomogeneous w.r.t. these weights, where $t_1, t_2, ..., t_k$ are coordinates on $\mathbf{C}^k$ and the variables of $\mathbf{K}[G]$ have weight* 0.
(a) The linearized action constructed by `LinearizeAction` *commutes with a $\mathbf{C}^*-$action, i.e., the polynomials $p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), ..., p_l(\mathbf{s}, \mathbf{t})$ are quasihomogeneous w.r.t. to the weights* $\deg(t_i)$ *for $1 \leq i \leq k$ and* $\deg(t_r) = \deg(t_i)$ *if t_r comes from the decomposition of $\alpha^* t_i$ for $r > k$.*
(b) The ideal of $\phi(T_-)$ is quasihomogeneous. w.r.t. the weights defined in (a).

Proof. (a) By assumption, the polynomials $p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), ..., p_k(\mathbf{s}, \mathbf{t})$ are quasihomogeneous w.r.t. $\mathbf{w}''$, hence the polynomials h_{ij} in the decomposition $\alpha^* t_i = \sum_{j=1}^{r_i} g_{ij} \otimes h_{ij} \in \mathbf{C}[G^{\mathbf{w}}_{f_0}] \otimes \mathbf{C}[t_1, t_2, ..., t_k]$ are quasihomogeneous w.r.t. $\mathbf{w}''$. If some h_{ij} is not contained in the vectorspace basis constructed so far, a new variable t_r is introduced. We extend the action α by

$$\alpha^* t_r := p_r(\mathbf{s}, \mathbf{t}) := h_{ij}(p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), ..., p_l(\mathbf{s}, \mathbf{t}))$$

and replace h_{ij} by t_r in the polynomials $p_1, p_2, ..., p_r$. By construction, p_1, p_2, ...,p_r are quasihomogeneous..
(b) By Proposition 2.3.53(c) the polynomials $\alpha(\sigma, t_i)$ are quasihomogeneous of degree $\deg(t_i)$. Suppose that $\phi \neq id$, i.e., the action is not linear (otherwise the claim is trivial). For $r > k$ we have $\phi_r = h_{ij}$ for some $i \leq k$ and $j \leq r_i$ which is quasihomogeneous by assumption.. ∎

Example 70 *(cont. of Example 3.5.68) Let* $f_0 = x^2z + x^2y - z^5 + y^5$. *We have to extend the* $\mathbf{C}-$*vectorspace* V*, spanned by* $\langle t_1, t_2, t_3 \rangle_{\mathbf{C}}$*, to a* $G_{f_0}^{\mathbf{w}}-$*invariant vectorspace* W*. The decomposition of the only nonlinear component of* α *equals*

$$\alpha^* t_3 = p_3(s_1, s_2, s_3, s_4, s_5, t_1, t_2, t_3) = \left(s_5^2 - s_4^2\right) \otimes t_2 + 2s_4^2 \otimes t_3^2.$$

Therefore we consider the vectorspace $W := \langle t_1, t_2, t_3, t_3^2 \rangle_{\mathbf{C}}$ *and we set* $t_4 = t_3^2$. *Note that* $\deg(t_4) = 2$. *We obtain*

$$\alpha^* t_4 = \alpha^* t_3^2 = (s_4 + s_5)^2 t_3^2 = \left(s_4^2 + 2s_4s_5 + s_5^2\right) t_4 = \left(s_4^2 + s_5^2\right) t_4$$

since s_4s_5 *is in the ideal of* $G_{f_0}^{\mathbf{w}}$. *Hence* W *is the desired* $G_{f_0}^{\mathbf{w}}-$*invariant vectorspace. The closed embedding* $\phi : T_- \hookrightarrow \mathbf{C}^4$ *is given by*

$$(t_1, t_2, t_3) \longmapsto \left(t_1, t_2, t_3, t_3^2\right)$$

and the representation $\rho : G_{f_0} \to GL_4(\mathbf{C})$ *by*

$$(s_1, s_2, s_3, s_4, s_5) \longmapsto \begin{pmatrix} s_5 - s_4 & 0 & 0 & 0 \\ 0 & s_5^2 - s_4^2 & 0 & 2s_4^2 \\ 0 & 0 & s_4 + s_5 & 0 \\ 0 & 0 & 0 & s_4^2 + s_5^2 \end{pmatrix}.$$

The ideal of the variety $\phi(T_-)$ *is generated by* $t_4 - t_3^2$ *and the generator is quasihomogeneous of type* $(2; 1, 2, 1, 2)$. *The image* $E'_{f_0} := \rho(G_{f_0}^{\mathbf{w}})$ *is a linearization of the action of* E_{f_0} *on* T_- *and its defining ideal is given by*

$$\mathcal{I}(E'_{f_0}) = \left\langle s_5^5 - 1, s_4^2 - s_5, s_3 + s_4s_5^3 - s_5, s_2 + s_3 - s_5, s_1 - s_5^3 \right\rangle$$

in the ring $\mathbf{C}[s_1, s_2, s_3, s_4, s_5]$. *Note that* $\left|G_{f_0}^{\mathbf{w}}\right| = 20$ *and* $\left|\rho(E'_{f_0})\right| = 10$. *The action is given by*

$$(s_1, s_2, s_3, s_4, s_5, t_1, t_2, t_3, t_4) \longmapsto (s_1t_1, s_3(t_4 - t_2) + s_5t_2, s_4t_3, s_5t_4).$$

3.7 Equations of the moduli space

Let $f \in \mathbf{C}\{x_1, x_2, ..., x_n\}$ be a quasihomogeneous power series of type $(d; \mathbf{w})$ with principal part $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ and let (ϕ, ρ) be an equivariant embedding of the action of the stabilizer $G^{\mathbf{w}}_{f_0}$ on T_{-}. Note that $E'_{f_0} := \rho(G^{\mathbf{w}}_{f_0})$ is the linearization of the action of E_{f_0} on T_{-}, in particular E'_{f_0} is finite. The categorical quotient $\phi(T_{-})/E'_{f_0}$, denoted by $\mathcal{M}$, is the desired coarse moduli space for the functor $Unf^{-}_{f_0}$. By Theorem 1.4.37 and Theorem 1.4.37 $\mathcal{M}$ carries the structure of an affine variety, hence there exists a closed subvariety $Y \subset \mathbf{C}^l$ for some l which is isomorphic to $\mathcal{M}$. Equations for the embedding Y of $\mathcal{M}$ can be computed in two steps.

1. Compute algebra generators $h_1, h_2, ..., h_m$ of $\mathbf{C}[x_1, x_2, ..., x_n]^{E'_{f_0}} =: S$. by means of Derksen's algorithm (cf. [Derksen 1997]). They provide a morphism
$$\Phi : \phi(T_{-}) \hookrightarrow \mathbf{C}^l \twoheadrightarrow \mathbf{C}^l/E'_{f_0} \hookrightarrow \mathbf{C}^m.$$
In general, one has to compute the Reynolds operator of E'_{f_0}. Note that $\mathbf{C}^l/E'_{f_0} \simeq spec(S)$.

2. Compute the ideal of the image of $\phi(T_{-})$ under the morphism Φ with `ImageVariety`.

For the computation of the invariant ring and invariant generators we apply the Reynolds operator, which plays an important role in invariant theory of reductive algebraic groups.

In the sequel let $\mathbf{K}$ be an algebraically closed field of characteristic 0 and G be a linearly reductive group which acts linearly on $\mathbf{K}^n$.

Lemma 71 *(a) Any $G-$invariant closed subvariety $V \subset \mathbf{K}^n$ has a defining ideal which is $G-$invariant.*
(b) For a $G-$invariant ideal $I \subset R$ we have $R^G/\left(I \cap R^G\right) \cong (R/I)^G$.

Proof. (a) For a defining ideal I of V we have $f \in I$ iff $f(v) = 0$ for all $v \in V$, so $0 = f(\sigma^{-1} \cdot v) = \sigma \cdot f(x)$ for all $v \in V$ and $\sigma \in G$. Hence $\sigma \cdot f \in I$.
(b) We refer, e.g., to [Kraft 1985], Ch. II, Section 3.2. ■

Proposition 72 *Let (Y, π) be a categorical quotient of the closed subvariety X by G, let $X' \subset X$ be a $G-$invariant subvariety and define $\pi' := \pi_{|X'}$. Then the pair $(\pi(X'), \pi')$ is a categorical quotient of X' by G and $\pi(X') \subset Y$ is a closed subvariety.*

Proof. By part (a) of Lemma 3.7.71 we have $\mathcal{O}(X') = \mathcal{O}(X)/I$ for some $G-$stable ideal I. Part (b) implies that $\mathcal{O}(X')^G = \mathcal{O}(X)^G/I \cap \mathcal{O}(X)^G$. By Theorem 1.4.37(b) $spec(\mathcal{O}(X')^G)$ is the categorical quotient of X' by G, therefore X'/G corresponds to the ideal $I \cap \mathcal{O}(X)^G$. Now the claim follows from Remark 1.1.1(d). ■

Remark 9 *For a $G-$invariant ideal $I \subset \mathcal{O}(X)$ it is, in general, wrong that $\langle I \cap \mathcal{O}(X)^G \rangle = I$. E.g., the ideal $I = \langle x_1, x_2, ..., x_n \rangle$ is always $G-$invariant, but $\langle I \cap \mathcal{O}(X)^G \rangle = I$ iff $G = \{id\}$.*

The projection map
$$\pi : \mathbf{K}^n \twoheadrightarrow \mathbf{K}^n/G$$
is given by the inclusion $\pi^* : \mathbf{K}[x_1, x_2, ..., x_n]^G \hookrightarrow \mathbf{K}[x_1, x_2, ..., x_n]$.

Proposition 73 *The morphism π is finite if G is finite. In particular, π is closed.*

Proof. The ring $\mathbf{K}[x_1, x_2, ..., x_n]$ is a finitely generated $\mathbf{K}[x_1, x_2, ..., x_n]^G-$module, which follows from the fact that the transcendence degree of the invariant ring $\mathbf{K}[x_1, x_2, ..., x_n]^G$ equals n, cf., e.g., [Sturmfels 1993], Proposition 2.1.1, hence π is finite. By Proposition 1.1.13(b) π is closed. ■

By choosing homogeneous generators $h_1, h_2, ..., h_m$ of $\mathbf{K}[x_1, x_2, ..., x_n]^G$ as a $\mathbf{K}-$algebra we obtain an embedding
$$\iota : \mathbf{K}^n/G \hookrightarrow \mathbf{K}^m$$
defined on the ring level by the morphism
$$\begin{array}{ccc} \iota^* & : & \mathbf{K}[y_1, y_2, ..., y_m] \longrightarrow \mathbf{K}[h_1, h_2, ..., h_m] \\ y_i & \longmapsto & h_i \end{array}$$
which is obviously surjective (for a proof we refer, e.g., to [CLS 1992]), i.e., $\mathbf{K}^n/G \simeq spec(\mathbf{K}[y_1, y_2, ..., y_m]/\ker \iota^*)$. By composing ι^* with π^* we obtain the morphism
$$\Phi : \mathbf{K}^n \twoheadrightarrow \mathbf{K}^n/G \hookrightarrow \mathbf{K}^m$$
defined on the ring level by
$$\Phi^* : \mathbf{K}[y_1, y_2, ..., y_m] \overset{\iota^*}{\twoheadrightarrow} \mathbf{K}[h_1, h_2, ..., h_m] \hookrightarrow \mathbf{K}[x_1, x_2, ..., x_n].$$
Let I be the defining ideal of a $G-$invariant variety $V \subset \mathbf{K}^n$. The quotient of V by G is given by $\Phi(V)$, and equations can be computed and set $I_G := I \cap \mathbf{K}[h_1, h_2, ..., h_m]$. The equations of the embedding
$$\Phi_{V/G} : V/G \hookrightarrow \mathbf{K}^n/G \hookrightarrow \mathbf{K}^m$$

are given as the kernel of the morphism

$$\Phi^*_{V/G} : \mathbf{K}[y_1, y_2, ..., y_m] \overset{\iota^*}{\twoheadrightarrow} \mathbf{K}[h_1, h_2, ..., h_m] \twoheadrightarrow \mathbf{K}[h_1, h_2, ..., h_m]/I_G.$$

Proposition 74 *Let $I = \langle f_1, f_2, ..., f_r \rangle \subset \mathbf{K}[x_1, x_2, ..., x_n]$ be the defining ideal of a $G-$invariant variety V. Then*

$$\operatorname{Im} \Phi_{V/G} = \Phi(V)$$

is a closed subvariety of $\mathbf{K}^m$ with defining ideal

$$\langle y_1 - h_1, y_2 - h_2, ..., y_m - h_m, f_1, f_2, ..., f_r \rangle \cap \mathbf{K}[y_1, y_2, ..., y_m].$$

Proof. Let $\iota_V : V \hookrightarrow \mathbf{K}^n$ be defined by $\mathbf{K}[x_1, x_2, ..., x_n] \twoheadrightarrow \mathbf{K}[x_1, x_2, ..., x_n]/I$. Hence $\Phi(V) = \operatorname{Im} \Phi \circ \iota_V$, and, on the ring level we have

$$\iota^* \circ \Phi^* : \mathbf{K}[Y] \overset{\iota^*}{\twoheadrightarrow} \mathbf{K}[h_1, h_2, ..., h_m] \hookrightarrow \mathbf{K}[x_1, x_2, ..., x_n] \twoheadrightarrow \mathbf{K}[x_1, x_2, ..., x_n]/I.$$

The kernel of

$$\mathbf{K}[h_1, h_2, ..., h_m] \twoheadrightarrow \mathbf{K}[h_1, h_2, ..., h_m]/I_G$$

equals the kernel of

$$\mathbf{K}[h_1, h_2, ..., h_m] \hookrightarrow \mathbf{K}[x_1, x_2, ..., x_n] \twoheadrightarrow \mathbf{K}[x_1, x_2, ..., x_n]/I$$

which proves the first claim. The morphism Φ is finite since it is the composition of the finite maps π and ι, hence $\Phi(V)$ is closed by Proposition 1.1.13(b). That the defining ideal of $\Phi(V)$ has the stated form follows from Proposition 3.2.64. ■

The finite group E'_{f_0} acts linearly $\mathbf{K}^l$, E'_{f_0} is the linearization of the action of E_{f_0} on T_- and let $I_\phi \subset \mathbf{K}[t_1, t_2, ..., t_l]$ be the ideal of the equivariant embedding of T_-. It follows from the above discussion that the moduli space $\mathcal{M}$ is isomorphic to the closed subvariety $\Phi(\phi(T_-))$ whose ideal can be computed by the algorithm `ImageVariety`.

For the computation of the invariant ring of the linear action of G on $\mathbf{K}^m$ we employ Derksen's algorithm, described below. In the algorithm the group G is given as an ideal, also denoted by G, of $\mathbf{K}[s_1, s_2, ..., s_r]$ and the linear action of G on $\mathbf{C}^n$ is given by the polynomials $p_1(\mathbf{s}, \mathbf{t}), p_2(\mathbf{s}, \mathbf{t}), ..., p_n(\mathbf{s}, \mathbf{t})$. If G is finite then the Reynolds operator is computed as explained in Section 3.8, otherwise, one has to provide a Reynolds operator.

Algorithm `QuotientEquations`$(G, \{p_1(\mathbf{s},\mathbf{t}), ..., p_l(\mathbf{s},\mathbf{t})\}, \{f_1, ..., f_r\})$
In: equations for G, $p_1, ..., p_l$ the linear action of G on $\mathbf{K}^l$,
equations $f_1, f_2, ..., f_r$ for a $G-$invariant variety V.
Out: $(g_1, g_2, ..., g_q)$ equations of the embedding of V/G.
begin
$\{h_1, h_2, ..., h_m\}$:=`NullCone`$(G, \{p_1(\mathbf{s},\mathbf{t}), p_2(\mathbf{s},\mathbf{t}), ..., p_l(\mathbf{s},\mathbf{t})\})$;
for $i := 1$ **to** m **do**
. **if** h_i is not invariant w.r.t. G **then** $h_i := \mathfrak{R}^G(h_i)$;
end;
$\{g_1, g_2, ..., g_q\}$:=`ImageVariety`$((f_1, f_2, ..., f_r), \{h_1, ..., h_m\})$;
return$(\{g_1, g_2, ..., g_q\})$;
end.

