Asymptotic Properties of H-free graphs

Habilitationsschrift

vorgelegt von

Angelika Steger

Forschungsinstitut für Diskrete Mathematik
Rheinische Friedrich-Wilhelms-Universität Bonn

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Introduction

Extremal graph theory is widely considered as one of the most elegant and deeply developed areas in all of graph theory. One of its very first roots, grown long before graph theory per se existed, is the solution to a problem raised by W. Mantel in the Dutch journal “Wiskundige Opgaven”: How many edges can a simple graph which does not contain a triangle (3-simplex or K_3) as a subgraph have? The answer, given by several people, including Mantel himself, shows that the complete bipartite graph with parts as equal as possible is extremal, i.e., is an edge maximal triangle-free graph. Mantel's result is usually considered as a kind of forerunner to extremal graph theory. Generalizing Mantel's observation, P. Turán, one of the famous Hungarian mathematicians of this century best known for his work in analytic number theory and real and complex analysis, characterized in 1941 the extremal graphs which do not contain the complete graph K_{l+1} on $l+1$ vertices as a subgraph. Turán's result, then, provided stimulation for a variety of problems and results, launching *extremal graph theory*. His theorem became the most important paradigm for one of the quintessential questions in extremal graph theory, viz. the problem of

“What does an extremal H-free graph look like?”,

i.e., how many edges can a graph have which does not contain a copy of a certain fixed graph H as a subgraph?

Random graph theory arose from the observation that simple counting arguments are often useful in tackling extremal problems in graph theory. Ramsey's theorem, or more precisely the Erdős-Szekeres proof of this result, shows that every graph on n vertices must either contain a complete subgraph on $1/2 \log n$ vertices or a stable set, a set of vertices not containing an edge, on $1/2 \log n$ vertices. Erdős (1947) used a straightforward counting argument to show that for each n there is a graph on n vertices which contains neither a complete subgraph on $2 \log n$ vertices nor a stable set on $2 \log n$ vertices. In fact, Erdős even showed that “almost every” graph on n vertices neither contains a complete subgraph nor a stable set of that size. Usually the appearance of the seminal paper “On the evolution of random graphs” by Erdős and Rényi (1960) is considered as the birth of random graph theory. They were the first to investigate random graphs for their own sake. From then on, stimulated by the appearance of theoretical computer science with its many new graph theoretic problems, numerous results were published in this very attractive part of graph theory. Throughout, one of the leading questions in this area remained

"What does a typical graph look like?",

i.e., what does "almost every" graph look like or which properties does a graph have with probability tending to one?

One major reason for the development and the beauty of random graph theory is the use of elements from probability theory such as martingale inequalities, discrete isoperimetric inequalities, eigenvalue techniques and rapidly mixing Markov chains, just to name a few. The application of these tools is heavily based on the fact that in most probability models used in random graph theory the edges can be chosen independently with some probability which only depends on the number of vertices of the graph. These models are no longer useful if classes of graphs are studied which are defined by, say, structural side constraints as happens quite frequently in various problems. One of the most important classes in this respect is the class of triangle-free graphs, as considered already by Mantel. Here, Erdős, Kleitman and Rothschild (1976) proved that "almost every" triangle-free graph is bipartite, or to link this result to extremal graph theory, that the structure of most of the triangle-free graphs is already determined by the extremal triangle-free graph. Surely, in contrary to the probabilistic methods mentioned before, the proof of the Erdős-Kleitman-Rothschild theorem had to use subtle enumeration techniques. In a sense, this result is the germ and the first paradigm of the theme of this thesis, viz.,

"What does a typical H-free graph look like?",

where, again, H is an arbitrary fixed graph.

The present work is organized as follows: In Part I, we review some basic facts from random graph theory (Chapter 1), extremal graph theory (Chapter 2) and some asymptotic structure results, including a proof of the Erdős-Kleitman-Rothschild theorem (Chapter 3).

The basic question of this thesis, namely, the study of the structure of a typical H-free graph, is then tackled in Chapter 4. The Erdős-Kleitman-Rothschild theorem was substantially generalized by Kolaitis, Prömel and Rothschild (1987), who proved that a typical K_{l+1}-free graph is already l-partite, i.e. has chromatic number l. This result relates to Turán's theorem on extremal K_{l+1}-free graphs as the Erdős-Kleitman-Rothschild theorem relates to Mantel's observation. In Chapter 4, we first extend Turán's result to forbidden color-critical graphs. Thereby, a graph H is called color-critical, if there exists a vertex in H such that its deletion decreases the chromatic number. Obviously, every complete graph K_{l+1} is color-critical. Then, in the second section of Chapter 4, we prove the asymptotic counterpart to this extremal graph theorem, characterizing the structure of a typical graph which does not contain a fixed color-critical graph as a (weak) subgraph. So far, for no other graphs H is the structure of a typical H-free graph completely known.

One of the sources for many questions on random graphs as well as on the asymptotic structure of graphs is computer science, in particular the design and analysis of algorithms. One of the most intrinsic problems in this area is the graph coloring problem: Given a graph G, decide its chromatic number $\chi(G)$. Unfortunately, even deciding whether $\chi(G) \leq l$ for some fixed $l > 2$ is $\mathcal{NP}$-complete. The importance of this problem has led many scientists to find algorithms which are, though exponential in the worst case, at least fast on average. In Chapter 5, inter alii, the result of Chapter 4 on the structure of almost every K_{l+1}-free graph are considerably strengthened in order to design an algorithm which colors every K_{l+1}-free graph with its chromatic number of colors (which may be arbitrarily large) and which has polynomial expected running time. This result extends earlier results of Turner (1988) and Dyer and Frieze (1989). It is worthwhile noticing that the algorithm presented can easily be converted to a parallel algorithm which colors every K_{l+1}-free graph with minimal number of colors using at most polynomially many processors such that the expected running time of this algorithm is bounded by a logarithmic function in the size of the input.

In random graph theory, another basic problem dating back to Erdős and Rényi, is the study of the evolution process of random graphs. Roughly speaking, this means that starting with edge-probability $p = 0$ one slowly increases p and asks for every p how a typical graph with edge-probability p looks like. Of particular interest are those p at which the structure of the typical graph changes, i.e., the phase transitions. For H-free graphs, one might study the analogous problem considering the edge model instead, i.e., starting with the empty graph (without any edges) one throws in edge by edge in such a way that no H-subgraph is created and asks for every edge-number m how a typical H-free graph with m edges looks like. Whereas for general graphs the evolution process is rather well understood, not much is known about the evolution process for H-free graphs. Even in the simple case of triangle-free graphs the picture is far from being complete. In Chapter 6, we study the evolution process for triangle-free graphs and present the results known so far.

In Chapter 7, then, we change the notion of "subgraph", switching from forbidden *weak* subgraphs to the less restrictive notion of forbidden *induced* subgraphs. Correspondingly, the notion of an extremal graph for a class of H-free graphs has to be redefined. Studying the asymptotic structure of H-free graphs where H is required to be a forbidden induced subgraph seems to be more difficult than the corresponding problem where H is supposed to be a weak subgraph. In fact, except for complete subgraphs where both notions coincide, only for the cases that H is the cycle of length 4 or the cycle of length 5 is the structure of a typical H-free graph known - the latter result has a rather nice consequence that will be discussed in Chapter 8. The main result of this chapter is a slightly weaker result, but one which applies for every graph H. We determine for every H a family of graphs such that almost every graph which does not contain H as an induced subgraph is "close" to some member of this family. This result, which is proved using a generalization of Szemerédi's uniformity lemma for hypergraphs, corresponds to a theorem of Erdős, Frankl and Rödl (1986) for excluded weak subgraphs. We also show that, as a consequence, the growth rate of every hereditary graph property can be determined quite easily.

Graph theory has grown along some leading problems. Most notably here was the four color conjecture. In Chapter 8 we apply methods from random graph theory, and respectively asymptotic graph theory, to shed some light on two classical conjectures in graph theory. These problems have been twins in the sense that both were offsprings of the same meeting on graph theory organized by H. Grötsch in 1960. At this meeting, the Hungarian mathematician G. Hajós conjectured that the chromatic number of every graph is bounded from above by its topological clique number, whereas the French mathematician C. Berge formulated his "Strong perfect graph conjecture" saying that the class of all perfect graphs and the class of all Berge graphs coincide. Hajós's conjecture became quite popular and stayed alive for nearly 20 years. Then, briefly after Catlin had constructed a counterexample to this conjecture, Erdős and Fajtlowicz (1981) showed that almost every graph is already a counterexample to the Hajós conjecture. This result is presented in the first section of Chapter 8. Berge's conjecture became even more popular than Hajós's and has already survived already more than 30 years. In the second section of this last chapter we prove that the Berge conjecture is at least almost always true. We even show something more. It is obvious that every perfect graph is a Berge graph. Moreover, firstly, by definition, every Berge graph does not contain a C_5, i.e., a cycle of length 5, as an induced subgraph and, secondly, by a theorem of Hayward (1985), every generalized split graph is perfect. We show that a typical graph which does not contain a C_5 as an induced subgraph is a generalized split graph implying that the class of perfect graphs is dense in the class of Berge graphs.

Basic Notations and Definitions

A *graph* G is a pair $(V(G), E(G))$, usually abbreviated as (V, E), such that E is a subset of the set of unordered pairs of V, i.e., $E \subseteq [{V \atop 2}]$. The set V is called the set of *vertices* and E the set of *edges*. Here we will consider only finite graphs, that is the sets V and E are always finite. We usually denote the cardinality of the vertex set V by n, and the cardinality of the edge set E by m. A vertex set of cardinality n is often denoted by V_n.

A graph H is a (weak) *subgraph* of G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G) \cap [{V(H) \atop 2}]$. If, in addition, $E(H) = E(G) \cap [{V(H) \atop 2}]$, then H is called an *induced* subgraph of G. Two vertices u and v are called *adjacent* in G, if $\{u, v\} \in E(G)$. The set of vertices which are adjacent to a vertex v is called the *neighbourhood* of v, denoted by $\Gamma(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$. The *degree* of v is $d(v) = |\Gamma(v)|$. The neighbourhood of a set $X \subseteq V$ is the set $\Gamma(X) = \bigcup_{x \in X} \Gamma(x)$. The complement $\overline{G}$ of a graph G is a graph on the same vertex set given by interchanging edges and non-edges, i.e., $\overline{G} = (V(G), [{V(G) \atop 2}] \setminus E(G))$.

A subset $S \subseteq V(G)$ is *stable*, if $E(G) \cap [{S \atop 2}] = \emptyset$. If, on the other hand, a subset $X \subseteq V(G)$ satisfies $E(G) \cap [{X \atop 2}] = [{X \atop 2}]$ then X is called a *clique*. A clique of size l is often called an l-clique, denoted by K_l. A clique covering of size k is a partition of the vertex set $V = W_1 \cup \cdots \cup W_k$, such that each of the k sets W_i induces a clique in G. The smallest natural number k for which G contains a clique covering of size k is called the *clique covering number* of G, denoted by $\sigma(G)$.

A (proper) *coloring* of a graph G with l colors is a mapping $\Delta : V \to \{1, \ldots, l\}$ such that every edge $\{u, v\} \in E$ satisfies $\Delta(u) \neq \Delta(v)$. G is called *l-colorable* if G has a coloring with l colors. The *chromatic number* of G is defined as $\chi(G) = \min\{l \mid \text{G is } l\text{-colorable}\}$. Observe that $\chi(G) = \sigma(\overline{G})$.

A graph $G = (V, E)$ is l-partite graph with (color) classes $V_1, \ldots, V_l$ if $V = V_1 \cup \cdots \cup V_l$, $V_i \cap V_j = \emptyset$ for all $1 \leq i < j \leq l$, and no edge of G joins two vertices in the same class. A *complete l-partite* graph is an inclusion maximal l-partite graph. The complete l-partite graph with t_i vertices in the ith class is denoted by $K_l(t_1, \ldots, t_l)$, which is abbreviated as $K_l(t)$ if $t_i \equiv t$. In particular, $K_l = K_l(1) = K_l(1, \ldots, 1)$. The *bipartite* graphs are exactly the 2-partite (or, equivalently, the 2-colorable) graphs.

If $G_1, \ldots, G_l$ is a family of graphs then we denote by $\mathsf{X}_{i=1}^{l} G_i$ the product of the graphs G_i, that is the graph obtained by taking the disjoint union of $G_1, \ldots, G_l$ and joining any two vertices belonging to different G_i's.

