Diophantine flavor of Kolmogorov complexity

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Plan of the talk:

I. *Some definitions and results in the theory of Kolmogorov complexity*

II. Main results in Diophantine computations

III. Diophantine flavor of Kolmogorov complexity

Part I. *Some definitions in the theory of Kolmogorov complexity*

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101010101010101010101010	010011100000100010001010
101010101010101010101010	011001111100110001110100
101010101010101010101010	000000100000101110111110
101010101010101010101010	101001100011101100010011
101010101010101010101010	100110110010001001010001
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101010101010101010101010	101001100011101100010011
101010101010101010101010	100110110010001001010001
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360 alternating “1” and “0”	The first 360 binary digits of the number π

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Other terminology/point of view:

D is an *archiving method*	D is a *compression method*
A is an *archive* of W	A is a *compressed form* of W

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- $\langle A, W' \rangle \in D \,\&\, \langle A, W'' \rangle \in D \Rightarrow W' = W''$ (the uniqueness of the decompression)

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An possible additional condition of *prefix-free* description mode can be imposed:

- The set $\mathcal{P} = \{A \,:\, \exists W \,\langle A, W \rangle \in D\}$ is prefix-free, i.e., no word from $\mathcal{P}$ is a prefix of another word from $\mathcal{P}$

Equivalently: if $\langle A, W \rangle \in D$ then $\langle AA', W' \rangle \notin D$ for every word W' and every non-empty word A'.

The *complexity* $K_D(W)$ of a word W with respect to a description mode D is defined as the length of the shortest compressed form of W:

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Proof. Enumerate all description modes

$$D_0, D_1, \ldots, D_m, \ldots$$

viewed as functions from B^* into B^* and construct a *universal set* U:

$$\langle m, A, W \rangle \in U \Leftrightarrow \langle A, W \rangle \in D_m.$$

Then define

$$D_{\text{opt}} = \{\langle 1^m 0 A, W \rangle \ : \ \langle m, A, W \rangle \in U\}.$$

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□

Optimal prefix-free description modes also exit.

We fix a particular optimal (prefix-free) description mode D_{opt} and define *Kolmogorov (prefix) entropy* of a word W as the complexity of the word W with respect to D_{opt}.

The complexity of an infinite word can be characterized by the (prefix) entropies of its initial fragments.

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If an infinite word is recursively enumerable (i.e., the set of positions of “1” is recursively enumerable), then the entropy of its initial fragment of length l grows also essentially as $\log(l)$.

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Gregory Chaitin introduced a particular method of constructing an infinite word of so big prefix-free complexity. This word can be defined as the sequence of binary digits of some real number Ω. This number can be interpreted as the probability that a randomly selected Turing machine stops. Of course, digits of this word do not form a recursively enumerable set. However, there is an effectively computable increasing sequence $\Omega_1, \Omega_2, \ldots,$... of rational numbers which converges to Ω.

Part II. *Main results in Diophantine computations*

A *Diophantine equation* is an equation of the form

$$P(x_1, \ldots, x_m) = 0,$$

where P is a polynomial with integer coefficients and *the unknowns* $x_1, \ldots, x_m$ are supposed to be natural numbers.

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A *family* of Diophantine equations is an equation of the form

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Consider the set $\mathfrak{M}$ such that

$$\langle a_1, \ldots, a_n \rangle \in \mathfrak{M} \Longleftrightarrow \exists x_1 \ldots x_m \{P(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0\}.$$

Sets which can be defined in this way are called *Diophantine*, the above equivalence being called *Diophantine representation* of the set $\mathfrak{M}$.