Example 75 *(cont. of Example 3.6.70) Let* $f_0 = x^2z + x^2y - z^5 + y^5$. *We compute the invariant ring of the variety* $t_4 - t_3^2$ *under the action of* $E'_{f_0} := \rho(G_{f_0}^{(2,1,1)})$ *given in Example 3.6.70. Generators* $h_1, h_2, ..., h_{21}$ *of* $\mathbf{C}[t_1, t_2, t_3, t_4]^{E'_{f_0}}$ *are given by*

$$\begin{aligned}
&t_2t_3t_4-t_3t_4^2, t_1t_4^2, t_1t_2^2-2t_1t_2t_4, t_2t_3^3-t_3^3t_4, t_1t_3^2t_4, t_1t_3^4,\\
&t_1^2t_2t_3-t_1^2t_3t_4, t_1^3t_4, t_4^5, t_2^2t_4^3-2t_2t_4^4, t_2^4t_4-4t_2^3t_4^2+8t_2t_4^4, t_1^3t_3^2,\\
&t_1^5, t_3^2t_4^4, t_2^4t_3^2, t_2^3t_3^2t_4+12t_2^2t_3^2t_4^2-16t_2t_3^2t_4^3+8t_3^2t_4^4, t_3^4t_4^3, t_3^6t_4^2,\\
&t_2^7t_3-7t_2^6t_3t_4+42t_2^5t_3t_4^2-140t_2^4t_3t_4^3+280t_2^3t_3t_4^4-336t_2^2t_3t_4^5+224t_2t_3t_4^6-64t_3t_4^7,\\
&t_3^8t_4, t_3^{10}, t_2^{10}-10t_2^9t_4+90t_2^8t_4^2-480t_2^7t_4^3+1680t_2^6t_4^4-4032t_2^5t_4^5+\\
&6720t_2^4t_4^6-7680t_2^3t_4^7+5760t_2^2t_4^8-2560t_2t_4^9+512t_4^{10}.
\end{aligned}$$

It remains to compute the image of the variety $\phi(T_\_) = \mathcal{V}(t_4 - t_3^2)$ *under the morphism*

$$\begin{aligned}\Phi^* : \mathbf{C}[y_1, y_2, ..., y_{21}] &\longrightarrow \mathbf{C}[h_1, h_2, ..., h_{21}]/\left\langle t_4 - t_3^2\right\rangle\\ y_i &\longmapsto h_i\end{aligned}$$

which corresponds to the embedding

$$\Phi : \phi(T_\_) \hookrightarrow \mathbf{C}^{21}.$$

The ideal of this embedding can be generated by 67 *elements with degrees ranging from* 1 *to* 5. *A Gröbner basis w.r.t. the lexicographical order consists of* 114 *elements with maximal degree* 19.

We conclude by computing equations of the moduli space of the singularity E_{19}.

Example 76 *We consider the singularity E_{19}, defined by the quasihomogeneous polynomial $f_0 = x^3 + xy^7$ of type $(21; 7, 2)$. The semiuniversal unfolding of negative weight with principal part f_0 is given by $F := f_0 + t_1x^2y^5 + t_2x^2y^4$. The stabilizer group $G_{f_0}^{(7,2)}$ has order 21 and is generated by diag(ξ^7, ξ^2). In this example the map Θ is bijective and the image $\Theta(G_{f_0}^{(7,2)}) = E_{f_0}$ is represented by the ideal $\langle s_2^{21} - 1, s_1 - s_2^3\rangle$. E_{f_0} acts linearly on $T_- = \mathbf{C}^2$ via $(t_1, t_2) \to (s_1t_1, s_2t_2)$ and the invariant ring $\mathbf{C}[t_1, t_2]^{E_{f_0}}$ can be generated by the 8 monomials, i.e.,*

$$\mathbf{C}[t_1, t_2]^{E_{f_0}} = \mathbf{C}[t_1^7, t_1^6t_2^3, t_1^5t_2^6, t_1^4t_2^9, t_1^3t_2^{12}, t_1^2t_2^{15}, t_1t_2^{18}, t_2^{21}].$$

An embedding $T_-/E_{f_0} \hookrightarrow \mathbf{C}^8$ is given by the ideal generated by

$$\begin{aligned} Y_6Y_8-Y_7^2, Y_5Y_8-Y_6Y_7, Y_5Y_7-Y_6^2, Y_4Y_8-Y_6^2, Y_4Y_7-Y_5Y_6, Y_4Y_6-Y_5^2,\\ Y_3Y_8-Y_5Y_6, Y_3Y_7-Y_5^2, Y_3Y_6-Y_4Y_5, Y_3Y_5-Y_4^2, Y_2Y_8-Y_5^2, Y_2Y_7-Y_4Y_5,\\ Y_2Y_6-Y_4^2, Y_2Y_5-Y_3Y_4, Y_2Y_4-Y_3^2, Y_1Y_8-Y_4Y_5, Y_1Y_7-Y_4^2, Y_1Y_6-Y_3Y_4,\\ Y_1Y_5-Y_3^2, Y_1Y_4-Y_2Y_3, Y_1Y_3-Y_2^2. \end{aligned}$$

3.8 Computing the invariant ring

There are several distinct algorithms for computing generators of the invariant ring of an algebraic group G over $\mathbf{K}$ which acts linearly on the polynomial ring $R := \mathbf{K}[x_1, x_2, ..., x_n]$ provided that G is finite and/or reductive, which depends also on $\mathbf{K}$. For a discussion we refer to [Bayer 1998], [Derksen 1997], [KS 1997], or [Sturmfels 1993]. We restrict ourselves linearly reductive groups over $\mathbf{C}$.

Derksen's algorithm

Derksen's algorithm (the algorithm `NullCone`) provides homogeneous generators for the ideal of the nullcone of a reductive algebraic group G which acts linearly on $\mathbf{C}^n$. These generators can be transformed to algebra generators of the invariant ring $\mathbf{C}[x_1, x_2, ..., x_n]^G$ by means of the Reynolds operator $\mathfrak{R}^G$, described below.

In 1890 Hilbert was able to prove the following theorem, now called Hilbert's Finiteness Theorem, by means of nonconstructive methods (he used his Basis Theorem!), cf. [Hilbert 1890].

Theorem 77 *(Hilbert 1890) If G is a reductive algebraic group acting linearly on $\mathbf{C}^n$ then $\mathbf{C}[x_1, x_2, ..., x_n]^G$ is finitely generated as a $\mathbf{C}-$algebra.*

More precisely, Hilbert proved that any ideal basis of the ideal (generated by all non-constant homogeneous invariants) $\left\langle \mathbf{C}[x_1, x_2, ..., x_n]_+^G \right\rangle \subset \mathbf{C}[x_1, x_2, ..., x_n]$, which consists of homogeneous invariants, generates the invariant ring $\mathbf{C}[x_1, x_2, ..., x_n]^G$ as a $\mathbf{C}-$algebra. At this time the proof was criticized because its non-constructive nature. In 1893 Hilbert came up with a constructive, but much more difficult proof, cf. [Hilbert 1893]. For a modern treatment we refer, e.g., [Bayer 1998], or [Sturmfels 1993].

In his PhD thesis H. Derksen provided a constructive proof of Hilbert's Finiteness Theorem (cf. [Derksen 1997]). Derksen proved that the Reynolds images of any ideal basis of $\left\langle \mathbf{C}[x_1, x_2, ..., x_n]_+^G \right\rangle$, which consists of not necessarily invariant homogeneous polynomials, still form an ideal basis of the ideal $\left\langle \mathbf{C}[x_1, x_2, ..., x_n]_+^G \right\rangle$, but now consisting of homogeneous invariants, and an algorithm for computing homogeneous ideal basis of $\left\langle \mathbf{C}[x_1, x_2, ..., x_n]_+^G \right\rangle$.

Let $\mathbf{K}[G] = \mathbf{C}[s_1, s_2, ..., s_r]/\left\langle g_1, g_2, ..., g_p \right\rangle$ with linear action on $\mathbf{C}^m$ given by

$$\sigma \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} a_{11}(\sigma) & a_{12}(\sigma) & \dots & a_{1m}(\sigma) \\ a_{21}(\sigma) & a_{22}(\sigma) & \dots & a_{2m}(\sigma) \\ \vdots & \vdots & & \vdots \\ a_{m1}(\sigma) & a_{m2}(\sigma) & \dots & a_{mm}(\sigma) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

For Derksen's algorithm we provide only the user interface.

Algorithm `NullCone`$(\{g_1, g_2, ..., g_p\}, \{a_{11}, a_{12}, ..., a_{mm}\})$
In: polynomials as above.
Out: $\{f_1, f_2, ..., f_t\}$ algebra generators for $\mathbf{C}[x_1, x_2, ..., x_n]^G$.

As noted above it might be necessary to apply the Reynolds operator in Derksen's algorithm.

Remark 10 *(a) The set of algebra generators for R^G obtained by Derksen's algorithm is not necessarily minimal.*
(b)We use Derksen's algorithm since the stabilizer $G_{f_0}^{\mathbf{w}}$ (or E_{f_0}) is given by means of equations, which fits perfectly with the interface. In addition, since the stabilizer contains roots of 1*, possibly of high order, the computation with the **Singular** library "**finvar.lib**" is not so efficient.*

Reynolds operator and invariance of polynomials

The Reynolds operator $\mathfrak{R}^G$ of a reductive algebraic group G is an R^G-linear projection of R to R^G which plays an important role in invariant theory of reductive algebraic groups. For the computation of algebra generators of the invariant ring R^G we apply the Reynolds operator to homogeneous generators of the ideal $\langle R_+^G \rangle$. For finite groups one has an explicit description of the Reynolds operator.

Definition 78 *The* ***Reynolds operator*** $\mathfrak{R}^G$ *of the finite group* G *which acts linearly on* R *via* $\bullet$ *is defined by*

$$\mathfrak{R}^G : R \longrightarrow R^G$$
$$f \longmapsto \frac{1}{|G|} \sum_{\sigma \in G} \sigma \bullet f.$$

If G is not finite but linearly reductive then the Reynolds operator can be computed by means of the action of some differential operators on the graded components R. We refer, e.g., to Ch.7.4 of [Vasconcelos 1997].

Remark 11 *For any reductive algebraic group* G *there exists a Reynolds operator* $\mathfrak{R}^G : R \to R^G$ *which is an* R^G-*linear projection. In many proofs only this property, and no explicit description, is used (cf. the proof of Hilbert's Finiteness Theorem below).*

Assume that G is finite and has defining ideal $I_G = \langle g_1, g_2, ..., g_p \rangle \subset \mathbf{K}[s_1, s_2, ..., s_m]$, i.e., the ideal I_G is zero-dimensional radical ideal. In order to apply the Reynolds operator we have to compute all solutions of I_G, which is done in Section 3.9.

Let $p_1(\mathbf{s}, \mathbf{x}), p_2(\mathbf{s}, \mathbf{x}), ..., p_n(\mathbf{s}, \mathbf{x})$ be polynomials defining a linear action of G on $\mathbf{K}[x_1, x_2, ..., x_n]$.

Proposition 79 *A polynomial* $f \in \mathbf{K}[x_1, x_2, ..., x_n]$ *is invariant w.r.t.* G *iff*

$$NormalForm(f(p_1(\mathbf{s}, \mathbf{x}), p_2(\mathbf{s}, \mathbf{x}), ..., p_n(\mathbf{s}, \mathbf{x})) - f(\mathbf{x}), I_G) = 0.$$

Proof. *cf. Proposition 7.4.3 in [Vasconcelos 1997].* ■

3.9 Computing the zero-set of a $0-$dimensional ideal

The computation of the solutions of a nontrivial zero-dimensional ideal $I \subset \mathbf{K}[s_1, s_2, ..., s_m]$ can be reduced to the computation of the solutions of a univariate polynomial by the elimination property of Gröbner bases, which says that a Gröbner basis of I contains at least one univariate polynomial. By the extension Theorem any solution of such a polynomial can be extended to an element of $\mathcal{V}(I)$. The roots of an univariate polynomial can be computed by iterative application of factorization and field extensions.

In the sequel let $I \subset \mathbf{K}[s_1, s_2, ..., s_m]$ be a proper zero-dimensional ideal. Assume that a Gröbner basis of I contains the nonconstant polynomial $f \in \mathbf{K}[s_m]$ and note that the ideal $I' \subset \mathbf{K}[s_1, s_2, ..., s_{m-1}]$, obtained from I by substituting any root of f in I, is again zero-dimensional.. We define the $k-$**th elimination ideal** of I to be $I_k := I \cap \mathbf{K}[s_{k+1}, s_{k+2}, ..., s_m]$.

Proposition 80 *Let $I \subset \mathbf{K}[s_1, s_2, ..., s_m]$ be a zero-dimensional ideal.*
(a) For $0 \leq k \leq m-1$ the ideal I_k is zero-dimensional in $\mathbf{K}[s_{k+1}, s_{k+2}, ..., s_m]$.
(b) Each solution of I_k can be lifted to a solution of I_{k-1}.

Proof. (a) Note that I_k is zero-dimensional iff $\mathbf{K}[s_{k+1}, s_{k+2}, ..., s_m]/I_k$ is Artinian (satisfies the descending chain condition). The projection

$$\pi : \mathbf{K}[s_1, s_2, ..., s_m]/I \to \mathbf{K}[s_{k+1}, s_2, ..., s_m]/I_k$$

is surjective, hence any strictly descending chain in $\mathbf{K}[s_{k+1}, s_2, ..., s_m]/I_k$ lifts to a strictly descending chain in the Artinian ring $\mathbf{K}[s_1, s_2, ..., s_m]/I$, hence $\mathbf{K}[s_{k+1}, s_2, ..., s_m]/I_k$ is Artinian.
(b) Let $\langle f_1, f_2, ..., f_r\rangle = I_{k-1}$ and $LT(f_i) = h_i(s_{k+1}, s_{k+2}, ..., s_m) \cdot s_k^{N_i}$. By the Extension Theorem (cf. [CLS 1992], Theorem 3.1.3) a partial solution $(a_{k+1}, a_{k+2}, ..., a_m)$ of I_k lifts to a solution of I_{k-1} iff there exists an i s.t. $h_i\left(a_{k+1}, a_{k+2}, ..., a_m\right) \neq 0$. Now (a) implies that $\sqrt{LT(I_k)} = \langle s_{k+1}, s_{k+2}, ..., s_m\rangle$ and $\sqrt{LT(I_{k-1})} = \langle s_k, s_{k+1}, ..., s_m\rangle$, hence some h_i must be constant. ■

In the sequel we provide two algorithms, one for the computation of the roots of an univariate polynomial and one for computing all solutions of a given zero-dimensional ideal. The algorithm `Roots`, described below, relies on factorization in $\mathbf{Q}(a)[s]$ and the computation of primitive elements for field extensions of the form $\mathbf{Q}(a, b)$, where minimal polynomials for a over $\mathbf{Q}$ and b over $\mathbf{Q}(a)$ are known. These algorithms are described, e.g., in Chapter 5.4 of [Winkler 1997]. Both algorithms are contained in the `Singular` library `zeroset.lib`.