Let H be a finite graph. Then $\mathcal{F}orb_n(H)$ denotes the set of all graphs on n vertices which do not contain H as a (weak) subgraph. Similarly, $\mathcal{F}orb_n^*(H)$ denotes the set of all graphs on n vertices which do not contain H as an induced subgraph. A graph $G \in \mathcal{F}orb_n(H)$ is called an *extremal graph* if G contains the maximum number of edges among all graphs in $\mathcal{F}orb_n(H)$. The cardinality of its edge set is denoted by $ex_n(H)$. (For the more complicated definition of an extremal graph with respect to the class $\mathcal{F}orb_n^*(H)$ we refer the reader to Chapter 7.) With $\mathcal{C}ol_n(l)$ we denote the set of all l-colorable graphs on n vertices.

The logarithm to base 2 is denoted by $\log n$, while $\ln n$ denotes the natural logarithm. The kth iteration of $\log n$ is denoted by $\log^{(k)} n$, i.e. $\log^{(k)} n = \log(\log^{(k-1)} n)$ and $\log^{(1)} n = \log n$. Throughout this thesis we will usually omit the explicit use of the Gauß-brackets $\lfloor\ \rfloor$ in denoting the cardinality of vertex sets, but instead tacitly assume them. For example, a subset $W \subseteq V_n$ of size $|W| = \log n$ is always meant to be a subset of size $|W| = \lfloor \log n \rfloor$.

PART I

Basics

Chapter 1

Random Graph Theory

Nowadays, the theory of random graphs is one of the most important and frequently used areas of Discrete Mathematics. Its origin, however, dates back only less than forty years ago to the seminal work of Erdős and Rényi in the late 50s, cf. [34], [35], [36], and [37]. A wide variety of methods have been developed and used since then, ranging from the methods of moments over correlation inequalities up to the use of martingale and isoperimetric inequalities. It seems impossible, even to only outline the most important results in a reasonable amount of space. We don't even try, but instead refer the interested reader to the excellent monograph of Bollobás [7] and to the more recent surveys [9] and [10]. Here we solely restrict out attention to the methods and results which are relevant for this thesis.

1.1 Basic models and results

The two most frequently used random graph models are the $G(n,m)$ and the $G(n,p)$-models. A random graph with respect to the first model, denoted as $G_{n,m}$, is obtained by choosing a graph uniformly at random from the set of all (labelled) graphs on n vertices and m edges. In the second model a random graph, usually denoted as $G_{n,p}$, is obtained by independently choosing the edges of the complete graph on n vertices with probability p. In other words, if G is a graph on n vertices with m edges, then $\mathrm{Prob}\,[G_{n,p} = G] = p^m(1-p)^{\binom{n}{2}-m}$. Typically, within the theory of random graphs, one considers only the asymptotic behaviour of $G_{n,p}$ and $G_{n,m}$ when n tends to infinity. In addition, the parameters m and p are usually allowed to vary as functions of n.

A fundamental problem in random graph theory is to determine (or at least to estimate) the probaility that a random graph $G_{n,p}$ or $G_{n,m}$ satisfies a certain property, e.g. is connected, contains a triangle, is 3-colorable. The reader is invited to observe that most naturally occuring graph properties are in some sense "monotone": they either devolve their property upon all subgraphs or upon all supergraphs.

A property $\mathcal{P}$ is said to be *monotone decreasing (increasing)* if whenever a graph G has $\mathcal{P}$ all subgraphs (supergraphs) of G have $\mathcal{P}$ as well. A typical example of a monotone decreasing property is the property "... is H-free". Trivially, for all (nonempty) graphs H, the empty graph is H-free, while the complete graph is not. The following proposition establishes the intuitively obvious fact that a monotone decreasing property is more likely to hold the smaller the number m of edges / the edge probability p is.

Proposition 1.1.1 *Let $\mathcal{P}$ be a monotone decreasing property and suppose $0 \leq m_1 \leq m_2 \leq \binom{n}{2}$ and $0 \leq p_1 \leq p_2 \leq 1$. Then*

$$\mathrm{Prob}\left[G_{n,m_1} \textit{ has property } \mathcal{P}\right] \geq \mathrm{Prob}\left[G_{n,m_2} \textit{ has property } \mathcal{P}\right]$$

and

$$\mathrm{Prob}\left[G_{n,p_1} \textit{ has property } \mathcal{P}\right] \geq \mathrm{Prob}\left[G_{n,p_2} \textit{ has property } \mathcal{P}\right].$$

Proof. Let $N_{n,m}$ denote the number of graphs on n vertices with m edges which have property $\mathcal{P}$. By the monotonicity of $\mathcal{P}$, we have $N_{n,m_1} \geq \binom{m_2}{m_1} \cdot N_{n,m_2} / \binom{\binom{n}{2}-m_1}{m_2-m_1}$. As $\mathrm{Prob}\left[G_{n,m} \text{ has property } \mathcal{P}\right] = N_{n,m}/\binom{\binom{n}{2}}{m}$, this inequality implies the first of the two claimed inequalities. To see the second, let $p = (p_2 - p_1)/(1 - p_1)$. As $p_2 = p_1 + p - p_1 p$, a random graph G_{n,p_2} can be obtained by considering the union of a random graph G_{n,p_1} and a random graph $G_{n,p}$. Again, the desired inequality is a consequence of the monotonicity of $\mathcal{P}$. $\square$

In a random graph $G_{n,p}$ the expected number of edges is $p\binom{n}{2}$. We would therefore expect that for $m = p\binom{n}{2}$ the two random graphs $G_{n,p}$ and $G_{n,m}$ behave very similarly. This is in fact the case, under very weak assumptions on p and n. The resulting interchangeability of the two random graph models turned out to be extremely useful and handy in random graph theory. For our purposes, however, the above heuristic observation suffices, and we refer the interested reader to Bollobás (1985) and Łuczak (1990) for a precise statement of the equivalence of the two models.

1.2 Thresholds for existence

Given a (monotone) property $\mathcal{P}$ a function $m^*(n)$ is called a *threshold function* for $\mathcal{P}$ if $m(n)/m^*(n) \to 0$ implies that almost every random graph $G_{n,m}$ has $\mathcal{P}$, while for $m(n)/m^*(n) \to \infty$ almost no $G_{n,m}$ has $\mathcal{P}$. (In the $G(n,p)$-model the threshold function is defined similarly.) Already Erdős and Rényi discovered that many natural properties have a threshold function. Much later Bollobás and Thomason (1986) showed that in fact every monotone property has a threshold function.

We are mainly interested in the property of "*is H-free*", for various graphs H. For "balanced" graphs H the threshold for being H-free was already determined by Erdős and Rényi (1960). (A graph H is called balanced if it contains no subgraph with higher density, i.e., if $d(H) = \max\{(L) \mid L \subseteq H\}$, where $d(G) = |E(G)|/|V(G)|$ is the *density* of the graph G. Typical examples of balanced graphs are cliques and trees.) Only twenty years later, Bollobás determined the threshold for general graphs.

Theorem 1.2.1 [Bollobás, 1981] *For every graph H let $m(H) = \max\{\frac{|E(L)|}{|V(L)|} \mid L \subseteq H\}$ denote the maximum subgraph density. Then*

$$\text{Prob}\,[G_{n,p} \in \mathcal{F}orb_n(H)] = \begin{cases} 1 - o(1), & \text{if } np^{m(H)} = o(1), \\ o(1), & \text{if } np^{m(H)} \to \infty. \end{cases}$$

Proof. For a graph L we denote with $X_{n,p}(L)$ the random variable which counts the number of copies of the graph L in the random graph $G_{n,p}$. Then

$$\mathbb{E}[X_{n,p}(L)] = \Theta(n^{|V(L)|}p^{|E(L)|}),$$

where the hidden constant depends only on the number of vertices of L and the size of its automorphism group. Let now $L_0 \subseteq H$ be a subgraph such that the density $d(L_0)$is maximal, i.e. such that $d(L_0) = m(H)$. Then,

$$\text{Prob}\,[X_{n,p}(H) > 0] \leq \text{Prob}\,[X_{n,p}(L_0) > 0] \leq \mathbb{E}[X_{n,p}(L_0)] = \Theta((np^{m(H)})^{|V(L_0)|}),$$

which proves the first half of the theorem.

For the second part observe that by Chebyshev's inequality

$$\text{Prob}\,[X_{n,p}(H) = 0] \leq \frac{\text{Var}[X_{n,p}(H)]}{(\mathbb{E}[X_{n,p}(H)])^2}.$$

It therefore suffices to bound $\text{Var}[X_{n,p}(H)]$. As $X_{n,p}(H)$ is simply the sum of indicator variables, say $I_{n,p}(H)^j$, we have

$$\begin{aligned}
\text{Var}[X_{n,p}(H)] &= \sum_{j,k} \text{Cov}[I_{n,p}(H)^j, I_{n,p}(H)^k] \\
&= \sum_{\substack{L \subseteq H \\ |E(L)|>0}} |\{(H_1, H_2) \mid H_1, H_2 \subseteq K_n,\ H_1, H_2 \cong H, H_1 \cap H_2 = L\}| \cdot \\
&\qquad (p^{2|E(H)|-|E(L)|} - p^{2|E(H)|}) \\
&\leq n^{2|V(H)|}p^{2|E(H)|} \cdot \sum_{\substack{L \subseteq H \\ |E(L)|>0}} n^{-|V(L)|}p^{-|E(L)|} \\
&\leq c \cdot (\mathbb{E}[X_{n,p}(H)])^2 \cdot \sum_{\substack{L \subseteq H \\ |E(L)|>0}} (np^{d(L)})^{-|V(L)|} \\
&\leq c' \cdot (\mathbb{E}[X_{n,p}(H)])^2 \cdot (np^{m(H)})^{-1},
\end{aligned}$$

for appropriate constants $c, c' > 0$. □

1.3 Janson inequalities

Our next goal is to estimate the probability that a random graph $G_{n,p}$ is H-free precisely. While it is easy to calculate the probability that $G_{n,p}$ does not contain H on a given, fixed set of vertices of cardinality $|V(H)|$, this is – due to the dependence of the corresponding events – not so in general. A way around this dilemma are the so-called Janson-inequalities, which we shall prove in this section.

Let $H_i \subset K_n$, $i = 1, \ldots, k$ be a family of subgraphs of the complete graph on n vertices, and let A_i denote the event that the graph H_i occurs in the random graph $G_{n,p}$. The quantity we are interested in is the probability that none of the events A_i occurs. (As a possible application, and in fact the one we have in mind, consider the case that the graphs H_i enumerate all copies of a fixed graph H – the probability that none of the events A_i occurs is then exactly the probability that $G_{n,p}$ is H-free.)

Let X_i denote the indicator random variables corresponding to the events A_i and let $X = \sum_{i=1}^k X_i$. Then

$$\mathrm{Prob}\,[\bigcap_{i=1} \bar{A}_i] = \mathrm{Prob}\,[X = 0].$$

Let us first recall trivial lower and upper bounds. The Markov inequality gives

$$\mathrm{Prob}\,[X = 0] \geq 1 - \mathbb{E}[X].$$

The Chebyshev inequality on the other hand implies

$$\mathrm{Prob}\,[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2}.$$

If the A_i are independent we can easily do better. Then

$$\mathrm{Prob}\,[X = 0] = \mathrm{Prob}\,[\bigcap_{i=1} \bar{A}_i] = \prod_{i=1}^{k} \mathrm{Prob}\,[\bar{A}_i]. \tag{1.3.1}$$

The aim of this section is to show that similar bounds also hold if the A_i are not totally independent – but not very dependent either. As a measure for the independence consider the function $\Delta = \sum_{i \sim j} \mathrm{Prob}\,[A_i \wedge A_j]$, where $\sim$ is the dependence relation on the events A_i, i.e., $i \sim j$ if and only if $\mathrm{Prob}\,[A_i \wedge A_j] \neq \mathrm{Prob}\,[A_i] \cdot \mathrm{Prob}\,[A_j]$. Clearly, the larger the number of independent pairs the smaller Δ. In particular, the 2-wise independence of the events A_i implies that Δ is zero. As we shall see, the function Δ is a good measure for how far away from the "truth" equation (1.3.1) is. More precisely, we shall prove the following result of Svante Janson.