Trivial fact. *Every Diophantine set is recursively enumerable.*

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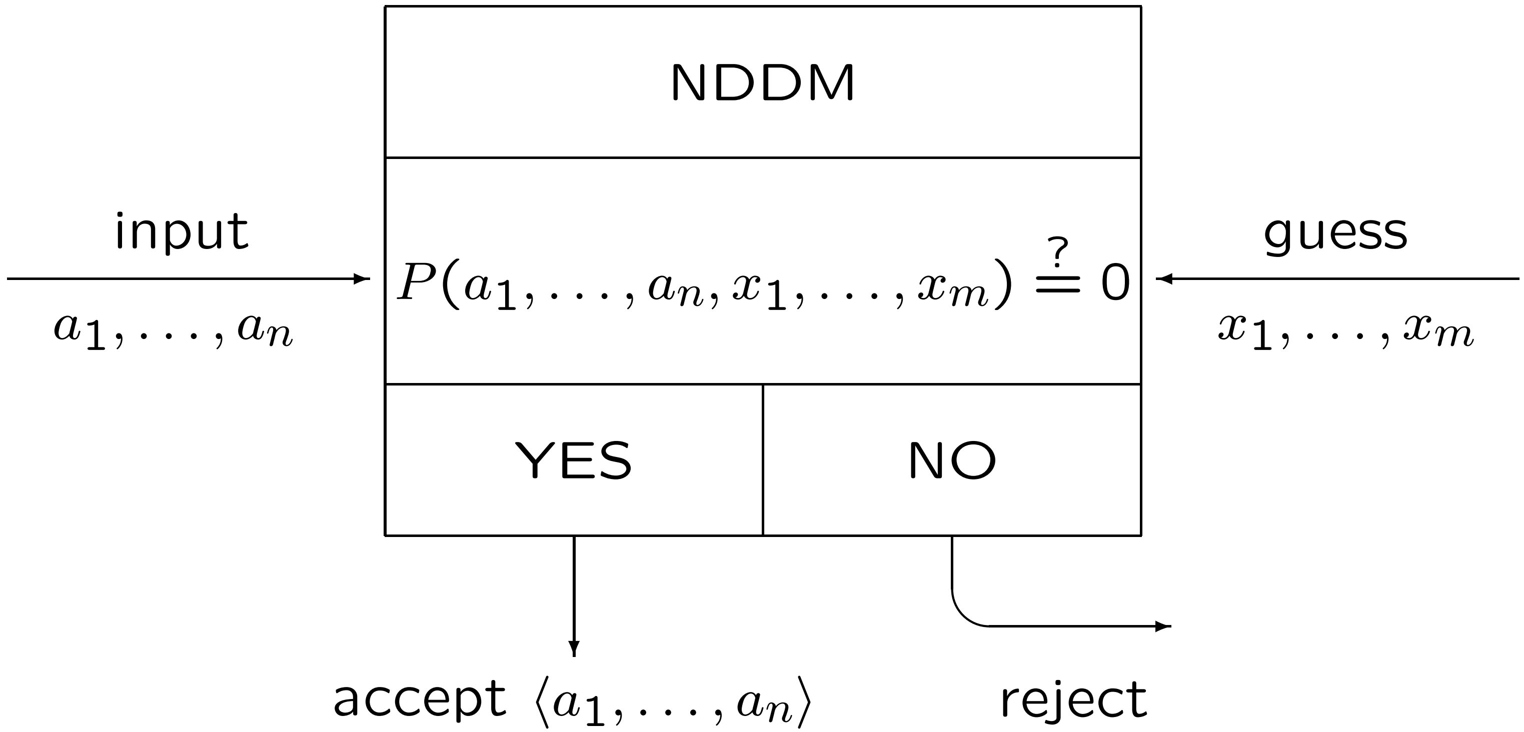
Main result (DPRM-theorem). *Every recursively enumerable set is Diophantine.*

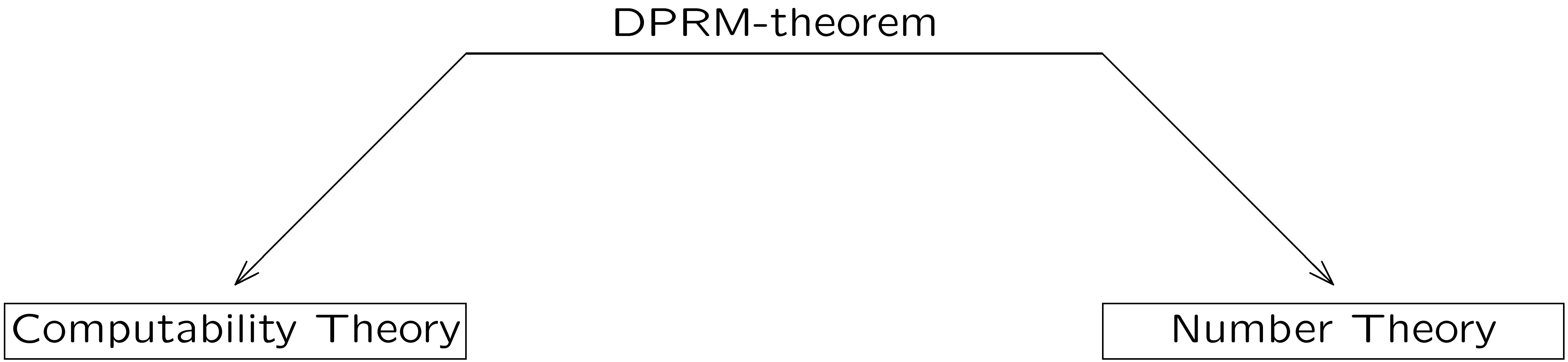
DPRM: after Davis-Putnam-Robinson-Matiyasevich

Non-Deterministic Diophantine Machine (NDDM for short) was introduced by Leonard Adleman and Kenneth Manders.