Algorithm `Roots`(f)

In: univariate polynomial $f \in \mathbf{Q}(a)[s]$, a an algebraic number over $\mathbf{Q}$ with minimal polynomial $p \in \mathbf{Q}[t]$

Out:$\{\{(a_1, n_1), (a_2, n_2), ..., (a_k, n_k)\}, q, \alpha\}$ roots of f with multiplicities, $a_i \in \mathbf{Q}[t]/q(t) =: \mathbf{Q}(b)$ may be a new extension with minimal polynomial q, and $\alpha(b)$ is the representation of a in $\mathbf{Q}(b)$. Note : $\sum_{i=1}^k n_i = \deg(f)$.

begin;

if $n = 0$ **then return**$(\{\{\}, p, id\})$;
$A := t$; // $A \in \mathbf{Q}[t]$;
factor f as $f = \prod_{i=1}^{r_1}(s - a_i)^{n_i} \cdot \prod_{i=1}^{r_2} f_i$; // f_i irreducible, $\deg(f_i) > 1$;
$roots := \{(a_1, n_1), (a_2, n_2), ..., (a_{r_1}, n_{r_2})\}$;
for $i := 1$ **to** r_2 **do**
. **if** f_i is irreducible **then begin**
. $\{q, \alpha, \beta\}$:=`PrimitiveElement`(p, f_i);
. replace a in $roots$ by $\alpha(a)$ and set $\mathbf{Q}(a) := \mathbf{Q}[t]/q(t)$;
. $p := q$;
. $A := \alpha(A(t))$;
. **end**;
. $\{\{(a'_1, n'_1), (a'_2, n'_2), ..., (a_m, n_m)\}, q, \alpha, \beta\} := Roots(f_i)$;
. **if** $p \neq q$ **then begin** // field has changed
. replace a in $roots$ by $\alpha(a)$ and set $\mathbf{Q}(a) := \mathbf{Q}[t]/q(t)$;
. $p := q$;
. $A := \alpha(A(t))$;
. **end**;
. $roots := roots \cup \{(a'_1, n'_1), (a'_2, n'_2), ..., (a_m, n_m)\}$;
end;
return$(\{roots, \}, q, A)$;
end;

We can state the main algorithm.

Algorithm `ZeroSet`$(\{g_1, g_2, ..., g_n\})$
In: generators $g_1, g_2, ..., g_n \in \mathbf{Q}(a)[s_1, s_2, ..., s_m]$ of a zero-dimensional ideal I,
a an algebraic number over $\mathbf{Q}$ with minimal polynomial $p \in \mathbf{Q}[t]$
Out:$\{\{(a_1^1, a_2^1, ..., a_m^1), ..., (a_1^r, a_2^r, ..., a_m^r)\}, q, \alpha\}$ all elements of the zero set of the ideal I, $\mathbf{Q}[t]/q(t) =: \mathbf{Q}(b)$ may be a new extension with minimal polynomial q, $\alpha(a)$ is the representation of a in $\mathbf{Q}(b)$, and $a_i^j \in \mathbf{Q}(b)$
Note $r = \dim_{\mathbf{Q}(b)} \mathbf{Q}(b)[s_1, ..., s_m]/I$.
begin;
if $n = 0$ **then return**$(\{\{\}, p, id\})$;
relabel the generators of I s.t. $g_{n-t}, g_{n-t+1}, ..., g_n \in \mathbf{Q}(a)[s_m]$;
$g := \gcd(g_{n-t}, g_{n-t+1}, ..., g_n)$;
$(solutions, p, \alpha)$:=`Roots`(g);
If necessary, change the minimal polynomial and replace a by $\alpha(a)$.
for $j := 1$ **to** $|roots|$ **do**
. // extend the partial solutions
. $\{pSol, q, \alpha\}$:=`ZeroSet`$(\{g_1(s_1, ..., s_{m-1}, a_m^j), ..., g_1(s_1, ..., s_{m-1}, a_m^j)\}$;

. **if** $p \neq q$ **then begin** // field has changed
. replace a in *solutions* by $\alpha(a)$ and set $\mathbf{Q}(a) := \mathbf{Q}[t]/q(t)$;
. $p := q$;
. $A := \alpha(A(t))$;
. **end**;
. $solutions := solutions \cup \{(a_1, a_2, ..., a_m^j) : (a_1, a_2, ..., a_{m-1}) \in pSol\}$;
end;
return($\{solutions\}$ $, q, A$);
end;

Example 81 *Let* $I = \langle x^2 - 2ay, y^2 - 2 \rangle \subset \mathbf{Q}(a)[x, y]$ *with minimal polynomial* $a^2 - 2$*. The elements of the zero set are contained in* $\mathbf{Q}(b)^2$ *where the field* $\mathbf{Q}(b)$ *has minimal polynomial* $b^4 + 4b^2 + 36$*. The algebraic number* a *is represented by* $-\frac{1}{12}b^3 + \frac{1}{6}b$*, i.e.,* $I = \left\langle x^2 + (\frac{1}{6}b^3 - \frac{1}{3}b)y, y^2 - 2 \right\rangle \subset \mathbf{Q}(b)[x, y]$*. The* 4 *elements of the zero set are given by*

$$\left(\frac{b^3}{12} + \frac{5b}{6}, \frac{b^3}{12} - \frac{b}{6}\right), \left(-\frac{b^3}{12} - \frac{5b}{6}, \frac{b^3}{12} - \frac{b}{6}\right),$$
$$\left(2, -\frac{b^3}{12} + \frac{b}{6}\right), \left(-2, -\frac{b^3}{12} + \frac{b}{6}\right).$$

Chapter 4

Specific singularities

We discuss semiquasihomogeneous power series with fixed part either a Brieskorn-Pham polynomial or a polynomial of inner modality ≤ 4

4.1 Brieskorn-Pham singularities

Brieskorn-Pham polynomials appeared, e.g., in the study of exotic spheres. A polynomial $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ is called a **Brieskorn-Pham** polynomial if $f_0 = \sum_{i=1}^{n} x_i^{a_i}$ for $a \in \mathbf{N}^n$. In this case the computation of the moduli space is significantly simpler because of the following two facts :

- Generators for $G_{f_0}^{\mathbf{w}}$ can be written down without computation.
- The image of Θ is contained in $GL_k(\mathbf{C})$, i.e., the induced group action of $G_{f_0}^{\mathbf{w}}$ on T_- is linear.

In order to prove the structure theorem on $G_{f_0}^{\mathbf{w}}$ we need the following Lemma, which appears (without proof) in [OW 1991].

Lemma 82 *(Orlik-Wagenreich 1991) Let $f \in \mathbf{C}[x_1, x_2, ..., x_n]$ be quasihomogeneous and $G = G_w^{f_0} \cap GL_n(\mathbf{C})$. Then*

$$\bigoplus_{i=1}^{n} \mathbf{C}\frac{\partial f}{\partial x_i}$$

is a $G-$module.

Proof. Let $\sigma \in G_w^{f_0} \cap GL_n(\mathbf{C})$. From $f_0 = f_0 \circ \sigma$ it follows that $\frac{\partial f_0}{\partial x_i}(x) = \frac{\partial f_0 \circ \sigma}{\partial x_i}(x) = \sum_{j=1}^{n} \frac{\partial f_0}{\partial x_j}(\sigma \cdot x) \cdot \frac{\partial \sigma_j}{\partial x_i}(x)$. Since $\sigma \in GL_n(\mathbf{C})$ the terms $\frac{\partial \sigma_j}{\partial x_i}(x)$ are constants which we denote by $\lambda_j \in \mathbf{C}$. We obtain $\frac{\partial f_0}{\partial x_i}(x) = \left(\sum_{j=1}^n \lambda_j \frac{\partial f_0}{\partial x_j}\right)(\sigma x)$. Now

$$\begin{aligned} \sigma \cdot \frac{\partial f_0}{\partial x_i}(x) &= \sigma \cdot \left(\sum_{j=1}^n \lambda_j \frac{\partial f_0}{\partial x_j}\right)(x) = \left(\sum_{j=1}^n \lambda_j \frac{\partial f_0}{\partial x_j}\right)(\sigma^{-1}x) \\ &= \sum_{j=1}^n \lambda_j \frac{\partial f_0}{\partial x_j}(\sigma\sigma^{-1} \cdot x) = \sum_{j=1}^n \lambda_j \frac{\partial f_0}{\partial x_j}(x). \quad \blacksquare \end{aligned}$$

We can state a structure theorem on the stabilizer group of a Brieskorn polynomial.

Theorem 83 *Let $f_0 \in \mathbf{C}[x_1, x_2, ..., x_n]$ be a Brieskorn-Pham polynomial.*
(a) For $f_0 = \sum_{i=1}^n x_i^d$ stabilizer group $G_{f_0}^{\mathbf{w}} \simeq S_n \times (\mathbf{Z}_d)^n$.
(b) Let $f_0 = \sum_{i=1}^r f_0^{(d_i)}$ where $f_0^{(d_i)} = \sum_{j=r_{i-1}+1}^{r_i} x_j^{d_i}$ and $r_i \in \mathbf{N}, r_0 = 0, r_i < r_{i+1}$. The stabilizer of f_0 is

$$G_{f_0}^{\mathbf{w}} = \bigoplus_{i=1}^r G_{f_0^{(d_i)}}^{\mathbf{w}}.$$

Proof. (a) Since all weights are equal the only quasihomogeneous automorphisms are matrices, i.e. $G_{f_0}^{\mathbf{w}} \subset GL_n(\mathbf{C})$. Let $M = \bigoplus_{i=1}^n \mathbf{C}\frac{\partial f_0}{\partial x_i} = \mathbf{C}x_1^{d-1} \oplus \mathbf{C}x_2^{d-1} \oplus ... \oplus \mathbf{C}x_n^{d-1}$. W.l.o.g.. let $A = (a_{ij}) \in G_{f_0}^{\mathbf{w}}$ be s.t. a_{11} and a_{12} are not equal to 0. By Lemma 4.1.82 M is a $G_w^{f_0}$ module, but $A^{-1} \cdot x_1^{d-1}$ contains terms like $a_{11}^{d-2}a_{12}x_1^{d-1}x_2$, and is therefore not contained in M, a contradiction. Any matrix in $G_{f_0}^{\mathbf{w}}$ contains precisely one nonzero entry per row, which must be a $d-$th root of 1. Since f_0 is invariant under permutations the claim follows.
(b) W.l.o.g. assume that $\deg_{\mathbf{w}}(x_1) \leq \deg_{\mathbf{w}}(x_2) \leq ... < \deg_{\mathbf{w}}(x_n)$. Note that for $r_{i-1} < k \leq r_i$ each $\varphi \in G_{f_0}^{\mathbf{w}}$ is of the form

$$\varphi(x_k) = \sum_{j=r_{i-1}+1}^{r_i} c_{kj}x_j + \sum c_{\boldsymbol{\alpha},k}p_{\boldsymbol{\alpha},k}(x_1, x_2, ..., x_{r_{i-1}})$$

for some $c_{kj}, c_{\boldsymbol{\alpha},k} \in \mathbf{C}$, and $p_{\boldsymbol{\alpha},k} \in \mathbf{C}[x_1, x_2, ..., x_{r_{i-1}}]$ with $\deg_{\mathbf{w}}(p_{\boldsymbol{\alpha},k}) = w_k$ s.t. $\det(\frac{\partial \varphi}{\partial x_i}(0)) \neq 0$. Choose $m \leq m'$ s.t. $\deg(x_{m-1}) < \deg(x_m) = \deg(x_{m'})$ and $\deg(x_{m'}) < \deg(x_{m'+1})$ or $m = m' = n$ s.t. each $\varphi \in G_{f_0}^{\mathbf{w}}$ satisfies the condition that $c_{\boldsymbol{\alpha},k} \neq 0$ implies $k \leq m$. For $k < m$ the polynomial $\varphi(x_k)$ dose not contain any of the variables $x_m, ..., x_n$. Let i be s.t. $r_i + 1 = m$ and consider the restriction of φ' of φ to $f_0^{(d_i)}$ where, in addition, each $c_{\boldsymbol{\alpha},k}$ is set to 0. Since φ' is an isomorphism and

$$\varphi\left(f_0^{(d_i)}\right) = \sum_{k=r_i+1}^{r_{i+1}} x_k^{d_i} + P(x_1, x_2, ..., x_{m'})$$

where P does not contain $x_k^{d_i}$ for $r_i+1 \leq k \leq r_{i+1}$, it follows that $\varphi' \in G^{\mathbf{w}}_{f_0^{(d_i)}}$. Now part (a) implies that precisely one $c_{kj} \neq 0$ for $_i + 1 \leq k \leq r_{i+1}$. Since any permutation of the variables occurring in $f_0^{(d_i)}$ is contained in $G^{\mathbf{w}}_{f_0}$ we may assume that $c_{kk} \neq 0$ for $r_i + 1 \leq k \leq r_{i+1}$. Applying φ to f_0 yields

$$\varphi(f_0) = \sum_{j=1}^{r_{i-1}} x_j^{d_{\rho(j)}} + \left(\sum_{k=r_i+1}^{r_{i+1}} c_{kk}x_k + \sum c_{\boldsymbol{\alpha},k}p_{\boldsymbol{\alpha},k}\right)^{d_i} + R(x_{m'+1}, x_{m'+2}, ..., x_n)$$

by the choice of m and m'. But the terms $(c_{kk}x_k)^{d_i-1} p_{\boldsymbol{\alpha},i}$ do not cancel, hence $\varphi(f_0) \neq f_0$, a contradiction. ■

Now we make use of the knowledge of the structure of $G^{\mathbf{w}}_{f_0}$ to show that the induced group-action of $G^{\mathbf{w}}_{f_0}$ on T_- is linear.

Theorem 84 *If f_0 is a Brieskorn-Pham polynomial then $E_{f_0} \subset GL_k(\mathbf{C})$.*

Proof. Let $f_0 = \sum_{i=1}^n x_i^{d_i}$ and $M_- = \{\mathbf{x}^{\boldsymbol{\alpha}} : \alpha \in B_-\}$ be an upper basis of $M_{f_0} = \mathbf{C}[x_1, x_2, ..., x_n]/ < x_i^{d_{i-1}} >$. It follows from Theorem 4.1.83 that $G_w^{f_0} \subset GL_n(\mathbf{C})$ is generated by diagonal and permutation matrices. For diagonal matrices $\sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n)$ it is clear that $\Theta(\sigma) \in E_{f_0}$ since

$$\sigma^{-1} \bullet \mathbf{x}^{\boldsymbol{\alpha}} = (\sigma_1 x_1)^{\alpha_1} \cdot (\sigma_2 x_2)^{\alpha_2} \cdot ... \cdot (\sigma_n x_n)^{\alpha_n}$$

induces a diagonal matrix in $Aut(T_-)$. Now note that any permutation is a product of transpositions. Any monomial in M_- can be written as $m = x_i^{\alpha_i} x_j^{\alpha_j} \cdot m'$, where m' does not contain x_i and x_j and $\alpha_i < d_i - 1, \alpha_j < d_j - 1$. If $\deg_{\mathbf{w}}(x_i) = \deg_{\mathbf{w}}(x_j)$ then the application of the transposition $\pi = (ij) \in G^{\mathbf{w}}_{f_0}$ to m yields

$$\pi \cdot \left(x_i^{\alpha_i} x_j^{\alpha_j} \cdot m'\right) = x_i^{\alpha_j} x_j^{\alpha_i} \cdot m' \in M_-$$

since $\alpha_i, \alpha_j < d_i - 1 = d_j - 1$. In both cases the map ψ from Proposition 2.3.53(a) is the identity. ■

Nonetheless the computation of moduli spaces for semiquasihomogeneous power series where the fixed principal part is a Brieskorn-Pham polynomial ranges from easy to intractable, even if we restrict ourselves to reasonable examples. We exclude examples of the form $\sum_{i=1}^n x_i^{a_i}$ with $n >> 1$ and/or $a_i >> 1$, e.g., $n = 2$ and $a_1 = a_2 = 1000$ (here T_- has dimension $\binom{2+a_1-5}{a_1-5} = 496506$ and $\left|G_{f_0}^{(1,1)}\right| = 2 \cdot 10^6$). We illustrate the situation with 4 examples.

Example 85 *(easy) Let* $f_0 = x^4 + y^8$ *be a Brieskorn-Pham polynomial of type* $(8; 2, 1)$. *The semiuniversal family of negative weight is given by* $F(x, y, t_1, t_2) = x^4 + y^8 + t_1 x^2 y^6 + t_2 x^2 y^5$. *We have* $\mu_{f_0}(0) = 21$, $G_{f_0}^{(2,1)} = \left\langle (12), diag(\xi^2, 1), diag(1, \xi) \right\rangle$ *where* $\xi := \exp(\frac{2\pi i}{8})$. *Note that* $\left| G_{f_0}^{(2,1)} \right| = |E_{f_0}| = 32$ *and* $T_- = \mathbf{C}^2$. *The group* E_{f_0} *is generated by* $diag(\xi^2, 1), diag(1, \xi)$ *where ... and the invariant ring* $\mathbf{C}[x, y]^{E_{f_0}} = \mathbf{C}[t_1^4, t_1^2 t_2^4, t_2^8]$. *The moduli space* $\mathcal{M} = T_-/E_{f_0}$ *is embedded in* $\mathbf{C}^3$ *and is given by*

$$\mathcal{V}(y_1 y_3 - y_2^2) = \mathcal{M}.$$

Example 86 *(hard)Let* $f_0 = x^3 + y^3 + z^7$ *be a Brieskorn-Pham polynomial of type* $(21; 7, 7, 3)$. *We have* $\mu_{f_0}(0) = 24$, $\left| G_{f_0}^{(7,7,3)} \right| = |E_{f_0}| = 126$ *and* $T_- = \mathbf{C}^5$. *The invariant ring* $\mathbf{C}[t_1, t_2, t_3, t_4, t_5]^{E_{f_0}}$ *is generated by 162 homogeneous invariants* $h_1, h_2, ..., h_{162}$ *with degrees ranging from .. to .. . In order to compute the equations one has to eliminate the variables* t_1, t_2, t_3, t_4, t_5 *in the ideal* $\langle y_1 - h_1, y_2 - h_2, ..., y_{162} - h_{162} \rangle$.