Theorem 1.3.1 (Janson-Inequality) *Let A_i be events as defined above. Then*

$$\prod_{i=1}^{k} \mathrm{Prob}\,[\bar{A}_i] \leq \mathrm{Prob}\,[\bigcap_{i=1}^{k} \bar{A}_i] \leq e^{\frac{1}{1-\alpha}\frac{\Delta}{2}} \cdot \prod_{i=1}^{k} \mathrm{Prob}\,[\bar{A}_i],$$

where $\Delta = \sum_{i \sim j} \mathrm{Prob}\,[A_i \wedge A_j]$ and $\alpha = \max\{\mathrm{Prob}\,[A_i] \mid 1 \leq i \leq k\}$.

The proof we present here is due to Boppana and Spencer (1989). It makes essential use of the so-called FKG-inequality due to Fortuin, Kasteleyn and Ginibre (1971).

Theorem 1.3.2 (FKG-Inequality) *Let S be a finite set and let μ, f and g be real functions defined on the power set $\mathcal{P}(S)$ of S such that*

$\mu \geq 0$ and μ log-supermodular, that is

$$\mu(X) \cdot \mu(Y) \leq \mu(X \cup Y) \cdot \mu(X \cap Y) \qquad \text{for all } X, Y \subseteq S, \text{ and}$$

f and g monotone increasing, that is

$$f(X) \leq f(Y) \quad \text{and} \quad g(X) \leq g(Y) \qquad \text{for all } X \subseteq Y \subseteq S.$$

Then

$$\left(\sum_{X \subseteq S} \mu(X) f(X)\right) \cdot \left(\sum_{X \subseteq S} \mu(X) g(X)\right) \leq \left(\sum_{X \subseteq S} \mu(X)\right) \cdot \left(\sum_{X \subseteq S} \mu(X) f(X) g(X)\right).$$

□

Proof of Theorem 1.3.1. First we show that the FKG-inequality implies

$$(1.3.2) \quad \mathrm{Prob}\,[A_i \mid \bigcap_{l \in I} \bar{A}_l] \leq \mathrm{Prob}\,[A_i] \qquad \text{for all } 1 \leq i \leq k \text{ and } I \subseteq \{1, \ldots, k\}.$$

and

$$(1.3.3) \quad \mathrm{Prob}\,[A_i \mid A_j \cap \bigcap_{l \in I} \bar{A}_l] \leq \mathrm{Prob}\,[A_i \mid A_j] \quad \text{for all } 1 \leq i, j \leq k,\ I \subseteq \{1, \ldots, k\}.$$

As (1.3.2) is a special case of (1.3.3) (let A_j denote the event that the empty graph E_n is a subgraph of $G_{n,p}$), it suffices to show the second ineqality. To do this let S denote the edge set of the complete graph on n vertices and let for $X \subseteq S$

$$\begin{aligned} \mu(X) &= \begin{cases} \mathrm{Prob}\,[G_{n,p} = X], & \text{if } E(H_j) \subseteq X \\ 0, & \text{otherwise,} \end{cases} \\ f(X) &= \begin{cases} 1, & \text{if } E(H_i) \subseteq X \\ 0, & \text{otherwise,} \end{cases} \\ g(X) &= \begin{cases} -1, & \text{if } E(H_l) \setminus X \neq \emptyset \text{ for all } l \in I \\ \ \ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

One easily checks that these functions satisfy the requirements of Theorem 1.3.2, and that for them the statement of the theorem corresponds exactly to the inequality (1.3.3).

The lower bound follows now easily. Indeed, (1.3.2) together with induction on k gives

$$\mathrm{Prob}\,[\bigcap_{j=1}^{k} \bar{A}_j] \;=\; \mathrm{Prob}\,[\bar{A}_k \mid \bigcap_{j=1}^{k-1} \bar{A}_j] \cdot \mathrm{Prob}\,[\bigcap_{j=1}^{k-1} \bar{A}_j]$$

$$\begin{aligned}
&= (1-\mathrm{Prob}\,[A_k \mid \bigcap_{j=1}^{k-1}\bar{A}_j])\cdot \mathrm{Prob}\,[\bigcap_{j=1}^{k-1}\bar{A}_j] \\
&\geq (1-\mathrm{Prob}\,[A_k])\cdot \prod_{j=1}^{k-1}\mathrm{Prob}\,[\bar{A}_j] \\
&= \prod_{j=1}^{k}\mathrm{Prob}\,[\bar{A}_j].
\end{aligned}$$

Now to the upper bound. Let $1 \leq i \leq k$ be arbitrary but fixed. We claim that

$$\mathrm{Prob}\,[A_i \mid \bar{A}_1 \cap \cdots \cap \bar{A}_{i-1}] \geq \mathrm{Prob}\,[A_i] - \sum_{\substack{j\sim i\\ j<i}} \mathrm{Prob}\,[A_i \cap A_j]. \tag{1.3.4}$$

To see (1.3.4) we assume without loss of generality that the ordering of the A_j is such that

$$\begin{aligned}
&i \sim j \quad \text{for } j = 1,\ldots,d \ \text{ and} \\
&i \not\sim j \quad \text{for } j = d+1,\ldots,i-1.
\end{aligned}$$

Recall furthermore that for any events A, B, and C we have

$$\mathrm{Prob}\,[A \mid B\cap C] \geq \mathrm{Prob}\,[A\cap B \mid C] = \mathrm{Prob}\,[A \mid C]\cdot \mathrm{Prob}\,[B \mid A\cap C].$$

If we now let $A = A_i$, $B = \bar{A}_1 \cap \cdots \cap \bar{A}_d$ and $C = \bar{A}_{d+1} \cap \cdots \cap \bar{A}_{i-1}$ we obtain

$$\begin{aligned}
&\mathrm{Prob}\,[A_i \mid \bar{A}_1 \cap \cdots \cap \bar{A}_{i-1}] \\
&\quad\geq \mathrm{Prob}\,[A_i \mid \bar{A}_{d+1} \cap \cdots \cap \bar{A}_{i-1}]\cdot \mathrm{Prob}\,[\bar{A}_1 \cap \cdots \cap \bar{A}_d \mid A_i \cap (\bar{A}_{d+1} \cap \cdots \cap \bar{A}_{i-1})]
\end{aligned}$$

As $i \not\sim j$ for all $j = d+1,\ldots,i-1$ we have

$$\mathrm{Prob}\,[A_i \mid \bar{A}_{d+1} \cap \cdots \cap \bar{A}_{i-1}] = \mathrm{Prob}\,[A_i].$$

With the help of (1.3.3) we obtain for the second factor

$$\begin{aligned}
&\mathrm{Prob}\,[\bar{A}_1 \cap \cdots \cap \bar{A}_d \mid A_i \cap (\bar{A}_{d+1} \cap \cdots \cap \bar{A}_{i-1})] \\
&\geq 1 - \sum_{j=1}^{d} \mathrm{Prob}\,[A_j \mid A_i \cap (\bar{A}_{d+1} \cap \cdots \cap \bar{A}_{i-1})] \\
&\geq 1 - \sum_{j=1}^{d} \mathrm{Prob}\,[A_j \mid A_i],
\end{aligned}$$

completing the proof of (1.3.4). Next observe that (1.3.4) is equivalent to

$$\mathrm{Prob}\,[\bar{A}_i \mid \bar{A}_1 \cap \cdots \cap \bar{A}_{i-1}] \leq \mathrm{Prob}\,[\bar{A}_i] + \sum_{\substack{j\sim i\\ j<i}} \mathrm{Prob}\,[A_i \cap A_j].$$

This immediately implies

$$\text{Prob}\,[\bar{A}_i \mid \bar{A}_1 \cap \cdots \cap \bar{A}_{i-1}] \leq \text{Prob}\,[\bar{A}_i](1 + \frac{1}{1-\alpha} \sum_{\substack{j \sim i \\ j < i}} \text{Prob}\,[A_i \cap A_j]).$$

Therefore

$$\begin{aligned}
\text{Prob}\,[\bigcap_{i=1}^{k} \bar{A}_i] &= \prod_{i=1}^{k} \text{Prob}\,[\bar{A}_i \mid \bar{A}_1 \cap \cdots \cap \bar{A}_{i-1}] \\
&\leq \prod_{i=1}^{k} \Big(\text{Prob}\,[\bar{A}_i] \cdot (1 + \tfrac{1}{1-\alpha} \sum_{\substack{j \sim i \\ j < i}} \text{Prob}\,[A_i \cap A_j])\Big) \\
&\leq e^{\frac{\Delta}{2(1-\alpha)}} \prod_{i=1}^{k} \text{Prob}\,[\bar{A}_i],
\end{aligned}$$

where the last inequality follows from the fact that $\ln(1+x) \leq x$ for all $x \geq 0$ and the definition of Δ. $\square$

Observe that for $\Delta \geq -(2-2\alpha) \sum_{i=1}^{k} \ln(1 - \text{Prob}\,[A_i])$ the upper bound of the Janson inequality is larger than 1 and provides no meaningful information. In this case, however, the following extension of Theorem 1.3.1 due to Spencer (1990) takes over.

Corollary 1.3.3 *If* $\text{Prob}\,[A_i] \leq \frac{1}{2}$ *for all* $i = 1, \ldots, k$ *and* $\Delta > \frac{1}{2} \sum_{i=1}^{k} \text{Prob}\,[A_i]$, *then*

$$\prod_{i=1}^{k} \text{Prob}\,[\bar{A}_i] \leq \text{Prob}\,[\bigcap_{i=1}^{k} \bar{A}_i] \leq e^{-(\sum_{i=1}^{k} \text{Prob}\,[A_i])^2/(4\Delta)}.$$

Proof. We only have to show the upper bound. Using $\ln(1-x) \leq -x$ for all $0 \leq x \leq 1$ we easily deduce that the upper bound of Theorem 1.3.1 implies that

$$-\ln(\text{Prob}\,[\bigcap_{i=1}^{k} \bar{A}_i]) \geq \sum_{i=1}^{k} \text{Prob}\,[A_i] - \Delta.$$

If we apply this inequality only to a subset $I \subseteq \{1, \ldots, k\}$, we obtain similarly

$$-\ln(\text{Prob}\,[\bigcap_{i \in I} \bar{A}_i]) \geq \sum_{i \in I} \text{Prob}\,[A_i] - \sum_{\substack{i \sim j \\ i,j \in I}} \text{Prob}\,[A_i \cap A_j].$$

Assuming, in addition, that I is a *random* subset with $\text{Prob}\,[i \in I] = \lambda$ for all $i = 1, \ldots, k$ independently of each other (λ is a parameter to be determined later), we deduce

$$\mathbb{E}[-\ln(\text{Prob}\,[\bigcap_{i \in I} \bar{A}_i])] \geq \lambda \cdot \sum_{i=1}^{k} \text{Prob}\,[A_i] - \lambda^2 \cdot \Delta.$$

That is, if we choose $\lambda = \frac{\sum_{i=1}^{k} \text{Prob}\,[A_i]}{2\Delta}$, we get

$$\mathbb{E}[-\ln(\text{Prob}\,[\bigcap_{i \in I} \bar{A}_i])] \geq \frac{1}{4} \frac{(\sum_{i=1}^{k} \text{Prob}\,[A_i])^2}{\Delta},$$

implying the existence of at least one set I_0 that satisfies

$$-\ln(\mathrm{Prob}\,[\bigcap_{i\in I_0} \bar{A}_i]) \geq \frac{1}{4}\frac{(\sum_{i=1}^k \mathrm{Prob}\,[A_i])^2}{\Delta}.$$

As trivially

$$\mathrm{Prob}\,[\bigcap_{i=1}^{k} \bar{A}_i] \leq \mathrm{Prob}\,[\bigcap_{i\in I_o} \bar{A}_i],$$

this concludes the proof of Corollary 1.3.3. □

1.4 Counting H-free graphs

In Section 1.2 we have seen that the property that a random graph $G_{n,p}$ contains a copy of a fixed graph G has a threshold function. Going one step further Janson, Luczak and Ruciński (1990) fully determined the order of magnitude of the logarithm of the probability that $G_{n,p}$ is H-free. In this section we will prove this results and subsequently transfer it to the $G(n,m)$-model. This will then allow us to obtain good bounds on the number of H-free graphs with a prescribed number of edges.