DPRM-theorem says that NDDMs are as powerful as, say, non-deterministic Turing machines.

Two-way bridge



Two-way bridge

DPRM-theorem

Computability Theory

Number Theory

To represent words by natural numbers, we can use two-letter alphabet $\{1, 2\}$ and consider words in this alphabet as numbers written in base 2 notation.

DPRM-theorem. Every r.e. set has an Diophantine representation

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DPR-theorem. Every r.e. set has an *exponential Diophantine representation*

$$\langle a_1, \ldots, a_n \rangle \in \mathfrak{M} \Longleftrightarrow \exists x_1 \ldots x_m \{E(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0\}$$

where E is constructed with addition, subtraction, multiplication and exponentiation.

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Every r.e. set has a *single-fold* exponential Diophantine representation, i.e., such that the x's are uniquely determined by the a's

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DPR-theorem. Every r.e. set has an *exponential Diophantine representation*

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Open Problem. Is it true that every r.e. set has a single-fold Diophantine representation?

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where E is constructed with addition, subtraction, multiplication and exponentiation.

Hilary Putnam's trick: Given a Diophantine equation

$$P(a, b, x_1, \ldots, x_m) = 0$$

we can construct another Diophantine equation of the form

$$Q(a, y_1, \ldots, y_n) = b$$

such that the former equation has a solution in $x_1, \ldots, x_m$ for the same values of the parameters a and b for which the latter equation has a solutions in $y_1, \ldots, y_n$.

This trick works provided that the parameter b assumes only non-negative integer values.

Part III. *Diophantine flavor of Kolmogorov complexity*

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$$P(w', x'_1, \ldots, x'_k) \geq 0$$

has a solution and

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Proof 2. Pass the bridge from right to left—use the existence of a universal Diophantine equation:

$$\exists y_1 \ldots y_n U(a, k, y_1, \ldots, y_n) = 0 \Leftrightarrow \exists x_1 \ldots x_{m_k} P_k(a, x_1, \ldots, x_{m_k}) = 0$$

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Theorem (Matiyasevich). Every optimal Diophantine description mode can be defined as

$$a = \min\{P_{\text{opt}}(w, x_1, \ldots, x_m)\}$$

where P_{opt} is a suitable polynomial assuming only non-negative values.

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There is an effectively computable increasing sequence $\Omega_1, \Omega_2, \ldots, \ldots$ of rational numbers which converges to Ω.	Toby Ord and Tien D. Kieu [2003] constructed particular exponential Diophantine equation $E(k, z_1, \ldots, z_m) = 0$ such that for every k this equation has finitely many solutions in $z_1, \ldots, z_m$ and the number of solutions is odd iff the k-th digit of Ω is equal to 1.

Theorem (Matiyasevich [2003]). Let $\mathfrak{C}$ be any infinite decidable set with infinite complement. Then we can construct an exponential Diophantine equation $E(k, z_1, \ldots, z_m) = 0$ such that

- for every k the equation has finitely many solutions in $z_1, \ldots, z_m$;
- the number of solutions of the equation for givel value of k belongs to the set $\mathfrak{C}$ iff the k-th digit of Ω is equal to 1.

Theorem (Matiyasevich [2003]). Let $\mathfrak{C}$ be any infinite decidable set with infinite complement. Then we can construct an exponential Diophantine equation $E(k, z_1, \ldots, z_m) = 0$ such that

- for every k the equation has finitely many solutions in $z_1, \ldots, z_m$;
- the number of solutions of the equation for givel value of k belongs to the set $\mathfrak{C}$ iff the k-th digit of Ω is equal to 1.

Open Problem. Could we improve any of the above mentioned definitions of Ω via the number of solutions of exponential Diophantine equations to similar definitions via the number of solutions of genuine (i.e., without exponentiation) Diophantine equations?