Example 87 *(hard)Let* $f_0 = x^8 + y^8$ *be a Brieskorn-Pham polynomial of type* $(8; 1, 1)$. *We have* $\mu_{f_0}(0) = 49$, *and* $\left| G_{f_0}^{(1,1)} \right| = |E_{f_0}| = 128$ *and* $T_- = \mathbf{C}^{10}$.

Example 88 *(tractable ?) Let* $f_0 = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$ *be a Brieskorn-Pham polynomial of type* $(5; 1, 1, 1, 1, 1)$. *This is the equation of the quintic threefold, which plays an important role in mirror symmetry. The Milnor number equals* $\mu_{f_0}(0) = 1024$, *the base space of the semiuniversal family is* $T_- = \mathbf{C}^{802}$ *and* $\left| G_{f_0}^{(1,1,1,1,1)} \right| = 375000$.

4.2 Semiquasihomogeneous singularities with inner modality ≤ 4.

We apply the algorithm to some principal parts with inner modality 2, 3 and 4. The principal parts with their names are taken from [YS 1979].

Note: We have omitted all quasihomogeneous polynomials which involve parameters since this case is not covered by the theory presented here. In the table μ denotes the Milnor number, t_- the dimension of T_-, $\left| G_{f_0}^{\mathbf{w}} \right|$ the order of $G_{f_0}^{\mathbf{w}}$, $|E_{f_0}|$ the order of E_{f_0}, l the dimension of the ambient space of the equivariant embedding of T_- (if the action is linear then $t_- = l$), #I the number of invariants delivered from `NullCone`, #E the number of equations delivered from `QuotientEquations` (`ImageVariety`), ub is the maximal degree occurring in these equations (except for modality 4) and time denotes number of seconds needed for the whole computation in seconds (up to 1/100 sec.). All examples were computed in Singular-1-3-7 under Ret Hat 5.2 Linux, running on a 200 MHz Pentium with 64 MB RAM.

Note: In each example was no need for computing the radical of the equations defining the stabilizer.

Some examples with modality 2

We apply the algorithm to some principal parts coming.

Name	Equation	μ	t_-	$\lvert G_{f_0}^{\mathrm{w}}\rvert$	$\lvert E_{f_0}\rvert$	l	#I	#E	ub	time
E_{18}	$y^{10}+x^3$	18	2	30	30	2	9	28	3	2.85
E_{19}	xy^7+x^3	19	2	21	21	2	8	21	2	2.45
W_{17}	xy^5+x^4	17	2	20	20	2	6	10	2	2.20
Z_{17}	y^8+x^3y	17	2	24	24	2	7	15	2	2.31
Q_{17}	$yz^2+xy^5+x^3$	17	2	30	15	2	6	10	2	2.69
S_{17}	$yz^2+y^6+x^2z$	17	2	24	12	2	5	6	2	2.49

Equations for the moduli spaces of these principal parts are as follows. The equations of E_{19} appear in Example 3.7.76.

$\mathbf{E}_{18}$ $R=\mathbf{C}[Y_1,Y_2,...,Y_9]$, equations : $Y_8^2-Y_7Y_9, Y_7Y_8-Y_6Y_9, Y_6Y_8-Y_5Y_9, Y_5Y_8-Y_3Y_9, Y_3Y_8-Y_2Y_9, Y_2Y_8-Y_1Y_9, Y_7^2-Y_6Y_8, Y_6Y_7-Y_5Y_8, Y_5Y_7-Y_3Y_8, Y_3Y_7-Y_2Y_8, Y_2Y_7-Y_1Y_8, Y_6^2-Y_5Y_7, Y_5Y_6-Y_3Y_7, Y_3Y_6-Y_2Y_7, Y_2Y_6-Y_1Y_7, Y_5^2-Y_3Y_6, Y_3Y_5-Y_2Y_6, Y_2Y_5-Y_1Y_6, Y_3^2-Y_2Y_5, Y_2Y_3-Y_1Y_5, Y_2^2-Y_1Y_3, Y_1^2Y_8-Y_4Y_9, Y_1^2Y_7-Y_4Y_8, Y_1^2Y_6-Y_4Y_7, Y_1^2Y_5-Y_4Y_6, Y_1^2Y_3-Y_4Y_5, Y_1^2Y_2-Y_3Y_4, Y_1^3-Y_2Y_4$.

$\mathbf{W}_{17}$ $R=\mathbf{C}[Y_1,Y_2,...,Y_6]$, equations : $Y_5^2-Y_4Y_6, Y_4Y_5-Y_3Y_6, Y_3Y_5-Y_2Y_6, Y_2Y_5-Y_1Y_6, Y_4^2-Y_3Y_5, Y_3Y_4-Y_2Y_5, Y_2Y_4-Y_1Y_5, Y_3^2-Y_2Y_4, Y_2Y_3-Y_1Y_4, Y_2^2-Y_1Y_3$.

$\mathbf{Z}_{17}$ $R=\mathbf{C}[Y_1,Y_2,...,Y_7]$, equations : $Y_6^2-Y_5Y_7, Y_5Y_6-Y_4Y_7, Y_4Y_6-Y_3Y_7, Y_3Y_6-Y_2Y_7, Y_2Y_6-Y_1Y_7, Y_5^2-Y_4Y_6, Y_4Y_5-Y_3Y_6, Y_3Y_5-Y_2Y_6, Y_2Y_5-Y_1Y_6, Y_4^2-Y_3Y_5, Y_3Y_4-Y_2Y_5, Y_2Y_4-Y_1Y_5, Y_3^2-Y_2Y_4, Y_2Y_3-Y_1Y_4, Y_2^2-Y_1Y_3$.

$\mathbf{Q}_{17}$ $R=\mathbf{C}[Y_1,Y_2,...,Y_6]$, equations : $Y_5^2-Y_4Y_6, Y_4Y_5-Y_3Y_6, Y_3Y_5-Y_2Y_6, Y_2Y_5-Y_1Y_6, Y_4^2-Y_3Y_5, Y_3Y_4-Y_2Y_5, Y_2Y_4-Y_1Y_5, Y_3^2-Y_2Y_4, Y_2Y_3-Y_1Y_4, Y_2^2-Y_1Y_3$.

$\mathbf{S}_{17}$ $R=\mathbf{C}[Y_1,Y_2,...,Y_5]$, equations : $Y_4^2-Y_3Y_5, Y_3Y_4-Y_2Y_5, Y_2Y_4-Y_1Y_5, Y_3^2-Y_2Y_4, Y_2Y_3-Y_1Y_4, Y_2^2-Y_1Y_3$.

Some examples with modality 3

We obtain the following table.

Name	Equation	μ	t_-	$\lvert G^{\mathbf{w}}_{f_0}\rvert$	$\lvert E_{f_0}\rvert$	l	#I	#E	ub	time
E_{24}	$y^{13}+x^3$	24	3	39	39	3	41	792	11	203.16
E_{25}	xy^9+x^3	25	3	27	27	3	39	666	4	149.56
E_{26}	$y^{14}+x^3$	26	3	42	42	3	22	216	6	11.93
Z_{23}	$y^{11}+x^3y$	23	3	33	33	3	35	673	5	121.56
Z_{24}	xy^8+x^3y	24	3	23	23	3	29	381	8	27.23
Z_{25}	$y^{12}+x^3y$	25	3	36	36	3	18	137	5	7.09
N_{19}	y^6+x^4y	19	3	24	24	3	35	513	2	52.99
N^1_{20}	xy^5+x^4y	20	3	19	19	3	33	507	5	68.23
N^2_{20}	y^6+x^5	20	3	30	30	3	18	125	5	6.93
N_{21}	xy^5+x^5	21	3	25	25	3	23	267	6	15.27
Q_{22}	$yz^2+y^{10}+x^3$	22	3	60	30	3	28	449	17	30.68
Q_{23}	$yz^2+xy^7+x^3$	23	3	42	21	3	24	236	6	14.12
Q_{24}	$yz^2+y^{11}+x^3$	24	3	66	33	3	25	336	8	22.10
V^{*1}_{18}	$z^4+y^5+x^2y$	18	3	40	20	3	27	291	4	14.19
V^{*2}_{18}	$y^3z+xz^3+x^2y$	18	3	19	19	3	30	388	7	40.96
V^{*1}_{19}	$z^4+y^4z+x^2y$	19	3	32	16	3	22	191	6	10.45
V^{*2}_{19}	$z^5+y^3z+x^2y$	19	3	30	15	3	19	137	3	6.85
V^{*3}_{19}	$y^4+xz^3+x^2y$	19	3	24	24	3	11	17	3	3.98
V^{*}_{20}	$z^5+y^4+x^2y$	20	3	40	20	3	15	100	6	7.04

Some examples with modality 4

We use `opt=4` because the computation of the nullcone (`NullCone`) is faster than in the case `opt=7`. Namely, the equations of $G^{\mathbf{w}}_{f_0}$ contain $2-3$ variables while the equations for E'_{f_0} contain 4 variables. Since the invariant rings are generated by more than 65 elements we set `opt=4` and omit the column #E.

Note that ub denotes the maximal degree of an invariant !

Name	Equation	μ	t_-	$\lvert G^{\mathbf{w}}_{f_0}\rvert$	$\lvert E_{f_0}\rvert$	l	#I	ub	time
E_{30}	$y^{16}+x^3$	30	4	48	48	4	143	48	24.53
E_{31}	$xy^{11}+x^3$	31	4	33	33	4	118	33	26.65
E_{32}	$y^{17}+x^3$	32	4	51	51	4	96	51	29.89
W_{24}	y^9+x^4	24	4	36	36	4	136	36	16.20
W_{25}	xy^7+x^4	25	4	28	28	4	105	28	21.79
Z_{29}	$y^{14}+x^3y$	29	4	42	42	4	94	42	29.65
Z_{30}	$xy^{10}+x^3y$	30	4	29	29	4	95	29	24.33
Z_{31}	$y^{15}+x^3y$	31	4	45	45	4	65	45	35.43
N^1_{24}	y^7+x^5	24	4	35	35	4	81	35	18.61
N^2_{24}	xy^6+x^4y	24	4	23	23	4	96	23	17.83
N_{25}	y^8+x^4y	25	4	32	32	4	68	32	10.17
Q_{28}	$yz^2+y^{13}+x^3$	28	4	78	39	4	104	39	44.20
Q_{29}	$yz^2+xy^9+x^3$	29	4	54	27	4	83	27	21.85
Q_{30}	$yz^2+y^{14}+x^3$	30	4	84	42	4	65	42	56.38
S_{23}	$yz^2+y^8+x^2z$	23	4	32	32	4	105	32	13.47
S_{24}	$yz^2+xy^6+x^2z$	24	4	25	25	4	83	25	68.80
V^*_{23}	$z^4+y^5z+x^2y$	23	4	40	20	4	73	20	9.94
V^*_{24}	$z^4+y^7+x^2y$	24	4	56	28	4	59	28	12.42
V'$_{20}$	$yz^3+xy^3+x^3$	20	4	27	27	4	82	27	104.64
V'$_{21}$	$y^4+xz^3+x^3$	21	4	36	36	4	42	36	11.64

Chapter 5

The Singular implementation - user manual

We provide a description of the implementation of the algorithms in the computer algebra system `Singular` (cf. [GPS 1997]). They are divided in the three libraries `qhmoduli.lib`, `rinvar.lib` and `zeroset.lib` where the first library contains algorithms for the computation of the stabilizer $G_{f_0}^{\mathbf{w}}$, the induced action of $G_{f_0}^{\mathbf{w}}$ on T_-, the image E'_{f_0} of the linearization of this action and the and equations for the quotient T_-/E'_{f_0}, provided that the invariant ring $\mathbf{C}[t_1, t_2, ..., t_n]^{E'_{f_0}}$ is given, the second contains algorithms for the computation of a linearization of an arbitrary (algebraic) group action, of the nullcone of any reductive group and the invariant ring (and the Reynolds operator), provided that the group is finite or that the Reynolds operator is not required, while the third library contains algorithms for finding the variety of a zerodimensional ideal over a finite extension of $\mathbf{Q}$ (which can be generated by a single polynomial due to the primitive element theorem, cf., e.g., Ch. 5.4 of [Winkler 1997]).

Algorithms are described in the form
`Procedure-Name(parameter-list[,optional-parameters])`

$>$ <`data-type`>

`parameter-list` description of the parameters.

`optional-parameter` description of optional parameters.

<`data-type`> a description of the result of the algorithm. The data type `ring` is described as follows.

`ring [ideal I,...]` denotes that the output is a ring which contains, among others, the ideal `I`.

Important Note The descriptions of the three libraries only contain the main procedures described in Chapter 3, whereas numerous sub- and auxiliary procedures have been omitted. They are described in the header-files, contained in the Appendix (cf. Chapter 6).

5.1 The `qhmoduli.lib` library

This library contains the algorithms for the computation of the moduli space of a semiquasihomogeneous power series with fixed part w.r.t. right equivalence. For the computation of invariant rings the library `rinvar.lib`, described below, is used. From a theoretical point of view the library can compute the moduli space of any quasihomogeneous fixed part, but in practical applications there may be constraints due to time and space, cf. Section 5.6.

5.1.1 Options and trace

Due to the fact that the computation of the moduli space of some quasihomogeneous polynomial may be hard there are three options for the control of the algorithm.
Note: Different options may be combined by adding their value to `opt`.

Equations of the moduli space (`opt + 1`)

If the invariant ring is generated by more than 50 elements, the computation of the algebraic relations (=equations) may take quite long or become intractable. Therefore one can omit the computation of the relations by using the corresponding option when calling the algorithm. In several examples in this thesis we used this option to avoid the computation of the relations because elimination w.r.t. more than 50 variables takes quite long (or is not tractable), cf. e.g., Example 4.1.86.

Primary decomposition of the ideal of $G_{f_0}^{\mathrm{w}}$ or E_{f_0} for the Reynolds operator (`opt + 2`)

In order to compute the Reynolds operator one has to know all elements of the corresponding *finite* group, i.e., the elements of a zero-dimensional variety over a finite algebraic extension of $\mathbf{Q}$. Since this process requires univariate factorization, it may be faster to compute the elements of more, but smaller varieties, which correspond to the primary components of the ideal defining the group.

Computing the image of $G_{f_0}^{\mathbf{w}}$ (opt + 4)

Note that $\left|G_{f_0}^{\mathbf{w}}\right| \geq |E_{f_0}|$ and in several occasions the inequality is proper, which may help if the Reynolds operator has to be computed. But it may be possible that the ring of the ideal of E'_{f_0} contains more variables than the ring of $G_{f_0}^{\mathbf{w}}$ and that the equations of E'_{f_0} are more complicated. For this reason one can decide to continue the computation with either $G_{f_0}^{\mathbf{w}}$ or E'_{f_0} by passing the corresponding option to the algorithm. E.g., the computation of the invariant ring for quasihomogeneous polynomials with inner modality 4,or for the quasihomogeneous polynomial of Example 4.1.86, is faster if one uses $G_{f_0}^{\mathbf{w}}$ instead of E'_{f_0}. A possible reason may be the fact, that in these examples the ideal of $G_{f_0}^{\mathbf{w}}$ contains less variables than the ideal of E'_{f_0}. In Example 4.1.86 the ideal of $G_{f_0}^{\mathbf{w}}$ is contained in $\mathbf{C}[s_1, s_2, s_3, s_4, s_5]$ while the ideal of E'_{f_0} is contained in $\mathbf{C}[s_1, s_2, s_3, s_4, s_5, s_6, s_7]$.