Theorem 1.4.1 [Janson, Luczak, Ruciński, 1990] *For every graph H with at least one edge there exist constants $c_1, c_2 > 0$ such that for all $p = p(n) < 1$ and all n*

$$e^{-\frac{c_1}{1-p}f(n,p)} \leq \mathrm{Prob}\,[G_{n,p} \textit{ is } H\textit{-free}] \leq e^{-c_2 f(n,p)},$$

where

$$f(n,p) = \min\{n^{|V(L)|}p^{|E(L)|} \mid L \subseteq H, |E(L)| > 0\}.$$

Proof. For every $0 < p < 1$ and $n \in \mathbb{N}$ let $H_{n,p} \subseteq H$ denote a subgraph of H satisfying $f(n,p) = n^{|V(H_{n,p})|}p^{|E(H_{n,p})|}$, and let $X(H)$ and $X(H_{n,p})$ be the random variables counting the number of copies of H and $H_{n,p}$ in the random graph $G_{n,p}$, respectively. In order to apply the Janson-inequalities of Section 1.3 we choose subgraphs H_i (and subgraphs H_i') in such a way that they enumerate all copies of H (of $H_{n,p}$). Then $\mathbb{E}[X(H)] = \sum_{i=1}^k \mathrm{Prob}\,[A_i]$, $\mathbb{E}[X(H_{n,p})] = \sum_{i=1}^k \mathrm{Prob}\,[A_i']$, and, using $-\frac{x}{1-x} \leq \ln(1-x) \leq -x$ for all $0 \leq x < 1$, one easily observes that the Janson-inequalities of Theorem 1.3.1 imply

$$\begin{aligned} (1.4.5)\quad e^{-\frac{1}{1-p^{|E(H_{n,p})|}}\cdot\mathbb{E}[X(H_{n,p})]} &\leq \mathrm{Prob}\,[G_{n,p} \text{ is } H_{n,p}\text{-free}]) \\ &\leq \mathrm{Prob}\,[G_{n,p} \text{ is } H\text{-free}]) \leq e^{-\mathbb{E}[X(H)]+\frac{1}{2(1-p)}\Delta}. \end{aligned}$$

As $\mathbb{E}[X(H_{n,p})] = \Theta(n^{|V(H_{n,p})|}p^{|E(H_{n,p})|})$ this proves the lower bound.

For the upper bound recall first (cf. Proposition 1.1.1) that

$$\mathrm{Prob}\,[G_{n,p} \text{ is } H\text{-free}] \leq \mathrm{Prob}\,[G_{n,p'} \text{ is } H\text{-free}] \qquad \text{whenever } p \geq p'.$$

So we may assume without loss of generality that $p \leq \frac{1}{2}$. Whenever $\Delta \leq \frac{1}{2}\mathbb{E}[X(H)]$, the desired bound is therefore a trivial consequence of (1.4.5) and the fact that $\mathbb{E}[X(H)] = \Omega(n^{|V(H_{n,p})|}p^{|E(H_{n,p})|})$. So assume that $\Delta \geq \frac{1}{2}\mathbb{E}[X(H)]$. By the definition of Δ we deduce that (in the notation of the previous section)

$$\begin{aligned}
\Delta &= \sum_{i \sim j} \mathrm{Prob}\,[A_i \wedge A_j] \\
&= \sum_{\substack{L \subseteq H \\ |E(L)|>0}} |\{(H_1, H_2) \mid H_1, H_2 \subseteq K_n,\ H_1, H_2 \cong H, H_1 \cap H_2 = L\}| \cdot p^{2|E(H)|-|E(L)|} \\
&\leq \sum_{\substack{L \subseteq H \\ |E(L)|>0}} n^{2|V(H)|-|V(L)|} \cdot p^{2|E(H)|-|E(L)|} \\
&\leq c \cdot (\mathbb{E}[X(H)])^2 \cdot \sum_{\substack{L \subseteq H \\ |E(L)|>0}} (\mathbb{E}[X(L)])^{-1} \\
&\leq c' \cdot (\mathbb{E}[X(H)])^2 \cdot (\mathbb{E}[X(H_{n,p})])^{-1},
\end{aligned}$$

for appropriate constants $c, c' > 0$. As $\mathbb{E}[X(H)] = \sum_{i=1}^{k} \mathrm{Prob}\,[A_i]$ Corollary 1.3.3 implies that

$$\mathrm{Prob}\,[G_{n,p} \text{ is } H\text{-free}] \leq e^{-(\mathbb{E}[X(H)])^2/(4\,\Delta)} \leq e^{-\frac{1}{4c'}\mathbb{E}[X(H_{n,p})]},$$

concludes the proof of Theorem 1.4.1. □

The proof of Janson, Łuczak, Ruciński is fundamentally based on the independence of appropriately defined events. In the $G(n, m)$-model this fact is not true. Nevertheless Theorem 1.4.1 carries over to the $G(n, m)$-model (cf. Prömel, Steger [70]). Before we state the result, we like to recall that *every* $G_{n,m}$ contains an H-subgraph whenever $m > ex(n, H)$. (Here and in the following $ex(n, H)$ denotes the number of edges of an extremal H-free graph, i.e. the maximum number of edges an H-free graph on n vertices may contain. For more details on extremal graph theory the reader is refered to the next chapter.) That is, for all $m > ex(n, H)$ we trivially have $\mathrm{Prob}\,[G_{n,m} \text{ is } H\text{-free}] = 0$.

Theorem 1.4.2 *For every graph H with chromatic number $\chi(H) \geq 3$ there exist positive constants c_1 and c_2 such that for all $n \in \mathbb{N}$ and all $m \leq ex(n, H)$*

$$e^{-c_1\, g(n,m)} \leq \mathrm{Prob}\,[G_{n,m} \textit{ is } H\textit{-free}] \leq e^{-c_2\, g(n,m)},$$

where

$$g(n, m) = \min\{n^{|V(L)|} \cdot (m/\tbinom{n}{2})^{|E(L)|} \mid L \subseteq H, |E(L)| > 0\}.$$

Theorem 1.4.3 *For every bipartite graph H there exist positive constants c_1, c_2 and d such that for all $n \in \mathbb{N}$ and all $m \leq d \cdot p_0\binom{n}{2}$:*

$$e^{-c_1\, g(n,m)} \leq \mathrm{Prob}\,[G_{n,m} \textit{ is } H\textit{-free}] \leq e^{-c_2\, g(n,m)},$$

where $g(n, m)$ is the function defined in Theorem 1.4.2 and

$$p_0 = sup\{p \mid g(n, p\tbinom{n}{2}) < n^2 p\}.$$

Theorem 1.4.3 is probably not best possible. As, however, for most bipartite graphs the value – and in many cases even the order of magnitude – of $ex(n, H)$ is not known (cf. Section 2.3), a theorem corresponding to Theorem 1.4.2 for bipartite graphs seems therefore currently out of reach.

We first prove an immediate counterpart of Theorem 1.4.1 in which additional terms in the lower and upper bounds account for the facts that every $G_{n,m}$ contains an H-subgraph whenever $m > ex(n, H)$, and that there are only $\binom{\binom{n}{2}}{m}$ different graphs with m edges.

Lemma 1.4.4 *For every graph H with at least one edge there exist positive constants d_1, d_2, d_3, and d_4 such that for all $n, m \in \mathbb{N}$*

$$e^{-\frac{d_1}{1-m/\binom{n}{2}} g(n,m)} - e^{-d_3 m} \le \mathrm{Prob}\,[G_{n,m} \textit{ is } H\textit{-free}] \le e^{-d_2\, g(n,m)},$$

where $g(n, m)$ is the function defined in Theorem 1.4.2.

Proof. We start with some notation. Let $N = \binom{n}{2}$,

$$P_p := \mathrm{Prob}\,[G_{n,p} \text{ is } H\text{-free}]$$

and (for $k \in \mathbb{N}$)

$$Q_k := \mathrm{Prob}\,[G_{n,k} \text{ is } H\text{-free}].$$

Obviously,

$$P_p = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} Q_k. \tag{1.4.6}$$

Observe first that we may assume without loss of generality that $m \ge C$ for any constant C. (We henceforth tacitly assume that C is sufficiently large so that all inequalities below hold.)

As the property of being H-free is monotone decreasing, (1.4.6) implies that for all sufficiently small $\epsilon > 0$ and all $0 < p < 1$

$$\begin{aligned} P_p &\le Q_{(1-\epsilon)pN} \cdot \sum_{k=(1-\epsilon)pN}^{(1+\epsilon)pN} \binom{N}{k} p^k (1-p)^{N-k} + \sum_{k\,:\,|k-pN|>\epsilon pN} \binom{N}{k} p^k (1-p)^{N-k} \\ &\le Q_{(1-\epsilon)pN} + \mathrm{Prob}\,[\,||E(G_{n,p})| - pN| > \epsilon pN] \\ &\le Q_{(1-\epsilon)pN} + e^{-\frac{\epsilon^2}{5} pN}, \end{aligned}$$

where the last inequality follows from the normal approximation of the binomial distribution (cf. Corollary B.2). In particular, this shows that for $m = pN$ we have (replace p by $\frac{p}{1-\epsilon}$)

$$Q_m \ge P_{\frac{p}{1-\epsilon}} - e^{-\frac{\epsilon^2}{5(1-\epsilon)} m}.$$

As the function f from Theorem 1.4.1 satisfies

$$f(n, \tfrac{p}{1-\epsilon}) \leq \left(\tfrac{1}{1-\epsilon}\right)^{|E(H)|} f(n,p) = \left(\tfrac{1}{1-\epsilon}\right)^{|E(H)|} g(n, pN), \tag{1.4.7}$$

this together with Theorem 1.4.1 concludes the proof of the lower bound. The upper bound follows similarly. From (1.4.6) we deduce

$$\begin{aligned} P_p &\geq Q_{(1+\epsilon)pN} \cdot \sum_{k=(1-\epsilon)pN}^{(1+\epsilon)pN} \binom{N}{k} p^k (1-p)^{N-k} \\ &\geq Q_{(1+\epsilon)pN} \cdot (1 - \text{Prob}\left[||E(G_{n,p})| - pN| > \epsilon pN\right]) \\ &\geq Q_{(1+\epsilon)pN} \cdot (1 - e^{-\frac{\epsilon^2}{5}pN}), \end{aligned}$$

and therefore (recall that $\frac{1}{1-x} \leq 1 + 2x$ for all $x \leq \frac{1}{2}$)

$$Q_m \leq P_{\frac{p}{1+\epsilon}} \cdot (1 - e^{-\frac{\epsilon^2}{5(1+\epsilon)}m})^{-1} \leq P_{\frac{p}{1+\epsilon}} + 2e^{-\frac{\epsilon^2}{5(1+\epsilon)}m}.$$

As $f(n, \frac{p}{1+\epsilon}) \geq (1 - \frac{\epsilon}{1+\epsilon})^{|E(H)|} g(n, pN)$ (consider (1.4.7) for "p" $= \frac{p}{1+\epsilon}$ and "ϵ" $= \frac{\epsilon}{1+\epsilon}$) and, with room to spare, $g(n,m) \leq 4m$ for all m and n this concludes the proof of the upper bound. □

Proof of Theorem 1.4.2. The upper bound follows immediately from Lemma 1.4.4. For the proof of the lower bound observe that, trivially, all $(\chi(H)-1)$-partite graphs are H-free. Considering the complete $(\chi(H)-1)$-partite graph with classes as equal as possible and counting all subgraphs with m edges, we deduce that there are at least $\binom{\frac{\chi(H)-2}{\chi(H)-1} \cdot \frac{n^2}{2}}{m}$ many $(\chi(H)-1)$-partite graphs on n vertices with m edges. Hence,

$$\text{Prob}\left[G_{n,m} \text{ is } H\text{-free}\right] \;\geq\; \binom{\frac{\chi(H)-2}{\chi(H)-1} \cdot \frac{n^2}{2}}{m} \cdot \binom{\binom{n}{2}}{m}^{-1} \;\geq\; e^{-cm},$$

for some constant $c > 0$ depending only on the chromatic number of H. □

Proof of Theorem 1.4.3. We again only have to prove the lower bound. Let $N = \binom{n}{2}$ and observe that the definition of p_0 implies that there exists a subgraph $H' \subseteq H$ such that $|E(H')| \geq 2$ and $g(n, p_0 N) = n^{|V(H')|} p_0^{|E(H')|} = n^2 p_0$. In particular, we therefore conclude that for every constant $0 < c \leq 1$ and $m_c = cp_0 N$ we have

$$g(n, m_c) \leq n^{|V(H')|} (cp_0)^{|E(H')|} \leq c^2 g(n, p_0 N) = c^2 p_0 n^2 = (1 + o(1)) \cdot c\, m_c.$$

As $p_0 = o(1)$ this together with Lemma 1.4.4 concludes the proof of Theorem 1.4.3. □

Let $\mathcal{F}orb_{n,m}(H)$ denote the class of all H-free graphs on n (labelled) vertices with m edges. As

$$|\mathcal{F}orb_{n,m}(H)| = \text{Prob}\left[G_{n,m} \text{ is } H\text{-free}\right] \cdot \binom{\binom{n}{2}}{m}, \tag{1.4.8}$$

Theorem 1.4.2 and Theorem 1.4.3 in fact determine the order of magnitude of the class $\mathcal{F}orb_{n,m}(H)$.

Chapter 2

Extremal Graph Theory

An old result of Mantel (1907) states that the complete bipartite graph is the extremal, i.e., edge-maximum, triangle-free graph. Mantel's theorem was a kind of forerunner of extremal graph theory. Usually considered as its starting point is Turán's celebrated generalization (Turán (1941)) of Mantel's result, characterizing the extremal graphs on n vertices which do not contain a clique on $l+1$ vertices as a subgraph. Turán's result stimulated a variety of deep results in graph theory. The reader is referred to Bollobás (1978) and Simonovits (1983), two excellent sources on these problems. Here we can give only a short and incomplete sketch, emphasizing those problems and results which will be crucial for the remaining part of this thesis.

2.1 Turán's Theorem

Given a family $\mathcal{H}$ of (finite) graphs, $ex_n(\mathcal{H})$ denotes the maximum number of edges a graph $G = (V_n, E)$ on n vertices can have without containing a subgraph from $\mathcal{H}$. Usually, we will only consider the case that $\mathcal{H}$ contains only a single element, say H. In theses case we write simply $ex_n(H)$ for the corresponding extremal graph problem.

Motivated by Ramsey's problem, which asks when a graph without stable set of size t is forced to contain a clique on r vertices, Paul Turán posed the following question: "How many edges ensure that a graph contains a K_r-subgraph?" Turán's problem has a nice solution, which he obtained shortly after posing the problem (cf. Turán [82] and [83]). Let $T_n(r)$ denote the complete r-partite graph on n vertices with all parts as equal as possible, and let $t_n(r) = |E(T_n(r))| = (1-\frac{1}{r})\frac{n^2}{2} + \mathcal{O}(n)$.

Theorem 2.1.1 (Turán's Theorem) *For every integer $r \geq 1$*

$$ex_n(K_{r+1}) \;=\; t_n(r)$$

and $T_n(r)$ is (up to isomophism) the unique K_{r+1}-free extremal graph.

Proof. We proceed by double induction on r and n. The cases $r = 1$ and $n \leq r$ are trivial. So assume that $G = (V_n, E)$ is a K_{r+1}-free graph on n vertices with $ex_n(K_{k+1})$

many edges and assume without loss of generality that G contains a subgraph $K = K_r$. Let $G' = G \setminus K$. Then, by the induction hypothesis

$$\begin{aligned} ex_n(K_{r+1}) = |E(G)| &= |E(K)| + |E(G')| + e(K, G') \\ &\leq \binom{r}{2} + ex_{n-r}(K_r) + (n-r)(r-1) \\ &\leq t_{n-r}(r-1) + (r-1)n - \binom{r}{2} \\ &= t_n(r). \end{aligned}$$

So, all that remains to be shown is that equality holds if and only if $G = T_n(r)$. This, however, follows immediately from the observation that $e(K, G') = (n-r)(r-1)$ if and only if $G' = T_{n-r+1}(r)$ and every class of G' is connected to exactly $r-1$ vertices of K. □

2.2 A theorem of Erdős and Stone

For many graphs H the problem of determining $ex_n(H)$ exactly seems too difficult. As a first step one therefore tried to determine at least the order of magnitude of this function. It turned out that this problem has in fact an easy and very attractive solution: results of Erdős, Stone and Simonovits tell us that the order of magnitude of $ex_n(H)$ depends only on the chromatic number of H. We will prove their results in this section.

Lemma 2.2.1 *For every graph H, $ex_n(H)/\binom{n}{2}$ is decreasing as $n \to \infty$.*

Proof. Choose $k \leq n$ arbitrarily and let $G = (V_n, E)$ be a H-free graph on n vertices with $|E| = ex_n(H)$ many edges. Consider all $\binom{n}{k}$ induced subgraphs $G_1, \ldots, G_{\binom{n}{k}}$ of G on k vertices. Then each edge of G belongs to exactly $\binom{n-2}{k-2}$ of these subgraphs. Hence,

$$\binom{n-2}{k-2} \cdot ex_n(H) = \sum_{i=1}^{\binom{n}{k}} |E(G_i)| \leq \binom{n}{k} \cdot ex_k(H).$$

□

Corollary 2.2.2 *For every graph H, $\lim_{n\to\infty} ex_n(H)/n^2$ exists.* □

Lemma 2.2.3 [56] *Let r, s be positive integers. Then*

$$ex_n(K_{r,s}) \leq \frac{1}{2}(s-1)^{\frac{1}{r}} n^{2-\frac{1}{r}} + \mathcal{O}(n).$$

Proof. Let $G = (V_n, E)$ be a graph without $K_{r,s}$-subgraph. Then the number of $K_{1,r}$-subgraphs is bounded from above by $(s-1)\binom{n}{r}$. As on the other hand the number of $K_{1,r}$-subgraphs is given by $\sum_{i=1}^{n}\binom{\deg(v_i)}{r}$ this shows that

$$(s-1)\binom{n}{r} \geq \sum_{i=1}^{n}\binom{\deg(v_i)}{r} \geq n \cdot \binom{2|E|/n}{r},$$

from which the claimed bound follows by straightforward calculations. □

Theorem 2.2.4 [Erdős, Stone, 1946] *For all integers $r \geq 2$ and $t \geq 1$*

$$ex_n(K_r(t)) \leq (1 - \frac{1}{r-1})\frac{n^2}{2} + o(n^2).$$

Proof. We proceed by induction on r. For $r = 2$ the theorem follows from Lemma 2.2.3. So assume now that the theorem is true for $r-1$ and all t and consider an arbitrary but fixed $t \in \mathbb{N}$. Define a constant $0 \leq c \leq 1$ by $1 - \frac{1}{r-1} + c = \lim_{n\to\infty} 2ex_n(K_r(t))/n^2$ (cf. Corollary 2.2.2). If $c = 0$ there is nothing to show, so assume $c > 0$. Let $q = 2t^2$ and choose $n_0 \geq 4q^{rt}$ sufficiently large so that $ex_n(K_{r-1}(q)) \leq (1 - \frac{1}{r-1})\frac{n^2}{2}$ for all $n \geq n_0/2$ (the existence of such an n_0 follows from the induction hypothesis) and

$$\left(1 - \frac{1}{r-1} + c(1 - \frac{1}{t(r-1)}\right) \cdot \frac{n^2}{2} < ex_n(K_r(t)) < \left(1 - \frac{1}{r-1} + c(1 + \frac{1}{t(r-1)}\right) \cdot \frac{n^2}{2}$$

for all $n \geq n_0/2$.

Let $G = (V_n, E)$ be a $K_r(t)$-free graph on $n \geq n_0$ vertices with $|E| \geq (1 - \frac{1}{r-1} + c(1 - \frac{1}{2t})) \cdot \frac{n^2}{2}$ many edges. By the choice of n_0 we know that G contains at least one $K_{r-1}(q)$-subgraph. Fix one such copy, say K, and let

$$X = \{v \in V_n \setminus K \mid |\Gamma(v) \cap K| \geq (r - 2 + \frac{1}{2t})q\}.$$

We claim that $|X| < q^{(r-1)t}$. To see this we count the $K_{r-1}(t)$-subgraphs of K in two ways. First, every vertex in X is connected to at least one such subgraph. Second, every such subgraph can be connected to at most $(t-1)$ vertices in X. Therefore, $|X| \leq (t-1)\binom{q}{t}^{r-1} \leq q^{(r-1)t}$.

We now remove K and all adjacent edges from G to obtain the graph $G^{(1)}$. Obviously, $|V(G^{(1)})| = n - (r-1)q$ and

$$\begin{aligned} |E(G^{(1)})| &\geq |E| - \binom{(r-1)q}{2} - |X|(r-1)q - (n - (r-1)q - |X|)(r - 2 + \frac{1}{2t})q \\ &\geq |E| - (r - 2 + \frac{1}{t})qn. \end{aligned}$$

In particular, $G^{(1)}$ is at least as dense as G, so $G^{(1)}$ also contains a $K_{r-1}(q)$-subgraph. By removing such a copy we obtain the graph $G^{(2)}$, and so on. Repeating this procedure $cn\,/\,(r-1)q$ many times, we obtain a graph $G' \subseteq G$ so that

$$|V(G')| = (1-c)n$$

and

$$\begin{aligned}
|E(G')| &\geq |E| - \left(1 - \frac{1}{r-1} - \frac{1}{t(r-1)}\right) \cdot cn^2 \\
&\geq \left(1 - \frac{1}{r-1} + c(1 - \frac{1}{t(r-1)})\right) \cdot \frac{n^2}{2} - \left(1 - \frac{1}{r-1} - \frac{1}{t(r-1)}\right) \cdot cn^2 \\
&\geq (1 - \frac{1}{r-1}) \cdot \frac{(1-c)^2 n^2}{2} + \left(1 + \frac{1}{t(r-1)} - c\right) \cdot \frac{cn^2}{2} \\
&= (1 - \frac{1}{r-1}) \cdot \frac{[(1-c)n]^2}{2} + \frac{1 + \frac{1}{t(r-1)(1-c)}}{1-c} \cdot \frac{c \cdot [(1-c)n]^2}{2}.
\end{aligned}$$

By choice of n_0 this implies that $\frac{1+\frac{1}{t(r-1)(1-c)}}{1-c} \leq 1 + \frac{1}{t(r-1)}$, which can only hold for $c > 1$. As by definition $c \leq \frac{1}{r-1}$ this is the desired contradiction. □

Corollary 2.2.5 [Erdős, Simonovits, 1966] *For every graph H,*

$$ex_n(H) = (1 - \frac{1}{\chi(H)-1})\frac{n^2}{2} + o(n^2).$$

Proof. As H is not contained in any $(\chi(H)-1)$-partite graph, we have $ex_n(H) \geq t_n(\chi(H)-1)$, which proves the lower bound. On the other hand every H-free graph is in particular $K_{\chi(H)-1}(|V(H)|)$-free. The Erdős-Stone Theorem 2.2.4 therefore implies the upper bound. □

Similarly, we obtain that excluding a family of graphs has almost the same effect as excluding a single graph.

Corollary 2.2.6 *Let $\mathcal{H}$ be a family of graphs with chromatic number at least three. Then there exists a graph $H \in \mathcal{H}$ such that*

$$\frac{ex_n(\mathcal{H})}{ex_n(H)} \to 1 \qquad \textit{for} \qquad n \to \infty.$$

□

Remark. In [42] Erdős and Stone proved in fact a slightly more general result than that stated in Theorem 2.2.4. Namely, they showed that for sufficiently large n every graph on n vertices with $(1 - \frac{1}{r} + \epsilon)\frac{n^2}{2}$ edges contains a $K_r((\ln^{(r-1)} n)^{1/2})$ subgraph, for every $\epsilon > 0$. This was later generalized, first by Bollobás, Erdős and Simonovits [12] and then by Chvátal and Szemerédi [20] (cf. also [21]), who showed that every such graph contains in fact already a $K_r(\log n/(500 \log \epsilon^{-1}))$ subgraph. The latter result is (up to a constant factor) essentially best possible (cf. Bollobás, Erdős [11]).

2.3 Degenerate graphs

An extremal problem is called *degenerate*, if the forbidden graph is bipartite, or, more general, if the set of forbidden graphs contains at least one bipartite graph. By Corollary 2.2.5 of Erdős and Simonovits we know that $ex_n(H) = o(n^2)$ for bipartite graphs H, while all graphs of chromatic number $\chi(H) \geq 3$ satisfy $ex_n(H) \geq \frac{1}{4}n^2$. This already indicates the increasing difficulty of degenerate extremal graph problems. In fact, despite of some results for special cases (some of which we will outline below), very little progress has been achieved so far. This is reflected in the following conjectures (cf. e.g. Erdős, Simonovits [39] and Erdős [28]), which at present seem to be completely out of reach.