Trace

Set the `Singular` variables `printlevel` to $1, 2$ or 3.

5.1.2 Algorithms

In the sequel f denotes a quasihomogeneous polynomials with an isolated singularity at the origin.

`ArnoldAction(poly f [, list` $G_f^{\mathbf{w}}$`, list` *B*`])`

$>$ `ring [ideal actionid, stabid]`

f semiquasihomogeneous polynomial with an isolated singularity at 0.

$G_f^{\mathbf{w}}$ is the stabilizer of f as returned from `StabEqn`.

B list of upper monomials.

`actionid` list of the polynomials which represent the action of $G_f^{\mathbf{w}}$ on T_-,

`stabid` is the ideal of the stabilizer G_f.

`ModEqn(poly f, [int opt])`

$>$ `ring [ideal modid]`

`modid` list of the generators of the ideal of the moduli space of f.

`opt = 1`: compute equations of the moduli space.

`opt = 2`: use a primary decomposition when computing the group elements.

`opt = 4` compute E_{f_0}, i.e., the image of G_{f_0}.

DEFAULT: `opt = 7`

`QuotientEquations(ideal` G`, action, embedding [, intvec` w`])`

$>$ `ring [ideal id, embedid]`

G ideal of a finite group, `action` ideal/list defining a linear action of G

w weights s.t. `action` and `embedding` are quasihomogeneous..

`id` ideal of the quotient variety if `opt = 1`, if `opt = 0, 2` then `id` contains generators of the coordinate ring R^G.

`embedid` is 0 if `opt = 1`, if `opt = 0, 2` it is the ideal defining the equivariant embedding.

`opt = 1`: compute equations of the quotient.

`opt = 2`: use a primary decomposition when computing the group elements.

`StabEqns(poly f)`

$>$ `list [ring R1, ring R2]`

`f` is the principal part.

`R1` ring containing equations of the stabilizer $G_f^{\mathbf{w}}$ (standard basis) (`stabid`) of the stabilizer

`R2` ring containing the ideal defining the action (`actionid`) of the stabilizer $G_f^{\mathbf{w}}$.

`StabEqnsId(ideal I, intvec w)`

$>$ `list [ring R1, ring R2]`

`I` is a quasihomogeneous ideal w.r.t. the weights w.

`R1` list of the equations of the stabilizer $\cap_{f \in I} G_f$ (standard basis) (`stabid`).

`R2` ring containing the ideal defining the action (`actionid`) of the stabilizer.

`StabOrder(poly f)`

`StabOrder(`$G_f^{\mathbf{w}}$`)`

$>$ `int order`

`f` is the principal part.

$G_f^{\mathbf{w}}$ is the result of `StabEqn`.

`order` the order of $G_f^{\mathbf{w}}$.

`UpperMonomials(poly f[, intvec` *`wt`*`])`

$>$ `list monomials`

f principal part.

`monomials` list/ideal of the monomials of a $\mathbf{C}-$base of the Milnor algebra with weighted degree $> \deg_{\mathbf{w}}(f)$.

`wt` weight of f (optional).

5.1.3 Implementation remarks

`APSusbtitution`: The application of the map constructed in the loop of the algorithm `ArnoldAction` may not be tractable. This procedure substitutes in each monomial separately and the substitution in a monomial is done step by step. After each step the terms of degree exceeding the upper bound are omitted and the result is reduced w.r.t. to the ideal of the group $G_{f_0}^{\mathbf{w}}$.

`QuotientEquations`: If weights are provided the algorithm uses the Hilbert-driven `std`.

5.2 The `rinvar.lib` library

This library contains the algorithms for the computation of invariant rings of reductive algebraic groups acting linearly by means of Derksen's algorithm (cf. [Derksen 1997]) and for linearizing a given algebraic action of an algebraic group. Note that Derksen's algorithm (the algorithm `NullCone`) provides homogeneous generators for the ideal of the nullcone. These generators can be transformed to algebra generators of the invariant ring by applying the Reynolds operator. The computation of the Reynolds operator is implemented only for finite groups over finite extensions of **Q** (by means of the `zeroset.lib`, see below).

Restrictions

`InvariantRing :` G must be finite and the ground field must be a finite extension of **Q**.

5.2.1 Algorithms

`LinearizeAction(`G`,action)`

$>$ `ring [ideal actionid, ideal embedid]`

G is the ideal of an algebraic group, `action` defines the action of G, both have type `ideal` or `list`.

`actionid` list of the polynomials which represent the linearized action of G. on T_-.

`embedid` list of the equations defining the equivariant embedding.

`ImageVariety(ideal `I`,ideal `F` [, intvec `w`])`

$>$ `ring [ideal imageid]`

I is the ideal of a variety X, F is an ideal defining a morphism $F : X \to Y$.

`imageid` ideal of the Zariski-closure of $F(X)$.

If I and F are quasihomogeneous w.r.t. w then the Hilbert-driven `std` is used.

`InvariantRing(ideal `G`, ideal action, poly f)`

$>$ `int`

G is the ideal of a finite algebraic group,

`action` list of polynomials defining a linear action of G on $\mathbf{K}^n$.

`f` a polynomial

0 if `f` is not invariant.
1 if `f` is invariant.

`InvariantRing(ideal` G`, ideal action)`
$>$ `ring [ideal invars, poly newA]`
G is the ideal of a finite algebraic group,
`action` list of polynomials defining a linear action of G on $\mathbf{K}^n$.
`invars` ideal of the algebra generators of the invariant ring $\mathbf{K}[x_1, x_2, ..., x_n]^G$.
`newA` if the minpoly changes this is the new representation of the algebraic number, otherwise it is set to `a`.

`NullCone(list` G`, ideal action)`
$>$ `ideal inv`
G is the ideal of a linearly reductive algebraic group over a field $\mathbf{K}$.
`action` list of the polynomials defining a linear action of G on $\mathbf{K}^n$.
both have type `ideal` or `list`.
`inv` ideal of the nullcone.

`ReynoldsImage(list reynolsdOp, ideal action, poly f)`
$>$ `poly f`*
`reynoldsOp` is the list representing the Reynolds operator of a finite algebraic group over a finite extension of $\mathbf{Q}$,
`action` defines the action of G.
`f`* image of `f` under the Reynolds operator.

`ReynoldsOperator(ideal` G`, ideal action [int opt])`
$>$ `ring [list ROelements, ideal id, poly newA]`
G is the ideal of an algebraic group, `action` defines the action of G,
`opt` cf. `ZeroSet`.
`ROelements` is the list containing all substitutions of group elements
`id` equations of G in the new ring
`newA` if the minpoly changes this is the new representation of the algebraic number, otherwise it is set to `a`.

5.2.2 Implementation remarks

`ImageVariety`: If the input is quasihomogeneous and the weights are provided then the Hilbert driven Gröbner basis algorithm is used. In this case the Hilbert function H_J of the ideal $\langle F_1 - y_1, ..., F_m - y_m \rangle$ can be computed directly and H_J provides an upper bound for the Hilbert function of $\langle F_1 - y_1, ..., F_m - y_m, I \rangle$, cf. [Traverso 1996].
`InvariantRing`: If the output of `NullCone` contains polynomials which are not invariant, the Reynolds operator has to be computed. This means that the zero set of the ideal of the group must be computed. The group must be

finite and it is of uttermost importance to keep the order of the group as low as possible, i.e., use the option for computing E_{f_0}.

5.3 The `zeroset.lib` library

This library contains algorithms for finding the variety of a zerodimensional radical ideal over a finite extension of $\mathbf{Q}$.

5.3.1 Algorithms

`Factor(poly` f`)`

$>$ `list [list factors, list multiplicities]`

f is a univariate polynomial over a finite algebraic extension of $\mathbf{Q}$.

`factors` is the list containing all nontrivial irreducible factors of f over the groundfield.

`multiplicities` is the list containing the multiplicities of the factors.

`Roots(poly` f`)`

$>$ `ring [list roots, poly g, poly newA]`

f is a univariate polynomial.

`roots` is the list containing all roots of f over a finite algebraic extension of the groundfield.

`g` the polynomial f in the new ring

`newA` if the minpoly changes this is the new representation of the algebraic number, otherwise it is set to `a`.

`ZeroSet(ideal` I `[, int opt])`

$>$ `ring [list zeroset, ideal id, poly newA]`

I is the radical ideal of a zerodimensional variety over $\mathbf{Q}$.

`opt = 0` no primary decomposition (default)

`opt` $>$ `0` primary decomposition.

`zeroset` is the list containing all elements of the variety

`id` equations of I in the new ring

`newA` if the minpoly changes this is the new representation of the algebraic number, otherwise it is set to `a`.

5.4 Performance

We provide 7 examples with increasing complexity and runtime. In the first table we provide information about the singularity, the group action and the invariant ring together with the embedding of the moduli space while the second table contains a profile of the computation. We consider the following examples.

1. E_{19} : $xy^7 + x^3$.
2. V^{*2}_{19} : $x^2y + y^3z + z^5$.
3. V^{*1}_{19} : $x^2y + y^4z + z^4$.
4. Q_{22} : $yz^2 + y^{10} + x^3$.
5. Z_{23} : $y^{11} + x^3y$.
6. E_{24} : $y^{13} + x^3$.
7. E_{31} : $x^3 + xy^{11}$, we set `opt` $= 6$.
8. the singularity $x^2y + x^2z + y^5 - z^5$, cf. Example 3.7.75.
9. the Brieskorn-Pham polynomial $x^3 + y^3 + z^7$, cf. Example 4.1.86. We set `opt` $= 6$.
10. The singularity $x^5y - xy^5$.
11. The K3 singularity $x_1^3 + x_2^3x_4 + x_3^4 + x_4^8$
12. The singularity $x^8 + 14x^4y^4 + y^8$ cf. 5.6.89.

13 The quintic threefold $x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$ 4.1.88.

Note that only the stabilizer of Example 7 and 8 contains matrices with more than one entry per row. A "?" denotes that the computation was aborted.

Note: In each example we did not compute the radical.

Nr.	μ	t_-	$\lvert G^{\mathbf{w}}_{f_0}\rvert$	$G^{\mathbf{w}}_{f_0}-$vars	$\lvert E'_{f_0}\rvert$	$E'_{f_0}-$vars	l	#I	#E	ub
1	19	2	21	2	21	2	2	8	21	2
2	19	3	30	3	15	3	3	19	137	3
3	19	3	32	3	16	3	3	22	191	6
4	22	3	60	3	30	3	3	28	449	17
5	23	3	33	2	33	3	3	35	673	5
6	24	3	39	2	39	3	3	41	792	11
7	31	4	33	2	33	4	4	118	–	33
8	24	3	20	5	10	5	4	21	67	5
9	24	5	126	5	126	?	5	?	?	?
10	25	3	144	4	?	?	3	?	?	?
11	102	49	288	4	?	?	?	?	?	?
12	49	10	192	4	?	?	?	?	?	?
13	1024	802	?	?	?	?	?	?	?	?

$G^{\mathbf{w}}_{f_0}-$vars and $E'_{f_0}-$vars denote the number of variables used for $G^{\mathbf{w}}_{f_0}$ and E'_{f_0} respectively, t_- indicates the dimension of T_- and l is the dimension of the ambient space of the embedding of T_- via ϕ. Note: $\dim T_- = l$ implies $\phi = id$. #I, #E and ub denote the number of generators of the invariant ring $\mathbf{K}[t_1, t_2, ..., t_l]^{E_{f_0}}$, the number of generators of the ideal I of the embedding of the quotient $\phi(T_-)/E'_{f_0} = \mathcal{M}_{hol}$ and the highest degree occurring in the basis of I. Note that this basis is a Gröbner basis w.r.t. degree-lexicographic order. A Gröbner basis w.r.t. lexicographical order contains, i.g., more elements with higher degrees, e.g., in example ? it contains 591 elements with highest degree = 20. Note: If $\dim T_- = l$ then $\phi = id$. The profile is given by the table below. A " – " denotes that the corresponding procedure was not executed. All examples were computed in Singular-1-3-7 under Ret Hat 5.2 Linux, running on a 200 MHZ Pentium with 64 MB RAM.

Nr.	B_-	$G^{\mathbf{w}}_{f_0}$	Θ	ρ	E'_{f_0}	NC	R1	R2	Eq.	tot.
1	0.12	0.19	0.89	0.08	0.11	0.34	0	0	0.36	2.54
2	0.13	0.30	1.90	0.12	0.14	0.75	0	0	2.80	7.04
3	0.13	0.33	1.92	0.11	0.17	0.89	0	0	6.20	10.84
4	0.16	0.48	2.29	0.11	0.16	1.97	0	0	23.88	31.01
5	0.14	0.22	1.48	0.10	0.14	1.91	0	0	122.58	129.65
6	0.15	0.23	1.44	0.11	0.13	3.70	0	0	193.87	203.16
7	0.19	0.22	1.75	0.14	0	21.01	0	0	–	26.20
8	0.16	0.58	8.81	0.96	0.20	2.05	3.49	0.07	33.21	50.47
9	0.17	0.33	2.49	0.19	?	?	?	?	?	?
10	0.15	0.42	8.34	0.13	?	?	?	?	?	?
11	0.71	0.73	33.71	1.56	?	?	?	?	?	?
12	0.29	0.42	?	?	?	?	?	?	?	?
13	108.79	?	?	?	?	?	?	?	?	?

Here we have the following list.
B_- is the time needed for the algorithm `UpperMonomials`
$G^{\mathbf{w}}_{f_0}$ time for `StabEqns`, including a std-computation,
Θ time for `ArnoldAction`. This step may cause problems if the action is not linear.
ρ time for `LinearizeAction`.
E'_{f_0} time for `ImageGroup`. May cause difficulty for large stabilizers.
NC time for `NullCone`.
R1 `ReynoldsOperator` (uses `ZeroSet`). Causes problems if the degree of the field-extension is > 8.
R2 `ReynoldsImage` applied to the non-invariant generators
Eq. time for `ImageIdeal`. Takes long if the number of invariants exceeds 50.

5.5 The example $f_0 = x^2z + x^2y - z^5 + y^5$.

We show how to compute the moduli space of semiquasihomogeneous hypersurface singularities w.r.t. the fixed part $f_0 = x^2z + x^2y - z^5 + y^5$ by means of the SINGULAR library qhmoduli.lib.

```
> LIB ''qhmoduli.lib'';
> ring R = 0,(x,y,z),ls;
> poly f = -z5+y5+x2z+x2y;
> list stab = StabEqn(f);
> def R1 = stab[1]; setring R1;
> stabid; stabid[1]=s(2)*s(3)
stabid[2]=s(1)^2*s(2)+s(1)^2*s(3)-1
stabid[3]=s(1)^2*s(3)^2-s(3)
stabid[4]=s(2)^4-s(3)^4+s(1)^2
stabid[5]=s(1)^4+s(2)^3-s(3)^3
stabid[6]=s(3)^5-s(1)^2*s(3)
> vdim(stabid); 20
> setring R;
> StabOrder(f);
20
> ideal B = UpperMonomials(f);
> B;
B[1]=y3z3
B[2]=x2y3
B[3]=x2y2
> def R2 = ArnoldAction(f,stab,B);
> setring R2;
> actionid;
actionid[1]=-s(2)*t(1)+s(3)*t(1)
actionid[2]=-s(2)^2*t(2)+2*s(2)^2*t(3)^2+s(3)^2*t(2)
actionid[3]=s(2)*t(3)+s(3)*t(3)
> nvars(stab[1]);
3
> def R3 = LinearizeAction(stabid, actionid, nvars(stab[1]));
> setring R3;
> actionid;
actionid[1]=-s(2)*t(1)+s(3)*t(1)
actionid[2]=-s(2)^2*t(2)+2*s(2)^2*t(4)+s(3)^2*t(2)
actionid[3]=s(2)*t(3)+s(3)*t(3)
actionid[4]=s(2)^2*t(4)+s(3)^2*t(4)
// 'ImageGroup' is not necessary
> def S = ImageGroup(groupid, actionid);
> setring S;
> groupid;
groupid[1]=s(2)+s(3)-s(5)
```

```
groupid[2]=s(4)^2-s(5)
groupid[3]=s(1)*s(5)+s(3)*s(4)-s(4)*s(5)
groupid[4]=s(1)*s(4)+s(3)-s(5)
groupid[5]=s(3)^2-2*s(3)*s(5)
groupid[6]=s(1)*s(3)-s(1)*s(5)+s(4)*s(5)
groupid[7]=s(1)^2+s(4)^2-2*s(5)
groupid[8]=-s(1)+s(5)^3
groupid[9]=s(3)*s(5)^2+s(4)-s(5)^3
groupid[10]=s(1)*s(5)^2-1
> ideal G = groupid,t(1),t(2),t(3),t(4);
> vdim(std(G));
10
// continue
setring R3;
> def R4 = QuotientEquations(groupid,actionid,embedid);
> setring R4;
> size(std(imageid)); // elimpart is not used !
72
> dim(std(imageid)); 3
// short version
> def T = ModEqn(f); setring T;
> size(modid); // 'elimpart' is used !
56
> dim(std(modid));
3
```

5.6 Practical limitations

There are no limitations from a theoretical point of view. Nonetheless, there are limitations w.r.t. time and memory in practical use. The following steps in the algorithm might be hard or even intractable. By intractable we mean either that a computation is not possible on a computer with 64 or 128 MB or elimination has to be performed with more than, say, 60 variables.