Conjecture 2.3.1 (Erdős, Simonovits) *For every bipartite graph H there exists a rational constant $\alpha \in [1, 2)$ such that $ex_n(H)/n^\alpha$ converges to a positive limit.*

Conjecture 2.3.2 (Erdős, Simonovits) *Let $\mathcal{H}$ be a finite family of bipartite graphs. Then there exists an $H \in \mathcal{H}$ such that*

$$\lim_{n\to\infty} \frac{ex_n(H)}{ex_n(\mathcal{H})} = 1.$$

Conjecture 2.3.3 (Erdős, Sós) *For every tree T_r on r vertices*

$$ex_n(T_r) \leq \frac{1}{2}(r-2)n.$$

For paths P_r the latter conjecture was earlier proven by Erdős and Gallai (1959). For trees T_r with at least one vertex adjacent to at least $\frac{1}{2}(r-2)$ leaves, the conjecture was recently shown to be true by Sidorenko (1989). In general one only knows the following sightly weaker bounds.

Lemma 2.3.4 *For all trees T_r on $r \geq 2$ vertices*

$$\frac{1}{2}(r-2)n - \mathcal{O}(1) \leq ex_n(T_r) \leq (r-2)n.$$

Proof. The lower bound is easily achieved by partitioning the vertex set into pairwise disjoint cliques of size $r-1$. For the upper bound consider a graph $G = (V_n, E)$ with at least $(r-2)n+1$ edges. Starting from G successively remove all vertices of degree at most $r-2$. Observe that the condition $|E| \geq (r-2)n+1$ implies that this process stops with a graph of minimal degree $r-1$ – which obviously contains every tree on r vertices. □

Corollary 2.3.5 $ex_n(H) = \mathcal{O}(n)$ *if and only if H is a tree.*

Proof. The sufficiency follows from Lemma 2.3.4. To see the necessity consider a random graph $G_{n,p}$ with edge probability $p = \log n/n$ and let X_r denote the random variable counting the number of edges minus the number of circuits of size at most r contained in $G_{n,p}$. Then

$$\mathbb{E}[X_r] = \binom{n}{2} \cdot p - \sum_{i=3}^{r} \binom{n}{i} \cdot \tfrac{1}{2}(i-1)! \cdot p^i = \frac{1}{2} n \log n - \mathcal{O}((\log n)^r).$$

This shows, for every sufficiently large n, the existence of a graph with, say, $\frac{1}{3} n \log n$ edges and girth at least $r+1$. Obviously, no such graph contains a subgraph on at most r vertices different from a tree. □

For even circuits the following upper bounds were obtained by Erdős (unpublished, cf. [28]; a proof of a more general result can be found in Bondy, Simonovits (1974)).

Theorem 2.3.6 *For all integers $k \geq 2$, $ex_n(C_{2k}) = \mathcal{O}(n^{1+\frac{1}{k}})$.* □

In fact, Erdős and Simonovits [41] conjectured that the bound from Theorem 2.3.6 has already the correct order of magnitude.

Conjecture 2.3.7 (Erdős, Simonovits) $ex_n(C_{2k}) = \frac{1}{2} n^{1+\frac{1}{k}} + o(n^{1+\frac{1}{k}})$.

While for paths and trees (if the Erdős-Sós Conjecture 2.3.3 is true) it is straightforward to construct extremal graphs, this is not the case for even circuits. From Theorem 1.4.3 we deduce that

$$ex_n(C_{2k}) \geq \Omega(n^{1+\frac{1}{2k-1}}),$$

which is the best know lower bound for all k except $k = 2, 3, 5$, for which constructions giving $\Omega(n^{1+\frac{1}{k}})$ are known (see Benson (1966) and Wenger (1991)).

2.4 Hypergraphs

An r-uniform hypergraph is a hypergraph $\mathcal{H} = (V, F)$ such that all $f \in F$ have cardinality exactly r. The class of 2-uniform hypergraphs corresponds therefore exactly to the class of all graphs. With $K_l^{(r)}(t)$ we denote the complete l-partite r-uniform hypergraph where every class has size t, i.e. a hypergraph $\mathcal{H} = (V, F)$ with $V = A_1 \dot\cup \cdots \dot\cup A_l$, $|A_i| = t$, and $F = \{X \subseteq V \mid |X| = r, |X \cap A_i| \leq 1 \text{ for all } i = 1, \ldots, l\}$.

The extremal graph problems of determining $ex_n(H)$ generalize to hypergraphs in the natural and obvious way. The following lemma is proven in exactly the same way as the corresponding result (Corollary 2.2.2) for graphs.

Lemma 2.4.1 *For every r-uniform hypergraph H, $\lim_{n \to \infty} ex_n(H)/n^r$ exists.* □

Let H be an r-uniform hypergraph. Similar as in the graph case we call the extremal hypergraph problem for H *degenerate*, iff $ex_n(H) = o(n^r)$. The following theorem will enable us to characterize those hypergraphs H, which give rise to degenerate extremal graph problems.

Theorem 2.4.2 *Let $r \geq 2$, t, and n be positive integers. Then*

$$ex_n(K_r^{(r)}(t)) \leq n^{r-\frac{1}{t^{r-1}}}.$$

For the proof of Theorem 2.4.2 we need an additional tool.

Proposition 2.4.3 *Let t, n be positive integers, and let $x \geq 1$ be a real number. Then every hypergraph $\mathcal{H} = (V_n, F)$ such that $|F| \geq 2\,t^2 x^t$ and $\sum_{f \in F} |f| \geq |F|\, n/x$ contains pairwise different edges $f_1, \dots, f_t$ such that*

$$|f_1 \cap \dots \cap f_t| \geq \frac{n}{2x^t}.$$

Proof. Assume $\mathcal{H} = (V_n, F)$ is a hypergraph contradicting the claim of the proposition. Then obviously

$$\sum_{\substack{X \subseteq F \\ |X| \leq t}} |\bigcap_{f \in X} f| \;\leq\; \tbinom{|F|}{t} \cdot \frac{n}{2x^t} + |F|^{t-1} t^2 \cdot n \;<\; |F|^t \cdot \frac{n}{2x^t} + |F|^{t-1} t^2 \cdot n.$$

As on the other side

$$\sum_{\substack{X \subseteq F \\ |X| \leq t}} |\bigcap_{f \in X} f| \;=\; \sum_{i=1}^{n} \deg(v_i)^t \;\geq\; n \left(\frac{\sum_{f \in F} |f|}{n} \right)^t, \;\geq\; n \left(\frac{|F|}{x} \right)^t,$$

where $V_n = \{v_1, \dots, v_n\}$ and $\deg(v_i) = |\{f \in F \mid v_i \in f\}|$ denotes the degree of v_i in F, this implies

$$|F|^t \cdot \frac{n}{2x^t} + |F|^{t-1} t^2 \cdot n > n \left(\frac{|F|}{x} \right)^t,$$

contradicting the assumption $|F| \geq 2t^2 x^t$. □

Proof of Theorem 2.4.2. For $t = 1$ there is nothing to be shown. So assume $t \geq 2$. We proceed by induction on r. The base of the induction ($r = 2$) is given be Lemma 2.2.3. It therefore suffices to show the induction step from $r-1$ to r. Let $\mathcal{H} = (V_n, F)$ be an r-uniform hypergraph with $|F| > n^{r-\frac{1}{t^{r-1}}}$. We need to show that H contains a $K_r^{(r)}(t)$. To do this let $V_n = \{v_1, \dots, v_n\}$ and let $X_i = \{Y \subseteq V_n \mid Y \cup \{v_i\} \in F, |Y| = r-1\}$. Then

$$\sum_{i=1}^{n} |X_i| = r \cdot |F| > r \cdot n^{r-\frac{1}{t^{r-1}}} \geq \binom{n}{r-1} \cdot r!\, n^{1-\frac{1}{t^{r-1}}}.$$

An application of Proposition 2.4.3 for "n" $= \binom{n}{r-1}$ and $x = \frac{1}{r!} n^{\frac{1}{t^{r-1}}}$ shows that we may assume without loss of generality that

$$|X_1 \cap \dots \cap X_t| \geq \frac{1}{2} \binom{n}{r-1} (r!)^t n^{-\frac{1}{t^{r-2}}} \geq n^{r-1-\frac{1}{t^{r-2}}}.$$

That is, there exist at least $n^{r-1-\frac{1}{t^{r-2}}}$ many $(r-1)$-sets, which form together with any vertex in $\{v_1, \dots, v_t\}$ an edge in F. By our induction hypothesis these $(r-1)$-edges contain a $K_{r-1}^{(r-1)}(t)$-subgraph, which together with $\{v_1, \dots, v_t\}$ forms the desired $K_r^{(r)}(t)$-subgraph in $\mathcal{H}$. □

Corollary 2.4.4 *Let K be an r-uniform hypergraph. Then the extremal graph problem for K is degenerate if and only if the vertices of K can be colored by r colors in such a way that each edge gets r different colors, i.e. if K is a subgraph of $K_r^{(r)}(t)$ for an appropriate chosen constant $t \in \mathbb{N}$.*

Proof. One direction is obvious from Theorem 2.4.2. The other follows from the fact that, if K does not have required form, then $K_r^{(r)}(n/r)$ does not contain K as a subhypergraph and has $\Omega(n^r)$ many edges. □

Extremal problems for hypergraphs are in general very difficult. This is indicated, for example, by the fact that one of the most natural questions, the Turán problem for hypergraphs, is still unsolved, even in its easiest case. To the memory of Turán, Erdős offers \$1,000 for resolving the following question.

Conjecture 2.4.5 (Erdős)

$$\lim_{n\to\infty} ex_n(K_4^{(3)}(1))/\tbinom{n}{3} \stackrel{?}{=} \tfrac{5}{9}.$$

Chapter 3

Asymptotic Graph Theory

A major reason for the beauty of random graph theory is the simplicity of the underlying probabilistic model: choosing edges independently of each other allows the application of a wide variety of tools and results from probability theory. Their use, however, is dramatically reduced if one moves on to more complicated combinatorial structures. A typical example for such structures are the graph classes central to this thesis, i.e., those defined by a forbidden subgraphs.

In this chapter we review some of the (few) methods and results which are known so far. The two most significant results to name here are those by Erdős, Kleitman and Rothschild (1976) and by Szemerédi (1978). The former was the first which completely determined the struture of a typical H-free graph for some graph H (namely for $H = K_3$). The essence of the latter is the observation that every graph in fact contains large parts which essentially behave like a random graph – allowing random graph methods to enter through the "back door".

3.1 Almost all triangle-free graphs are bipartite

In 1976 Erdős, Kleitman and Rothschild succeeded in determining the structure of a typical triangle-free graph, i.e. of a traingle-free graph drawn uniformly at random from the class of all triangle-free graphs. While their result is nice and clear-cut, to prove it they had to substitute the probabilistic methods known from random graph theory by subtle enumeration techniques.

Theorem 3.1.1 [Erdős, Kleitman, Rothschild, 1976] *Almost all triangle-free graphs are bipartite, that is*

$$\text{Prob}\left[G \in \mathcal{C}ol_n(2) \mid G \in \mathcal{F}orb_n(K_3)\right] = 1 - o(1).$$

The proof of Theorem 3.1.1 will consist of three parts:

I. Define sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ such that

$$\mathcal{F}orb_n(K_3) \subseteq \mathcal{C}ol_n(2) \cup \mathcal{A}(n) \cup \mathcal{B}(n) \cup \mathcal{C}(n).$$

II. Bound the cardinalities of the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ in terms of those of $\mathcal{F}orb_{n-x}(K_3)$ for appropriate $x \in \mathbb{N}$.

III. Show inductively that each of the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ is negligible compared to $\mathcal{C}ol_n(2)$.

Figure 3.1 outlines the desired partition of $\mathcal{F}orb_n(K_3)$ and indicates how the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ are defined. Note however, that the drawing is not according to scale.