- The computation of the induced group action of $G^{\mathbf{w}}_{f_0}$ on T_-, i.e., the algorithm `ArnoldAction`, cf. Example 5.6.89 below.

- The computation of the invariant ring $\mathbf{K}[t_1, t_2, ..., t_l]^{E_{f_0}}$ (`NullCone`), cf. Example 5.6.90.

- The computation of the algebraic relations among the generators of $\mathbf{K}[t_1, t_2, ..., t_l]^{E_{f_0}}$ (`ImageVariety`), cf. e.g., Example 4.1.86, or any example where the invariant ring has more than 60 generators.

Specific classes of singularities.

As a general rule, examples where the dimension of the moduli space is > 3 and the group action is not linear, may be intractable. We provide ... examples .

Example 89 *Consider the homogeneous polynomial* $f_0 = x^8 + 14x^4y^4 + y^8$. *We have* $\mu_{f_0}(0) = 49$ *and* $T_- = T_- = \mathbf{C}^{10}$. *The stabilizer* $G_{f_0}^{(1,1)}$ *is generated by the three matrices*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and has order $\left| G_{f_0}^{(1,1)} \right| = 192$. *The upper monomials have degrees ranging from* 9 *to* 12. *The computation of the action of* $G_{f_0}^{\mathbf{w}}$ *on* T_- *exceeds the memory (64 MB + 20 MB swap-space) while processing the terms of degree* 11. *In addition, the resulting action will be quite complicated and the computation of the nullcone is the next challenging problem.*

Example 90 *The quasihomogeneous polynomial* $f = x_1^3 + x_2^3x_4 + x_3^4 + x_4^8$ *(hypersurface K3 singularity). Since* $\dim(T_-) = 49$ *the computation of* E'_{f_0} *is very hard. The computation of the nullcone seems to be out of reach (at the moment).*

Chapter 6

Appendix

We provide the headers of all three libraries and procedures.

6.1 QHMODULI.LIB

```
// Moduli Spaces for Semiquasihomogeneous Hypersurface Singularities
// w.r.t.  right-equivalence
// Bayer Thomas
// Last change 27.6.2000
//
//
///////////////////////////////////////////////////////////////////////////////
version=''Id:  qhmoduli.lib,v 1.0 2000/06/27 12:32:15 Singular Exp $'';
info=''
LIBRARY: qhmoduli.lib PROCEDURES FOR MODULI SPACES OF SEMIQUASIHOMOGENOUS SINGULARITIES
AUTHOR: Thomas Bayer, email:  tbayer@mathematik.uni-kl.de
PROCEDURES:
 ArnoldAction(f, [G, w]) Induced action of G_f on T_.
 ModEqn(f) Equations of the moduli space for principal part f
 QuotientEquations(G,A,I) Equations of Variety(I)/G w.r.t.  action 'A'
 StabEqn(f) Equations of the stabilizer of f.
 StabEqnId(I, w) Equations of the stabilizer of the qhom.  ideal I.
 StabOrder(f) Order of the stabilizer of f.
 UpperMonomials(f, [w]) Upper basis of the Milnor algebra of f.
 SUBPROCEDURES:
 Max(data) maximal integer contained in 'data'
 Min(data) minimal integer contained in 'data'
 Table(cmd, i, lb, ub) list, i-th entry is cmd(i), lb <= i <= ub
```

```
'';
LIB ''rinvar.lib'';
///////////////////////////////////////////////////////////////////////////////
proc ModEqn(poly f, list #)
''USAGE: ModEqn(f [, opt]); poly f; int opt;
PUROPSE: compute equations of the moduli space of semiquasihomogenos hyper-
 surface singularity with principal part f w.r.t.  right equivalence
ASSUME: f quasihomogeneous polynomial with an isolated singularity at 0
RETURN: polynomial ring over a simple extension of the groundfield of the
 basering, containing the ideal 'modid'
 - 'modid' is the ideal of the moduli space if opt is even (> 0).
 otherwise it contians generators of the coordinate ring R of the
 moduli space (note :  Spec(R) is the moduli space)
OPTIONS: 1 compute equations of the mod.  space,
 2 use a primary decomposition
 4 compute E_f0, i.e., the image of G_f0
 To combine options, add their value.
DEFAULT: opt = 7
EXAMPLE: example ModEqn; shows an example
///////////////////////////////////////////////////////////////////////////////
proc QuotientEquations(ideal G, ideal Gaction, ideal embedding, list#)
''USAGE: QuotientEquations(G,action,emb [, opt]); ideal G,action,emb;int opt
PUROPSE: compute the quotient of the variety given by the parameterization
 'emb' by the linear action 'action' of the algenraic group G.
ASSUME: 'action' is linear, G must be finite if the Reynolds operator is
 needed (i.e., NullCone(G,action) returns some non-invariant polys)
RETURN: polynomial ring over a simple extension of the groundfield of the
 basering, containing the ideals 'id' and 'embedid'.
 - 'id' contians the equations of the quotient if opt = 1,
 if opt = 0, 2 'id' contains generators of the coordiante ring R
 of the quotient (Spec(R) is the quotient)
 - 'embedid' = 0 if opt = 1,
 if opt = 0, 2 it is the ideal defining the equivariant embedding
OPTIONS: 1 compute equations of the quotient,
 2 use a primary decomposition when computing the Reynolds operator
 To combine options, add their value.
DEFAULT: opt = 3
EXAMPLE: example QuotientEquations; shows an example
///////////////////////////////////////////////////////////////////////////////
proc UpperMonomials(poly f, list #)
''USAGE: UpperMonomials(poly f, [intvec w])
```

```
PUROPSE: compute the upper monomials of the milnor algebra of f.
ASSUME: f is quasihomogeneous (w.r.t.  w)
RETURN: ideal
EXAMPLE: example UpperMonomials; shows an example
///////////////////////////////////////////////////////////////////////////////
proc ArnoldAction(poly f, list #)
''USAGE: ArnoldAction(f, [Gf, B]); poly f; list Gf, B;
 'Gf' is a list of two rings (coming from 'StabEqn')
PUROPSE: compute the induced action of of the stabilizer G of f on T_, where
 T_ is given by the upper monomials B of the Milnor algebra of f.
ASSUME: f is quasihomogeneous
RETURN: polynomial ring over the same groundfield, containing the ideals
 'actionid' and 'stabid'.
 - 'actionid' is the ideal defining the induced action of Gf on T_
 - 'stabid' is the ideal of the stabilizer Gf in the new ring
EXAMPLE: example ArnoldAction; shows an example
///////////////////////////////////////////////////////////////////////////////
proc StabOrder(list #)
''USAGE: StabOrder(f); poly f;
PUROPSE: compute the order of the stabilizer group of f.
ASSUME: f quasihomogeneous polynomial with an isolated singularity at 0
RETURN: int
GLOBAL: varSubsList
///////////////////////////////////////////////////////////////////////////////
proc StabEqn(poly f)
''USAGE: StabEqn(f); f polynomial
PUROPSE: compute the equations of the isometry group of f.
ASSUME: f semiquasihomogeneous polynomial with an isolated singularity at 0
RETURN: list of two ring 'S1', 'S2'
 - 'S1' contians the equations of the stabilizer (ideal 'stabid')
 - 'S2' contains the action of the stabilizer (ideal 'actionid')
EXAMPLE: example StabEqn; shows an example
GLOBAL: varSubsList, contains the index j s.t.  x(i) -> x(i)t(j) ...
///////////////////////////////////////////////////////////////////////////////
proc StabEqnId(ideal data, intvec wt)
''USAGE: StabEqn(I, w); I ideal, w intvec
PUROPSE: compute the equations of the isometry group of the ideal I
 each generator of I is fixed by the stabilizer.
ASSUME: I semiquasihomogeneous ideal w.r.t.  'w' with an isolated singularity at 0
RETURN: list of two ring 'S1', 'S2'
 - 'S1' contians the equations of the stabilizer (ideal 'stabid')
```

```
 - 'S2' contains the action of the stabilizer (ideal 'actionid')
EXAMPLE: example StabEqnId; shows an example
GLOBAL: varSubsList, contains the index j s.t.  t(i) -> t(i)t(j) ...
///////////////////////////////////////////////////////////////////////////////
static
proc ArnoldFormMain(poly f, B, poly Fs, ideal reduceIdeal, int nrs, int nrt)
''USAGE: ArnoldFormMain(f, B, Fs, rI, nrs, nrt);
 poly f,Fs; ideal B, rI; int nrs, nrt
PUROPSE: compute the induced action of 'G_f' on T_, where f is the principal part
 and 'Fs' is the semiuniversal unfolding of 'f' with x_i substituted by
 actionid[i], 'B' is a list of upper basis monomials for the milnor
 algebra of 'f', 'nrs' = number of variables for 'G_f' and
 'nrt' = dimension of T_
ASSUME: f is quasihomogeneous with an isolated singularity at 0,
 s(1..r), t(1..m) are parameters of the basering
RETURN: poly
EXAMPLE: example ArnoldAction; shows an example
///////////////////////////////////////////////////////////////////////////////
proc MonosAndTerms(poly f, wt, int ub)
''USAGE: MonosAndTerms(f, w, ub); poly f, intvec w, int ub
PUROPSE: returns a list of all monomials and terms occuring in f of
 weighted degree < ub
RETURN: list
 _[1] list of monomials
 _[2] list of terms
EXAMPLE: example MonosAndTerms shows an example
///////////////////////////////////////////////////////////////////////////////
proc SelectMonos(parts, intvec wt, int d)
''USAGE: SelectMonos(parts, w, d); list/ideal parts, intvec w, int d
PUROPSE: returns a list of all monomials and terms occuring in f of
 weighted degree = d
RETURN: list
 _[1] list of monomials
 _[2] list of terms
EXAMPLE: example SelectMonos; shows an example
///////////////////////////////////////////////////////////////////////////////
proc Expand(substitution, degVec, ideal reduceI, intvec w1, int ub, list truncated)
''USAGE: Expand(substitution, degVec, reduceI, w, ub, truncated);
 ideal/list substitution, list/intvec degVec, ideal reduceI, intvec w,
 int ub, list truncated
PUROPSE: substitute 'substitution' in the monomial given by the list of
```

```
  exponents 'degVec', omit all terms of weighted degree > ub and reduce
  the result w.r.t.  'reduceI'. If truncated[i] = 0 then the result is
  stored for later use.
 RETURN: poly
 NOTE: used by APSubstitution
 GLOBAL: computedPowers
 ///////////////////////////////////////////////////////////////////////////////
 proc PolyProduct(poly g1, poly h1, ideal reduceI, intvec wt, int ub)
 ''USAGE: PolyProduct(g, h, reduceI, wt, ub); poly g, h; ideal reduceI, intvec wt, int ub.
 PUROPSE: compute g*h and reduce it w.r.t 'reduceI' and omit terms of weighted
  degree > ub.
 RETURN: poly
 NOTE: used by 'Expand'
 ///////////////////////////////////////////////////////////////////////////////
 proc PolyPower1(int varIndex, poly f, int e, ideal reduceI, intvec wt, int truncated, int ub)
 ''USAGE: PolyPower1(i, f, e, reduceI, wt, truncated, ub); int i, e, ub; poly f;
  ideal reduceI; intvec wt; list truncated;
 PUROPSE: compute f^e, use previous computations if possible, and reduce it
  w.r.t reudecI and omit terms of weighted degree > ub.
 RETURN: poly
 NOTE: used by 'Expand'
 GLOBAL: 'computedPowers'
 ///////////////////////////////////////////////////////////////////////////////
 proc Num(int i)
 // Type :  void
 // Purpose :  set basering R, poly f and weight w to ...
 ///////////////////////////////////////////////////////////////////////////////
 proc RingVarsToList(list @index)
 ///////////////////////////////////////////////////////////////////////////////
 proc APSubstitution(poly f, ideal substitution, ideal reduceIdeal, intvec wt, int ub, int nrs,
int nrt)
 ''USAGE: APSubstitution(f, subs, reduceI, w, ub, int nrs, int nrt); poly f
  ideal subs, reduceI, intvec w, int ub, nrs, nrt;
  nrs = number of parameters s(1..nrs),
  nrt = number of parameters t(1..nrt)
 PUROPSE: substitute 'subs' in f, omit all terms with weighted degree > ub and
  reduce the result w.r.t.  'reduceI'.
 RETURN: poly
 GLOBAL: 'computedPowers'
 ///////////////////////////////////////////////////////////////////////////////
 proc StabVar(intvec wt)
```

```
''USAGE: StabVar(w); intvec w
PUROPSE: compute the indicies for quasihomogeneous substitutions of each
 variable.
ASSUME: f semiquasihomogeneous polynomial with an isolated singularity at 0
RETURN: list
 _[i] list of combinations for var(i) (i must be appended to each comb)
GLOBAL: 'varSubsList', contains the index j s.t.  x(i) -> x(i)t(j) ...
///////////////////////////////////////////////////////////////////////////////
proc StabVarComb(intvec wt)
''USAGE: StabVarComb(w); intvec w
PUROPSE: list all possible indices of indeterminates for a quasihom.  subs.
RETURN: list
 _[1] linear substitutions
 _[2] nonlinear substiutions
GLOBAL: 'varSubsList', contains the index j s.t.  x(i) -> x(i)t(j) ...
///////////////////////////////////////////////////////////////////////////////
proc AllCombinations(list partition, list indices)
''USAGE: AllCombinations(partition,indices); list partition, indices)
PUROPSE: all compbinations for a given partititon
RETURN: list
GLOBAL: varSubsList, contains the index j s.t.  x(i) -> x(i)t(j) ...
///////////////////////////////////////////////////////////////////////////////
proc AllSingleCombinations(int n, list index)
''USAGE: AllSingleCombinations(n index); int n, list index
PUROPSE: all combinations for var(n)
RETURN: list
///////////////////////////////////////////////////////////////////////////////
proc AllCombinationsAux(list parts, list index)
''USAGE: AllCombinationsAux(parts ,index); list parts, index
PUROPSE: all compbinations for nonlinear substituiton
RETURN: list
///////////////////////////////////////////////////////////////////////////////
proc Partitions(int n, list nr)
''USAGE: Partitions(n, nr); int n, list nr
PUROPSE: partitions of n consisting of elements from nr
RETURN: list
///////////////////////////////////////////////////////////////////////////////
proc NumFactor(int a, int b)
'' USAGE: NumFactor(a, b); int a, b
PUROPSE: if b divides a then return b/a, else return 0
RETURN: int
```

```
///////////////////////////////////////////////////////////////////////////////
proc Table(string cmd, string iterator, int lb, int ub)
'' USAGE: Table(cmd,i, lb, ub); string cmd, i; int lb, ub
PUROPSE: generate a list of size ub - lb + 1 s.t.  _[i] = cmd(i)
RETURN: list
///////////////////////////////////////////////////////////////////////////////
proc Flatten(list data)
'' USAGE: Flatten(n, nr); list data
PUROPSE: flatten the list (one level) 'data', which is a list of lists
RETURN: list
///////////////////////////////////////////////////////////////////////////////
proc Intersection(list l1, list l2)
// Type :  list
// Purpose :  Intersection of l1 and l2
///////////////////////////////////////////////////////////////////////////////
proc FirstEntry(data, elem)
// Type :  int
// Purpose :  position of first entry equal to elem in data (from left to right)
///////////////////////////////////////////////////////////////////////////////
proc Max(data)
''USAGE: Max(n, nr); intvec/list data
PUROPSE: find the maximal integer contained in 'data'
RETURN: list
ASSUME: 'data' contians only integers and is not empty
///////////////////////////////////////////////////////////////////////////////
proc Min(data)
''USAGE: Min(n, nr); intvec/list data
PUROPSE: find the minimal integer contained in 'data'
RETURN: list
ASSUME: 'data' contians only integers and is not empty
///////////////////////////////////////////////////////////////////////////////
```