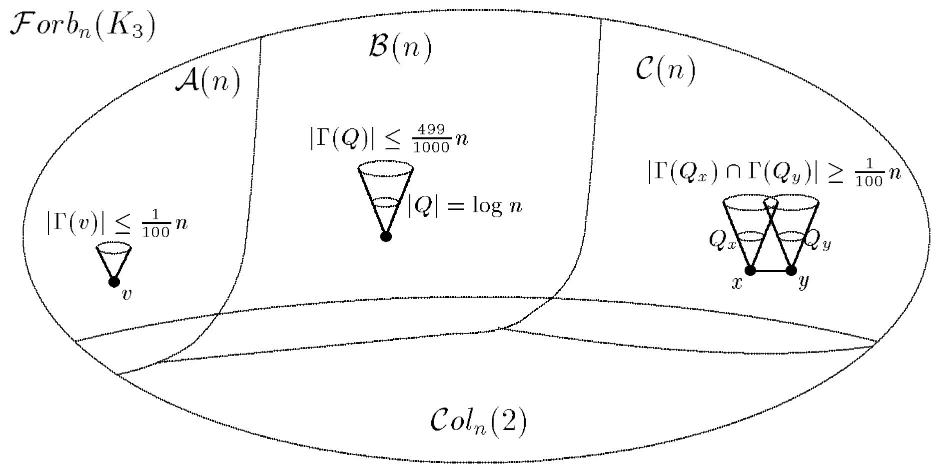

Figure 3.1: Partition of the set $\mathcal{F}orb_n(K_3)$

Part I. The three sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ will reflect properties which hold almost surely in a (random) bipartite graph on n vertices. As we shall see in the proof of III, the color classes in a random bipartite graph are of approximately the same size. For the time being it therefore suffices to view a random bipartite graph as a bipartite graph with color classes as equal as possible, in which the edges are inserted randomly with probability 1/2. It should be intuitively clear that with very high probability in such a graph every vertex as degree at least, say, $\log n$, and that every set of size $\log n$ has at least, say, $(\frac{1}{2} - \frac{1}{1000})n$ neighbors. The definition of $\mathcal{A}(n)$ and $\mathcal{B}(n)$ captures these two properties, while the definition of $\mathcal{C}(n)$ reflects the property that in bipartite graphs all edges $\{x, y\}$ satisfy $\Gamma(\Gamma(x)) \cap \Gamma(\Gamma(y)) = \emptyset$. Let

$\mathcal{A}(n)$ denote the set of all graphs in $\mathcal{F}orb_n(K_3)$ which contain a vertex v such that $|\Gamma(v)| \leq \log n$,

$\mathcal{B}(n)$ denote the set of all graphs in $\mathcal{F}orb_n(K_3) \setminus \mathcal{A}(n)$ which contain a vertex v and a set $Q \subseteq \Gamma(v)$ of size $\log n$ such that $|\Gamma(Q)| \leq (\frac{1}{2} - \frac{1}{1000})n$, and

$\mathcal{C}(n)$ denote the set of all graphs in $\mathcal{F}orb_n(K_3) \setminus [\mathcal{A}(n) \cup \mathcal{B}(n)]$ which contain an edge $\{x, y\}$ and sets $Q_x \subseteq \Gamma(x)$ and $Q_y \subseteq \Gamma(y)$ of size $|Q_x| = |Q_y| = \log n$ such that $|\Gamma(Q_x) \cap \Gamma(Q_y)| \geq \frac{1}{100} n$.

It is not difficult to show that these sets have the required properties.

Lemma 3.1.2

$$\mathcal{F}orb_n(K_3) \subseteq \mathcal{C}ol_n(2) \cup \mathcal{A}(n) \cup \mathcal{B}(n) \cup \mathcal{C}(n).$$

Proof. Let $G = (V_n, E)$ be an arbitrary but fixed graph in $\mathcal{F}orb_n(K_3) \setminus [\mathcal{A}(n) \cup \mathcal{B}(n) \cup \mathcal{C}(n)]$. We need do show that $G \in \mathcal{C}ol_n(2)$. To see this fix for every vertex $v \in V_n$ a set $Q_v \subseteq \Gamma(v)$ of size $\log n$ and let $R_v = \Gamma(Q_v)$. Observe that by the definition of the set $\mathcal{A}(n)$ such sets Q_v exist and that by the definition of $\mathcal{B}(n)$ the sets R_v satisfy $|R_v| \geq (\frac{1}{2} - \frac{1}{1000})n$. Furthermore, by the definition of the set $\mathcal{C}(n)$, we have $|R_x \cap R_y| \leq \frac{1}{100}$ for all $\{x, y\} \in E$. One easily checks that this implies that G can neither contain a C_5 nor a C_7 or C_9.

Choose now an arbitrary edge $\{x, y\} \in E$. As G contains no C_5 we conclude that R_x and R_y are stable sets and that $R_x \cap R_y = \emptyset$. Let $S = V_n \setminus (Q_x \cup Q_y \cup R_x \cup R_y)$ and

$$S_x = \{v \in S \mid R_v \cap R_x \neq \emptyset\} \qquad \text{and} \qquad S_y = \{v \in S \mid R_v \cap R_y \neq \emptyset\}.$$

Observe that certainly $S_x \cup S_y = S$ and that, as G contains no C_7 or C_9, we also know that $R_x \cup S_x$ and $R_y \cup S_y$ are stable and $S_x \cap S_y = \emptyset$. At this point a proper two coloring of G is easily defined (cf. Figure 3.2). $\square$

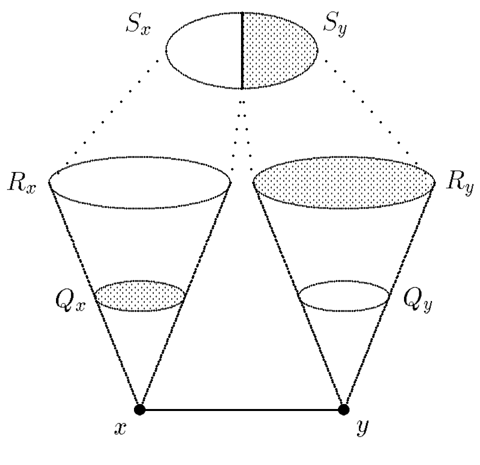

Figure 3.2: 2-coloring of a graph G in $G \in \mathcal{F}orb_n(K_3) \setminus [\mathcal{A}(n) \cup \mathcal{B}(n) \cup \mathcal{C}(n)]$

Part II. We now bound the cardinalities of the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ in terms of those of $\mathcal{F}orb_{n-x}(K_3)$ for an appropriate $x \in \mathbb{N}$. This is done by first closing an appropriate subset of the vertex set, in case of $\mathcal{A}(n)$, for example, the vertex v, a K_3-free graph on the remaining vertices and then connecting the vertices of the special set with the remaining vertices.

Lemma 3.1.3 *For all sufficiently large n*

$$\log \frac{|\mathcal{A}(n)|}{|\mathcal{F}orb_{n-1}(K_3)|} \leq 2(\log n)^2.$$

Proof. Construct all graphs in $\mathcal{A}(n)$ as follows. First choose the vertex v and a triangle-free graph on $V_n \setminus \{v\}$ (in at most $n \cdot |\mathcal{F}orb_{n-1}(K_3)|$ ways). Then choose the set $\Gamma(v)$. As there are exactly

$$\sum_{i=0}^{\log n} \binom{n-1}{i} \leq n^{\log n}$$

ways to do this, the desired bound follows immediately. □

Lemma 3.1.4 *For all sufficiently large n*

$$\log \frac{|\mathcal{B}(n)|}{|\mathcal{F}orb_{n-\log n}(K_3)|} \leq (\frac{1}{2} - \frac{1}{2000})n \log n.$$

Proof. Construct all graphs in $\mathcal{B}(n)$ as follows. First choose the set Q and a triangle-free graph on $V_n \setminus Q$ (in at most $\binom{n}{\log n} \cdot |\mathcal{F}orb_{n-\log n}(K_3)|$ ways). Then choose the set $R = \Gamma(Q)$ (less than 2^n ways) and connect Q to the set R (at most $2^{|Q|\cdot|R|} \leq 2^{\log n \cdot (\frac{1}{2} - \frac{1}{1000})n}$ ways). Together this gives

$$\begin{aligned} \log \frac{|\mathcal{B}(n)|}{|\mathcal{F}orb_{n-\log n}(K_3)|} &\leq (\log n)^2 + n + (\frac{1}{2} - \frac{1}{1000})n \log n \\ &\leq (\frac{1}{2} - \frac{1}{2000})n \log n, \end{aligned}$$

for n sufficiently large. □

Remark. Observe that in the proof of Lemma 3.1.4 we did not use the fact that the set Q is a subset of the neighbourhood of some vertex v. That is, we could also have let $\mathcal{B}(n)$ denote the (larger) set of all graphs in $\mathcal{F}orb_n(K_3)$ which contain a set $Q \subseteq \Gamma(v)$ of size $\log n$ such that $|\Gamma(Q)| \leq (\frac{1}{2} - \frac{1}{1000})n$.

Lemma 3.1.5 *For all sufficiently large n*

$$\log \frac{|\mathcal{C}(n)|}{|\mathcal{F}orb_{n-2}(K_3)|} \leq (1 - \frac{1}{2000})n.$$

Proof. Construct all graphs in $\mathcal{C}(n)$ as follows. First choose the two vertices x and y, a triangle-free graph on $V_n \setminus \{x, y\}$, and appropriate sets Q_x, Q_y (in less than $n^2 \cdot |\mathcal{F}orb_{n-2}(K_3)| \cdot n^{2\log n}$ ways). Let for conciseness of notation $R_x = \Gamma(Q_x)$ and $R_y = \Gamma(Q_y)$ and observe that R_x and R_y are determined by the choice of Q_x and Q_y. Finally, connect x and y to $V_n \setminus \{x, y\}$. As no vertex in R_x (R_y) may be connected to

y (x) and no vertex in $V_n \setminus (R_x \cup R_y)$ may be connect to both x and y there are at most

$$2^{|R_x \setminus R_y|+|R_y \setminus R_x|} \cdot 3^{n-|R_x \cup R_y|} \le 2^{\frac{7}{4}n-\frac{3}{4}(|R_x|+|R_y|)-\frac{1}{4}|R_x \cap R_y|} \le 2^{n-\frac{1}{1000}n}.$$

ways to do this (recall that by definition of $\mathcal{B}(n)$ we have $|R_x|, |R_y| \ge (\frac{1}{2} - \frac{1}{1000})n$ and that by assumption $|R_x \cap R_y| \ge \frac{1}{100}n$). Together this gives

$$\begin{aligned} \log \frac{|\mathcal{C}(n)|}{|\mathcal{F}orb_{n-2}(K_3)|} &\le 2\log n + 2(\log n)^2 + n - \frac{1}{1000}n \\ &\le (1 - \frac{1}{2000})n, \end{aligned}$$

for n sufficiently large. □

Part III. To show inductively that the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ are negligible compared to $\mathcal{C}ol_n(2)$ we first establish bounds on the size of the set $\mathcal{C}ol_n(2)$.

Lemma 3.1.6 *For all sufficiently large n*

$$2^{\frac{n^2}{4}+n-\log n} \le |\mathcal{C}ol_n(2)| \le 2^{\frac{n^2}{4}+n}.$$

Proof. The upper bound follows immediately from the fact that there are at most 2^n ways to partition the vertex set in two classes and that for each such partition there are at most $2^{n^2/4}$ ways to choose edges between these classes.

For the lower bound we count only those partitions which split the vertex set in two classes of size $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$, respectively. Observe that this counts all connected graphs at most once. To make up for those graphs which are counted more than once, we subtract all bipartite graphs with at least two components. Together with the upper bound on $|\mathcal{C}ol_n(2)|$ this gives

$$\begin{aligned} |\mathcal{C}ol_n(2)| &\ge \binom{n}{\lfloor \frac{n}{2} \rfloor} \cdot 2^{\lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor} - \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{i} \cdot |\mathcal{C}ol_i(2)| \cdot |\mathcal{C}ol_{n-i}(2)| \\ &\ge \frac{2^n}{\sqrt{n}} \cdot 2^{\frac{n^2}{4}-\frac{1}{4}} - \sum_{i=1}^{n/2} n^i \cdot 2^{\frac{i^2}{4}+i} \cdot 2^{\frac{(n-i)^2}{4}+n-i} \\ &\ge \frac{2^n}{\sqrt{n}} \cdot 2^{\frac{n^2}{4}-\frac{1}{4}} - 2^{\frac{n^2}{4}+n} \underbrace{\sum_{i=1}^{n/2} 2^{i\log n - \frac{in}{2}}}_{=\, o(2^{-\log n})}, \end{aligned}$$

which concludes the proof of Lemma 3.1.6. □

Remark. With some more care in the calculations it is in fact not too difficult to show that $|\mathcal{C}ol_n(2)| = (1 + o(1)) \cdot 2^{\frac{1}{4}n^2+n-\frac{1}{2}\log n}$. We will prove this asymptotic – in a more general setting – in Chapter 5.