6.2 RINVAR.LIB

```
// Invariants of reductive groups
// Bayer Thomas
// Last change 14.8.2000
//
///////////////////////////////////////////////////////////////////////////////
version=''Id:  rinvar.lib,v 1.0 2000/08/14 17:32:15 Singular Exp $'';
```

```
info=''
LIBRARY: rinvar.lib PROCEDURES FOR INVARIANT RINGS OF REDUCTIVE GROUPS
AUTHOR: Thomas Bayer, email:  tbayer@mathematik.uni-kl.de
PROCEDURES:
 HilbertSeries(I, w) Hilbert series of the ideal I w.r.t.  weight w
 HilbertWeights(I, w) weighted degrees of the generators of I
 ImageVariety(I, F) ideal of the image variety F(variety(I))
 ImageGroup(G, F) ideal of G w.r.t.  the induced representation
 InvariantRing(G, Gaction) generators of the invariant ring of G
 InvariantQ(f, G, Gaction) decide if f is invariant w.r.t.  G
 LinearizeAction(G, Gaction) linearization of the action 'Gaction' of G
 LinearActionQ(action,s,t) decide if action is linear in var(s..nvars)
 LinearCombinationQ(base, f) decide if f is in the linear hull of 'base'
 MinimalDecomposition(f,s,t) minimal decomposition of f (like coef)
 NullCone(G, act) ideal of the nullcone of the action 'act' of G
 ReynoldsImage(RO,f) image of f under the Reynolds operator 'RO'
 ReynoldsOperator(G, Gaction) Reynolds operator of the group G
 SimplifyIdeal(I[,m,s]) simplify the ideal I (try to reduce variables)
 TransferIdeal(R,name,nA) transfer the ideal 'name' from R to basering
;
''
LIB ''presolve.lib'';
LIB ''elim.lib'';
LIB ''zeroset.lib'';
///////////////////////////////////////////////////////////////////////////////
proc EquationsOfEmbedding(ideal embedding, int nrs)
USAGE: EquationsOfEmbedding(embedding, s); ideal embedding; int s;
PUROPSE: compute the ideal of the variety parameterized by 'embedding' by
 implicitation and change the variables to the old ones.
RETURN: ideal
ASSUME: nvars(basereing) = n, size(embedding) = r and s = n - r.
 The polynomials of embedding contain only var(s + 1 ..  n).
NOTE: the result is the Zariski closure of the parameterized variety
EXAMPLE: example EquationsOfEmbedding; shows an example
///////////////////////////////////////////////////////////////////////////////
proc ImageGroup(ideal Grp, ideal Gaction)
USAGE: ImageGroup(G, action); ideal G, action;
PUROPSE: compute the ideal of the image of G in GL(m,K) induced by the linear
 action 'action', where G is an algebraic group and 'action' defines
 an action of G on K^m (size(action) = m).
RETURN: ring, a polynomial ring over the same ground field as the basering,
```

```
 containing the ideals 'groupid' and 'actionid'.
 - 'groupid' is the ideal of the image of G (order <= order of G)
 - 'actionid' defines the linear action of 'groupid' on K^m.
NOTE: 'action' and 'actionid' have the same orbits
 all variables which give only rise to 0's in the m x m matrices of G
 have been omitted.
ASSUME: basering K[s(1..r),t(1..m)] has r + m variables, G is the ideal of an
 algebraic group and F is an action of G on K^m.  G contains only the
 variables s(1)...s(r).  The action 'action' is given by polynomials
 f_1,...,f_m in basering, s.t.  on the ring level we have
 K[t_1,...,t_m] --> K[s_1,...,s_r,t_1,...,t_m]/G
 t_i --> f_i(s_1,...,s_r,t_1,...,t_m)

EXAMPLE: example ImageGroup; shows an example
///////////////////////////////////////////////////////////////////////////////
proc HilbertWeights(ideal I, wt)
USAGE: HilbertWeights(I, w); ideal I, intvec wt
PUROPSE: compute the weights of the slack varaibles needed for the
 computation of the algebraic relations of the generators of 'I' s.t.
 the Hilbert driven 'std' can be used.
RETURN: intvec
ASSUME: basering = K[t_1,...,t_m,...], 'I' is quasihomogeneous w.r.t.  'w' and
 contains only polynomials in t_1,...,t_m
///////////////////////////////////////////////////////////////////////////////
proc HilbertSeries(ideal I, wt)
USAGE: HilbertSeries(I, w); ideal I, intvec wt
PUROPSE: compute the polynomial p of the Hilbert Series,represented by p/q, of
 the ring K[t_1,...,t_m,y_1,...,y_r]/I1 where 'w' are the weights of
 the variables, computed, e.g., by 'HilbertWeights', 'I1' is of the
 form I[1] - y_1,...,I[r] - y_r and is quasihomogeneous w.r.t.  'w'
RETURN: intvec
NOTE: the leading 0 of the result does not belong to p, but is needed in
 the hilbert-driven 'std'.
///////////////////////////////////////////////////////////////////////////////
proc ImageVariety(ideal I, F, list #)
USAGE: ImageVariety(ideal I, F [, w]);ideal I; F is a list/ideal, intvec w.
PUROPSE: compute the Zariski closure of the image of the variety of I under
 the morphism F.
NOTE: if 'I' and 'F' are quasihomogeneous w.r.t.  'w' then the Hilbert-driven
 'std' is used.
RETURN: polynomial ring over the same groundfield, containing the ideal
```

```
 'imageid'.  The variables are Y(1),...,Y(k) where k = size(F)
 - 'imageid' is the ideal of the Zariski closure of F(X) where
 X is the variety of I.
EXAMPLE: example ImageVariety; shows an example
///////////////////////////////////////////////////////////////////////////////
proc LinearizeAction(ideal Grp, Gaction, int nrs)
USAGE: LinearizeAction(G,action,r); ideal G, action; int r
PUROPSE: linearize the group action 'action' and find an equivariant embedding
 of K^m where m = size(action).
ASSUME: G contains only variables var(1..r)
basering = K[s(1..r),t(1..m)], K = Q or K = Q(a) and minpoly != 0.
RETURN: polynomial ring contianing the ideals 'actionid', 'embedid', 'groupid'
 - 'actionid' is the ideal defining the linearized action of G
 - 'embedid' is a parameterization of an equivariant embedding (closed)
 - 'groupid' is the ideal of G in the new ring
NOTE: set printlevel > 0 to see a trace
EXAMPLE: example LinearizeAction; shows an example
///////////////////////////////////////////////////////////////////////////////
proc LinearActionQ(Gaction, int nrs)
USAGE: LinearActionQ(action,nrs,nrt); ideal action, int nrs
PUROPSE: check if the action defined by 'action' is linear w.r.t.  the variables
 var(nrs + 1...nvars(basering)).
RETURN: 0 action not linear
 1 action is linear
EXAMPLE: example LinearActionQ; shows an example
///////////////////////////////////////////////////////////////////////////////
proc LinearCombinationQ(ideal I, poly f)
USAGE: LinearCombination(I, f); ideal I, poly f
PUROPSE: test if f can be written as a linear combination of the generators of I.
RETURN: 0 f is not a linear combination
 1 f is a linear combination
///////////////////////////////////////////////////////////////////////////////
proc InvariantRing(ideal G, ideal action, list #)
USAGE: InvariantRing(G, Gact [, opt]); ideal G, Gact; int opt
PUROPSE: compute generators of the invariant ring of G w.r.t.  the action 'Gact'
ASSUME: G is a finite group and 'Gact' is a linear action.
RETURN: polynomial ring over a simple extension of the groundfield of the
 basering (the extension might be trivial), containing the ideals
 'invars' and 'groupid' and the poly 'newA'
 - 'invars' contains the algebra-generators of the invariant ring
 - 'groupid' is the ideal of G in the new ring
```

```
 - 'newA' if the minpoly changes this is the new representation of the
 algebraic number, otherwise it is set to 'a'.
NOTE: the delivered ring might have a different minimal polynomial
EXAMPLE: example InvariantRing; shows an example
///////////////////////////////////////////////////////////////////////////////
proc InvariantQ(poly f, ideal G, action)
USAGE: InvariantQ(f, G, action); poly f; ideal G, action
PUROPSE: check if the polynomial f is invariant w.r.t.  G where G acts via
 'action' on K^m.
ASSUME: basering = K[s_1,...,s_m,t_1,...,t_m] where K = Q of K = Q(a) and
 minpoly != 0, f contains only t_1,...,t_m, G is the ideal of an
 algebraic group and a standardbasis.
RETURN: int;
 0 if f is not invariant,
 1 if f is invariant
NOTE: G need not be finite
EXAMPLE: example InvariantQ; shows an example
///////////////////////////////////////////////////////////////////////////////
proc MinimalDecomposition(poly f, int nrs, int nrt)
USAGE: MinimalDecomposition(f,a,b); poly f; int a, b.
PUROPSE: decompose f as a sum M[1,1]*M[2,1] + ...  + M[1,r]*M[2,r] where M[1,i]
 contains only s(1..a), M[2,i] contains only t(1...b) s.t.  r is minimal
ASSUME: f polynomial in K[s(1..a),t(1..b)], K = Q or K = Q(a) and minpoly != 0
RETURN: 2 x r matrix M s.t.  f = M[1,1]*M[2,1] + ...  + M[1,r]*M[2,r]
EXAMPLE: example MinimalDecomposition;
///////////////////////////////////////////////////////////////////////////////
proc NullCone(ideal G, action)
USAGE: NullCone(G, action); ideal G, action
PUROPSE: compute the ideal of the nullcone of the linear action of G on K^n,
 given by 'action', by means of Deksen's algorithm
ASSUME: basering = K[s(1..r),t(1..n)], K = Q or K = Q(a) and minpoly != 0,
 G is an ideal of a reductive algebraic group in K[s(1..r)],
 'action' is a linear group action of G on K^n (n = ncols(action))
RETURN: ideal of the nullcone of G.
NOTE: the generators of the nullcone are homogeneous, but i.g.  not invariant
EXAMPLE: example NullCone; shows an example
///////////////////////////////////////////////////////////////////////////////
proc ReynoldsOperator(ideal Grp, ideal Gaction, list #)
USAGE: ReynoldsOperator(G, action [, opt); ideal G, action; int opt
PUROPSE: compute the Reynolds operator of the group G which act via 'action'
RETURN: polynomial ring R over a simple extension of the groundfield of the
```

```
 basering (the extension might be trivial), containing a list
 'ROelements', the ideals 'id', 'actionid' and the polynomial 'newA'.
 R = K(a)[s(1..r),t(1..n)].
 - 'ROelements' is a list of ideal, each ideal represents a
 substitution map F : R -> R according to the zero-set of G
 - 'id' is the ideal of G in the new ring
 - 'newA' is the new representation of a' in terms of a.  If the
 basering does not contain a parameter then 'newA' = 'a'.
ASSUME: basering = K[s(1..r),t(1..n)], K = Q or K = Q(a') and minpoly != 0,
 G is the ideal of a finite group in K[s(1..r)], 'action' is a linear
 group action of G
EXAMPLE: example ReynoldsOperator; shows an example
///////////////////////////////////////////////////////////////////////////////
proc ReynoldsImage(list reynoldsOp, poly f)
USAGE: ReynoldsImage(RO, f); list RO, poly f
PUROPSE: compute the Reynolds image of the polynomial f where RO represents
 the Reynolds operator
RETURN: poly
///////////////////////////////////////////////////////////////////////////////
static proc SimplifyCoefficientMatrix(matrix coefMatrix)
USAGE: SimplifyCoefficientMatrix(M); M matrix coming from coef(...)
PUROPSE: simplify the matrix, i.e.  find linear dependencies among the columns
RETURN: matrix M, f = M[1,1]*M[2,1] + ...  + M[1,n]*M[2,n]
///////////////////////////////////////////////////////////////////////////////
proc SimplifyIdeal(ideal I, list #)
USAGE: SimplifyIdeal(I [,m, name]); ideal I; int m, string name
PURPOSE: simplify ideal I to the ideal I', do not change the names of the
 first m variables, new ideal I' might contain less variables.
 I' contains variables var(1..m)
RETURN: list
 _[1] ideal I'
 _[2] ideal representing a map phi to a ring with probably less vars.  s.t.
 phi(I) = I'
 _[3] list of variables
 _[4] list from 'elimpart'
NOTE:
////////////////////////////////////////////////////////////////////////////////
proc TransferIdeal(R, string name, poly newA)
 USAGE: TransferIdeal(R, name, newA); ring R, string name, poly newA
PUROPSE: Maps an ideal with name 'name' in R to the basering, s.t.  all
 variables are fixed but par(1) is replaced by 'newA'.
```

```
RETURN: ideal
NOTE: this is used to transfor an ideal if the minimal polynomial has changed
///////////////////////////////////////////////////////////////////////////////
proc RingVarProduct(index)
///////////////////////////////////////////////////////////////////////////////
```

6.3 ZEROSET.LIB

```
// Algorithms for finding the zero-set of a zero-dim.  ideal in Q(a)[x_1,..x_n],
// Roots and Factorization of univariate polynomials over Q(a)[t]
// where a is an algebric number
// Last change 19.5.2000
//
///////////////////////////////////////////////////////////////////////////////
version=''Id:  zeroset.lib,v 1.0 2000/05/19 12:32:15 Singular Exp $'';
info=''
LIBRARY: zeroset.lib PROCEDURES FOR ROOTS AND FACTORIZATION
AUTHOR: Thomas Bayer, email:  tbayer@mathematik.uni-kl.de
PROCEDURES:
 EGCD(f, g) gcd over an algebraic extension field of Q
 Factor(f) factorization of f over an algebraic extension field
 InvertNumber(c) inverts an element of an algenraic extension field
 LinearZeroSet(I) find the linear (partial) solution of I
 Quotient(f, g) quotient q of f w.r.t.  g (in f = q*g + remainder)
 Remainder(f,g) remainder of the division of f by g
 Roots(f) computes all roots of f in an extension field of Q
 SQFRNorm(f) norm of f (f must be squarefree)
 ZeroSet(I) zero-set of the 0-dim.  ideal I

 SUBPROCEDURES:
 EGCDMain(f, g) gcd over an algebraic extension field of Q
 FactorMain(f) factorization of f over an algebraic extension field
 InvertNumberMain(c) inverts an element of an algenraic extension field
 QuotientMain(f, g) quotient of f w.r.t.  g
 RemainderMain(f,g) remainder of the division of f by g
 RootsMain(f) computes all roots of f, might extend the groundfield
 SQFRNormMain(f) norm of f (f must be squarefree)
 ContainedQ(data, f) f in data ?
 SameQ(a, b) a == b (list a,b)
 TransferRing() create a new basering (to use ...Main())
```

```
 NewBaseRing() create a new basering (return from ...Main())
 SimplifyData(data) reduces entries of data w.r.t.  mpoly
 SimplifyPoly(f) reduces coefficients of f w.r.t.  mpoly
 SimplifySolution(sol) reduces the entries of sol w.r.t.  the ideal mpoly
 NOTE: this library is meant as a preliminary extension of the functionality
 of Singular for univariate factorization of polynomials over simple algebraic
 extensions in characteristic 0.
'';
LIB primitiv.lib;
LIB primdec.lib;
// note :  return a ring :  ring need not be exported !!!
// Artihmetic in Q(a)[x] without built-in procedures
// assume basering = Q[x,a] and minpoly is represented by mpoly(a).
// the algorithms are taken from Polynomial Algorithms in Computer Algebra,
// F. Winkler, Springer Verlag Wien, 1996.
// To do :
// squarefree factorization
// multiplicities
// Improvement :
// a main problem is the growth of the coefficients.  Try Roots(x7 - 1)
// retrurn ideal mpoly !
// mpoly is not monic, comes from primitive_extra
// IMPLEMENTATION
//
// In procedures with name 'proc-name'Main a polynomial ring over a simple
// extension field is represented as Q[x...,a] together with the ideal
// 'mpoly' (attribute isSB). The arithmetic in the extension field is
// implemented in the procedures in the procedures 'MultPolys' (multiplication)
// and 'InvertNumber' (inversion).  After addition and substraction one should
// apply 'SimplifyPoly' to the result to reduce the result w.r.t.  'mpoly'.
// This is done by reducing each coefficient seperately, which is more
// efficient for polynomials with many terms.
///////////////////////////////////////////////////////////////////////////////
proc Roots(poly f)
USAGE: Roots(f); poly f
PUROPSE: compute all roots of f in a finite extension of the groundfield
 without multiplicities.
RETURN: ring, a polynomial ring over an extension field of the ground field,
 containing a list 'roots', poly 'newA', poly 'f':
 - 'roots' is the list of roots of the polynomial f (no multiplicities)
 - if the groundfield is Q(a') and the extension field is Q(a), then
```

```
 'newA' is the representation of a' in Q(a).  If the basering
 contains a parameter 'a' and the minpoly remains unchanged then
 'newA' = 'a'.  If the basering does not contain a parameter then
 'newA' = 'a' (default).
 - 'f' is the polynomial f in Q(a) (a' being substituted by newA)
ASSUME: ground field to be Q or a simple extension of Q given by a minpoly
EXAMPLE: example Roots; shows an example
///////////////////////////////////////////////////////////////////////////////
proc RootsMain(poly f)
USAGE: RootsMain(f); poly f
PUROPSE: compute all roots of f in a finite extension of the groundfield
 without multiplicities.
RETURN: list, all entries are polynomials
 _[1] = roots of f, each entry is a polynomial
 _[2] = 'newA' - if the groundfield is Q(a') and the extension field is Q(a), then
 'newA' is the representation of a' in Q(a)
 _[3] = minpoly of the algebraic extension of the groundfield
ASSUME: basering = Q[x,a]
 ideal mpoly must be defined, it might be 0 !
NOTE: might change teh ideal mpoly !!
EXAMPLE: example Roots; shows an example
///////////////////////////////////////////////////////////////////////////////
proc ZeroSet(ideal I, list #)
USAGE: ZeroSetMain(ideal I [, int opt]); ideal I, int opt
PUROPSE: compute the zero-set of the zero-dim.  ideal I, in a finite extension
 of the groundfield.
RETURN: ring, a polynomial ring over an extension field of the ground field,
 containing a list 'zeroset', a polynomial 'newA', and an ideal 'id'.
 - 'zeroset' is list of the zeros of the ideal 'I', each zero is
 an ideal.
 - if the groundfield is Q(a') and the extension field is Q(a), then
 'newA' is the representation of a' in Q(a).  If the basering
 contains a parameter 'a' and the minpoly remains unchanged then
 'newA' = 'a'.  If the basering does not contain a parameter then
 'newA' = 'a' (default).
 - 'id' is the ideal 'I' in Q(a)[x_1,...]  (a' substituted by 'newA')
ASSUME: dim(I) = 0, and ground field to be Q or a simple extension of Q given
 by a minpoly.
OPTIONS: opt = 0 no primary decomposition (default)
 opt > 0 primary decomposition
NOTE: If I contains an algebraic number (parameter) then 'I' must be transformed
```

```
 w.r.t.  'newA' in the new ring.
EXAMPLE: example ZeroSet; shows an example
///////////////////////////////////////////////////////////////////////////////
proc InvertNumberMain(poly f)
USAGE: InvertNumberMain(f); poly f
PURPOSE: compute 1/f if f is a number in Q(a) i.e., f is represented by a
 polynomial in Q[a].
RETURN: poly 1/f
ASSUME: basering = Q[x_1,...,x_n,a], ideal 'mpoly' must be defined and != 0 !
///////////////////////////////////////////////////////////////////////////////
proc MultPolys(poly f, poly g)
USAGE: MultPolys(f, g); poly f,g
PURPOSE: multiply the polynomials f and g and reduce them w.r.t.  mpoly
RETURN: poly f*g
ASSUME: basering = Q[x,a], ideal mpoly must be defined, it might be 0 !
///////////////////////////////////////////////////////////////////////////////
proc LeadTerm(poly f, int i)
USAGE: LeadTerm(f); poly f, int i
PUROPSE: compute the leading coef and term of f w.r.t var(i), where the last
 ring variable is treated as a parameter.
RETURN: list of polynomials
 _[1] = leading term
 _[2] = leading monomial
 _[3] = leading coefficient
ASSUME: basering = Q[x_1,...,x_n,a]
///////////////////////////////////////////////////////////////////////////////
proc Quotient(poly f, poly g)
USAGE: Quotient(f, g); poly f, g;
PUROPSE: compute the quotient q and remainder r s.t.  f = g*q + r, deg(r) < g
RETURN: list of polynomials
 _[1] = quotient q
 _[2] = remainder r
ASSUME: basering = Q[x] or Q(a)[x]
EXAMPLE: example Quotient; shows an example
///////////////////////////////////////////////////////////////////////////////
proc Remainder(poly f, poly g)
USAGE: Remainder(f, g); poly f, g
PUROPSE: compute the remainder of the division of f by g, i.e.  a polynomial r
 s.t.  f = g*q + r, deg(r) < g.
RETURN: poly
ASSUME: basering = Q[x] or Q(a)[x]
```

```
///////////////////////////////////////////////////////////////////////////////
proc EGCD(poly f, poly g)
USAGE: EGCDMain(f, g); poly f, g
PUROPSE: compute the polynomial gcd of f and g over Q(a)[x]
RETURN: poly h s.t.  h is a greatest common divisor of f and g (not nec.  monic)
ASSUME: basering = Q(a)[t]
EXAMPLE: example EGCD; shows an example
///////////////////////////////////////////////////////////////////////////////
proc MEGCD(poly f, poly g, int varIndex)
USAGE: MEGCD(f, g, i); poly f, g; int i
PUROPSE: compute the polynomial gcd of f and g in the i'th variable
RETURN: poly
ASSUME: f, g are polynomials in var(i), last variable is the algebraic number
EXAMPLE: example MEGCD; shows an example
///////////////////////////////////////////////////////////////////////////////
proc SQFRNorm(poly f)
USAGE: SQFRNorm(f); poly f
PUROPSE: compute the norm of the squarefree polynomial f in Q(a)[x].
RETURN: list
 _[1] = squarefree norm of g (poly)
 _[2] = g (= f(x - s*a)) (poly)
 _[3] = s (int)
ASSUME: f must be squarefree, basering = Q(a)[x] and minpoly != 0.
NOTE: the norm is an element of Q[x]
EXAMPLE: example SQFRNorm; shows an example
///////////////////////////////////////////////////////////////////////////////
proc Factor(poly f)
USAGE: Factor(f); poly f
PUROPSE: compute the factorization of the squarefree poly f over Q(a)[t], minpoly = p(a).
RETURN: list
 _[1] = factors (monic), first entry is the leading coefficient
 _[2] = multiplicities (not yet)
ASSUME: basering must be the univariate polynomial ring over a field which is Q
 or a simple extension of Q given by a minpoly.
NOTE: if basering = Q[t] then this is the built-in 'factorize'.
EXAMPLE: example Factor; shows an example
///////////////////////////////////////////////////////////////////////////////
proc FactorMain(poly f)
USAGE: FactorMain(f); poly f
PUROPSE: compute the factorization of the squarefree poly f over Q(a)[t], minpoly = p(a).
RETURN: list
```

```
 _[1] = factors, first is a constant
 _[2] = multiplicities (not yet)
ASSUME: basering = Q[x,a], represents Q(a)[x] and minpoly.
 ideal mpoly must be defined, it might be 0 !
EXAMPLE: example Factor; shows an example
///////////////////////////////////////////////////////////////////////////////
proc ZeroSetMain(ideal I, int primDecQ)
USAGE: ZeroSetMain(ideal I, int opt); ideal I, int opt
PUROPSE: compute the zero-set of the zero-dim.  ideal I, in a simple extension
 of the groundfield.
RETURN: list
 - 'f' is the polynomial f in Q(a) (a' being substituted by newA)
 _[1] = zero-set (list), is the list of the zero-set of the ideal I,
 each entry is an ideal.
 _[2] = 'newA'; if the groundfield is Q(a') and the extension field
 is Q(a), then 'newA' is the representation of a' in Q(a).
 If the basering contains a parameter 'a' and the minpoly
 remains unchanged then 'newA' = 'a'.  If the basering does not
 contain a parameter then 'newA' = 'a' (default).
 _[3] = 'mpoly' (ideal), the minimal polynomial of the simple extension
 of the ground field.
ASSUME: basering = K[x_1,x_2,...,x_n] where K = Q or a simple extension of Q
 given by a minpoly.
 dim(I) = 0.
NOTE: opt = 0 no primary decomposition
 opt > 0 use a primary decomposition
EXAMPLE: example ZeroSet; shows an example
///////////////////////////////////////////////////////////////////////////////
proc ZeroSetMainWork(ideal id, intvec wt, int sVars)
USAGE: ZeroSetMainWork(I, wt, sVars);
PUROPSE: compute the zero-set of the zero-dim.  ideal I, in a finite extension
 of the groundfield (without multiplicities).
RETURN: list, all entries are polynomials
 _[1] = zeros, each entry is an ideal
 _[2] = newA; if the groundfield is Q(a') this is the rep.  of a' w.r.t.  a
 _[3] = minpoly of the algebraic extension of the groundfield (ideal)
 _[4] = name of algebraic number (default = 'a')
ASSUME: basering = Q[x_1,x_2,...,x_n,a]
 ideal mpoly must be defined, it might be 0 !
NOTE: might change 'mpoly' !!
EXAMPLE: example IdealSolve; shows an example
```

```
///////////////////////////////////////////////////////////////////////////////
proc LinearZeroSetMain(ideal I, intvec wt)
USAGE: LinearZeroSetMain(I, wt)
PURPOSE: solve the univariate linear polys in I
ASSUME: basering = Q[x_1,...,x_n,a]
RETURN: list
 _[1] = partial solution of I
 _[2] = index of solved vars
 _[3] = new generators (standardbasis)
///////////////////////////////////////////////////////////////////////////////
proc NonLinearZeroSetMain(ideal I, intvec wt)
USAGE: ZeroSetMainWork(I, wt, sVars);
PUROPSE: solves the (nonlinear) univariate polynomials in I
 of the groundfield (without multiplicities).
RETURN: list, all entries are polynomials
 _[1] = list of solutions
 _[2] = newA
 _[3] = minpoly
 _[4] - index of solved variables
 _[5] = new representation of I
ASSUME: basering = Q[x_1,x_2,...,x_n,a], ideal 'mpoly' must be defined,
 it might be 0 !
NOTE: might change 'mpoly' !!
///////////////////////////////////////////////////////////////////////////////
proc ExtendSolutions(list solutions, list newSolutions)
USAGE: ExtendSolutions(sols, newSols); list sols, newSols;
PUROPSE: extend the entries of 'sols' by the entries of 'newSols',
 each entry of 'newSols' is a number.
RETURN: list
ASSUME: basering = Q[x_1,...,x_n,a], ideal 'mpoly' must be defined,
 it might be 0 !
NOTE: used by 'NonLinearZeroSetMain'
///////////////////////////////////////////////////////////////////////////////
proc MergeSolutions(list sol1, list index1, list sol2, list index2)
USAGE: MergeSolutions(sol1, index1, sol2, index2); all parameters are lists
RETURN: list
PURPOSE: create a list of solutions of size n, each entry of 'sol2' must
 have size n.  'sol1' is one partial solution (from 'LinearZeroSetMain')
 'sol2' is a list of partial solutions (from 'NonLinearZeroSetMain')
ASSUME: 'sol2' is not empty
NOTE: used by 'ZeroSetMainWork'
```

```
/////////////////////////////////////////////////////////////////////////////
proc SubsMapIdeal(list sol, list index, int opt)
USAGE: SubsMapIdeal(sol,index,opt); list sol, index; int opt;
PUROPSE: built an ideal I as follows.
 if i is contained in 'index' then set I[i] = sol[i]
 if i is not contained in 'index' then
 - opt = 0:  set I[i] = 0
 - opt = 1:  set I[i] = var(i)
 if opt = 1 and n = nvars(basering) then set I[n] = var(n).
RETURN: ideal
ASSUME: size(sol) = size(index) <= nvars(basering)
/////////////////////////////////////////////////////////////////////////////
proc SimplifyZeroset(data)
USAGE: SimplifyZeroset(data); list data
PUROPSE: reduce the entries of the elements of 'data' w.r.t.  the ideal 'mpoly'
 'data' is a list of ideals/lists.
RETURN: list
ASSUME: basering = Q[x_1,...,x_n,a], order = lp
 'data' is a list of ideals
 ideal 'mpoly' must be defined, it might be 0 !
/////////////////////////////////////////////////////////////////////////////
proc Variables(poly f, int n)
USAGE: Variables(f,n); poly f; int n;
PUROPSE: list of variables among var(1),...,var(n) which occur in f.
RETURN: list
ASSUME: n <= nvars(basering)
/////////////////////////////////////////////////////////////////////////////
proc ContainedQ(data, f, list #)
USAGE: ContainedQ(data, f [, opt]); list data; f is of any type, int opt
PUROPSE: test if 'f' is an element of 'data'.
RETURN: int
 0 if 'f' not contained in 'data'
 1 if 'f' contained in 'data'
OPTIONS: opt = 0 :  use '==' for comparing f with elements from data
 opt = 1 :  use 'SameQ' for comparing f with elements from data
////////////////////////////////////////////////////////////////////////////
proc SameQ(a, b)
USAGE: SameQ(a, b); list/intvec a, b;
PUROPSE: test a == b elementwise, i.e., a[i] = b[i].
RETURN: int
 0 if a != b
```

```
 1 if a == b
///////////////////////////////////////////////////////////////////////////////
proc SimplifyPoly(poly f)
USAGE: SimplifyPoly(f); poly f
PUROPSE: reduces the coefficients of f w.r.t.  the ideal 'moly' if they contain
 the algebraic number 'a'.
RETURN: poly
ASSUME: basering = Q[x_1,...,x_n,a]
 ideal mpoly must be defined, it might be 0 !
///////////////////////////////////////////////////////////////////////////////
proc SimplifyData(data)
USAGE: SimplifyData(data); ideal/list data;
PUROPSE: reduces the entries of 'data' w.r.t.  the ideal 'mpoly' if they contain
 the algebraic number 'a'
RETURN: ideal/list
ASSUME: basering = Q[x_1,...,x_n,a]
 ideal 'mpoly' must be defined, it might be 0 !
///////////////////////////////////////////////////////////////////////////////
proc TransferRing(R)
USAGE: TransferRing(R);
PUROPSE: creates a new ring containing the same variables as R, but without
 parameters.  If R contains a parameter then this parameter is added
 as the last variable and 'minpoly' is represented by the ideal 'mpoly'
 If the basering does not contain a parameter then 'a' is added and
 'mpoly' = 0.
RETURN: ring
ASSUME: R = K[x_1,...,x_n] where K = Q or K = Q(a).
NOTE: Creates the ring needed for all prodecures with name 'proc-name'Main
///////////////////////////////////////////////////////////////////////////////
proc NewBaseRing()
USAGE: NewBaseRing();
PUROPSE: creates a new ring, the last variable is added as a parameter.
 minpoly is set to mpoly[1].
RETURN: ring
ASSUME: basering = Q[x_1,...,x_n, a], 'mpoly' must be defined
///////////////////////////////////////////////////////////////////////////////
// remove later
///////////////////////////////////////////////////////////////////////////////
```

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