Corollary 3.1.7 *For all sufficiently large n*

$$\log \frac{|\mathcal{C}ol_{n-1}(2)|}{|\mathcal{C}ol_n(2)|} \le -\frac{1}{2}n + \log n \qquad \text{for all } n \in \mathbb{N}. \qquad \square$$

Proof of Theorem 3.1.1. Since $\mathcal{C}ol_n(2) \subseteq \mathcal{F}orb_n(K_3)$ it suffices to show that there exist constants $c \ge 0$ and $\gamma > 1$ such that

$$(*) \qquad |\mathcal{F}orb_n(K_3)| \le (1 + c\gamma^{-n})|\mathcal{C}ol_n(2)|$$

holds for all $n \in \mathbb{N}$. Let $\gamma = 2^{-\frac{1}{3000}}$ and choose n_0 large enough so that all lemmas above hold for all $n \ge n_0$ and such that for all $n \ge n_0$:

$$\max\{2^{-\frac{1}{2}n+3(\log n)^2},\ 2^{-\frac{1}{2000}n\log n+2(\log n)^2},\ 2^{-\frac{1}{2000}n+3\log n}\} \le \gamma^{-n}.$$

Subsequently, choose $c \ge 1$ such that $(*)$ holds for all $n \le n_0$. We conclude the proof by induction on n. So assume $(*)$ holds for some $n-1 \ge n_0$. By Lemma 3.1.2 we conclude that

$$|\mathcal{F}orb_n(K_3)| \le |\mathcal{A}(n)| + |\mathcal{B}(n)| + |\mathcal{C}(n)| + |\mathcal{C}ol_n(2)|.$$

Hence, it suffices to show that the ratio of each of the set $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ with $\mathcal{C}ol_n(2)$ is at most $\frac{c}{3}\gamma^{-n}$. By Lemma 3.1.3, induction hypothesis, and Corollary 3.1.7 we conclude

$$\begin{aligned}
\frac{|\mathcal{A}(n)|}{|\mathcal{C}ol_n(2)|} &\le \frac{|\mathcal{A}(n)|}{|\mathcal{F}orb_{n-1}(K_3)|} \cdot \frac{|\mathcal{F}orb_{n-1}(K_3)|}{|\mathcal{C}ol_{n-1}(2)|} \cdot \frac{|\mathcal{C}ol_{n-1}(2)|}{|\mathcal{C}ol_n(2)|} \\
&\le 2^{2(\log n)^2} \cdot \underbrace{(1 + c\gamma^{-n+1})}_{\le 2c} \cdot 2^{-\frac{1}{2}n+\log n} \le \frac{c}{3} \cdot 2^{-\frac{1}{2}n+3(\log n)^2} \le \frac{c}{3}\gamma^{-n},
\end{aligned}$$

by choice of γ. Similarly, we obtain

$$\begin{aligned}
\frac{|\mathcal{B}(n)|}{|\mathcal{C}ol_n(2)|} &\le \frac{|\mathcal{B}(n)|}{|\mathcal{F}orb_{n-\log n}(K_3)|} \cdot \frac{|\mathcal{F}orb_{n-\log n}(K_3)|}{|\mathcal{C}ol_{n-\log n}(2)|} \cdot \prod_{i=1}^{\log n} \frac{|\mathcal{C}ol_{n-i}(2)|}{|\mathcal{C}ol_{n-i+1}(2)|} \\
&\le 2^{(\frac{1}{2}-\frac{1}{2000})n\log n} \cdot (1 + c\gamma^{-n+\log n}) \cdot 2^{\sum_{i=1}^{\log n}(-\frac{1}{2}(n-i)+\log(n-i))} \\
&\le \frac{c}{3} \cdot 2^{-\frac{1}{2000}n\log n+2(\log n)^2} \le \frac{c}{3}\gamma^{-n}
\end{aligned}$$

and

$$\begin{aligned}
\frac{|\mathcal{C}(n)|}{|\mathcal{C}ol_n(2)|} &\le \frac{|\mathcal{C}(n)|}{|\mathcal{F}orb_{n-2}(K_3)|} \cdot \frac{|\mathcal{F}orb_{n-2}(K_3)|}{|\mathcal{C}ol_{n-2}(2)|} \cdot \frac{|\mathcal{C}ol_{n-2}(2)|}{|\mathcal{C}ol_{n-1}(2)|} \cdot \frac{|\mathcal{C}ol_{n-1}(2)|}{|\mathcal{C}ol_n(2)|} \\
&\le 2^{(1-\frac{1}{2000})n} \cdot (1 + c\gamma^{-n+2}) \cdot 2^{-n+2\log n+\frac{1}{2}} \\
&\le \frac{c}{3} \cdot 2^{-\frac{1}{2000}n+3\log n} \le \frac{c}{3}\gamma^{-n}
\end{aligned}$$

$\square$

3.2 Excluding cliques – a first step

Comparing Theorem 3.1.1 with Theorem 2.1.1 one is tempted to speculate that in fact the following generalization might me true: For every integer $l \geq 2$ almost all K_{l+1}-free graphs are l-colorable. As it turned out, this is indeed this case.

Theorem 3.2.1 [Kolaitis, Prömel, Rothschild, 1987] *For every integer $l \geq 2$*

$$\mathrm{Prob}\left[G \in \mathcal{C}ol_l(n) \mid G \in \mathcal{F}orb_n(K_{l+1})\right] = 1 - o(1).$$

The proof of Theorem 3.2.1 is quite lengthy and we postpone it to Section 4.2, where will obtain it as a corollary of a more general result. Here we instead only prove the following, slightly weaker result of Erdős, Kleitman, Rothschild (1976).

Theorem 3.2.2 *For every integer $l \geq 2$*

$$\mathcal{F}orb_n(K_{l+1}) = 2^{(1-\frac{1}{l})\frac{n^2}{2}+o(n^2)}.$$

Theorem 3.2.2 is an immediate consequence of the following lemma, which bounds the number of graphs containg not "too many" $(l+1)$-cliques.

Lemma 3.2.3 *For all integers $l \geq 1$ let $\mathcal{F}_l(n,\epsilon)$ denote the class of graphs on n vertices containing at most ϵn^{l+1} many $(l+1)$-cliques. Then there exist functions $\xi_l(\epsilon)$ and $\zeta_l(\epsilon)$ with $\lim_{\epsilon\to 0}\xi_l(\epsilon) = 0$ such that for all $n \in \mathbb{N}$:*

$$F_l(n,\epsilon) \leq 2^{(1-\frac{1}{l}+\xi_l(\epsilon))\frac{n^2}{2}+\zeta_l(\epsilon)}. \tag{3.2.1}$$

Proof of Theorem 3.2.2. Trivially, $|\mathcal{F}orb_n(K_{l+1})| \geq 2^{ex_n(K_{l+1})} = 2^{(1-\frac{1}{l})\frac{n^2}{2}-\mathcal{O}(n)}$. This shows the lower bound. The upper bound follows from Lemma 3.2.3 by letting $\epsilon \to 0$. □

Proof of Lemma 3.2.3. We proceed by double induction on l and n. The case $l = 1$ follows easily from the fact that for all sufficiently large n:

$$|\mathcal{F}_1(n,\epsilon)| = \sum_{i=0}^{\epsilon n^2}\binom{\binom{n}{2}}{i} \leq 2^{-(\epsilon\log\epsilon-(1-\epsilon)\log(1-\epsilon))\frac{n^2}{2}}.$$

For the induction step choose $n_0 = n_0(l,\epsilon)$ sufficiently large so that

$$\zeta_{l-1}(c) + ln < \tfrac{1}{4l(l-1)}n^2 - 1 \qquad \text{and} \qquad lt\log n + \tfrac{1}{2}l^2t^2 < n\log l$$

for all $n \geq n_0$. Subsequently, choose $\zeta_l(\epsilon) > 0$ in such a way that (3.2.1) holds for all $n \leq n_0$. Assume $n \geq n_0$ and let $0 < \epsilon < \frac{1}{2}$ be arbitrary, but fixed. Let

$\mathcal{A}_l(n,\epsilon)$ denote the subset of all those graphs in $\mathcal{F}_l(n,\epsilon)$ containing at most cn^l many l-cliques, where $0 < c < \frac{1}{2}$ is a constant such that $\xi_{l-1}(c) \leq \frac{1}{2l(l-1)}$, and let

$\mathcal{B}_l(n,\epsilon)$ denote the subset of all those graphs in $\mathcal{F}_l(n,\epsilon)$ containing a $K_l(t)$-subgraph K so that less than αn of the remaining vertices have a neighbor in each of the l classes of K, where $t = 2\epsilon^{-1/(l+1)}$ and $\alpha = 4(l+1)t^l\epsilon/c$.

We claim that for all sufficiently large n:

$$\mathcal{F}_l(n,\epsilon) = \mathcal{A}_l(n,\epsilon) \cup \mathcal{B}_l(n,\epsilon). \tag{3.2.2}$$

To see this, assume there exists a graph $G \in \mathcal{F}_l(n,\epsilon) \setminus [\mathcal{A}_l(n,\epsilon) \cup \mathcal{B}_l(n,\epsilon)]$. Then G contains at least cn^l many l-cliques. Consider these as the edges of a l-uniform hypergraph $\mathcal{H}$. By Theorem 2.4.2 this hypergraph contains a $K_l^{(l)}(t)$-subgraph K, whenever n is sufficiently large. Viewed as a subgraph in G this graph forms a $K_l(t)$-subgraph K. As $G \notin \mathcal{B}_l(n,\epsilon)$ there exist at least αn many vertices outside of K that are connected (in G) to at least one vertex in each class of K. By averaging this implies that G contains a l-clique which is contained in at least $\alpha n/t^l$ many $(l+1)$-cliques. Remove this clique from $\mathcal{H}$ and repeat the above argument. $\frac{1}{2}cn^l$ iterations of this procedure guarantee the existence of

$$\tfrac{1}{l+1} \cdot \tfrac{1}{2}cn^l \cdot \alpha n/t^l \;\geq\; 2\epsilon n^{l+1}$$

many $(l+1)$-cliques in G (note that every $(l+1)$-clique may be counted at most $l+1$ times and recall the choice of α) — contradicting the definition of $\mathcal{F}_l(n,\epsilon)$.

Observe now, that (3.2.2), the fact that $|\mathcal{F}_l(n,\epsilon)| \geq 2^{ex_n(K_{l+1})} \geq 2^{(1-\frac{1}{l})\frac{n^2}{2}-ln}$, the induction hypothesis on $l-1$, and the choice of c imply

$$\begin{aligned}
|\mathcal{F}_l(n,\epsilon)| &\leq |\mathcal{A}_l(n,\epsilon)| + |\mathcal{B}_l(n,\epsilon)| \\
&\leq |\mathcal{F}_{l-1}(n,c)| + |\mathcal{B}_l(n,\epsilon)| \\
&\leq 2^{(1-\frac{1}{l-1}+\xi_{l-1}(c))\frac{n^2}{2}+\zeta_{l-1}(c)} \cdot |\mathcal{F}_l(n,\epsilon)| \cdot 2^{-(1-\frac{1}{l})\frac{n^2}{2}+ln} + |\mathcal{B}_l(n,\epsilon)| \\
&\leq \tfrac{1}{2}|\mathcal{F}_l(n,\epsilon)| + |\mathcal{B}_l(n,\epsilon)|.
\end{aligned} \tag{3.2.3}$$

On the other hand, by the definition of $\mathcal{B}_l(n,\epsilon)$ and induction hypothesis (on n):

$$\begin{aligned}
|\mathcal{B}_l(n,\epsilon)| &\leq \binom{n}{t}^l \cdot |\mathcal{F}_l(n-lt,\epsilon)| \cdot \sum_{i=0}^{\alpha n} \binom{n}{i} 2^{lti} \cdot l^{n-i} 2^{(l-1)t(n-i)} \\
&\leq n^{lt} \cdot 2^{(1-\frac{1}{l}+\xi_l(\epsilon))\frac{(n-lt)^2}{2}+\zeta_l(\epsilon)} \cdot 2^n \cdot l^n 2^{ltn-t(1-\alpha)n} \\
&\leq 2^{(1-\frac{1}{l}+\xi_l(\epsilon))\frac{n^2}{2}+\zeta_l(\epsilon)} \cdot n^{lt} 2^{\frac{1}{2}l^2t^2} \cdot 2^{-(lt\cdot\xi_l(\epsilon)-\alpha t-\log l-1)n}
\end{aligned}$$

Letting $\xi_l(\epsilon) = (\alpha t + 3\log l)/(lt)$, this together with (3.2.3) concludes the proof of Lemma 3.2.3. (Note that $\xi_l(\epsilon) \to 0$ for $\epsilon \to 0$ is satisfied by the choice of α and t). □

3.3 Szemerédi's Uniformity Lemma

In 1978 E. Szemerédi published a theorem, which since then has found numerous applications in various branches of graph theory and combinatorics. It became known as the "Uniformity Lemma". Very roughly, it states that every graph can be partitioned into a (small) number of classes such that the arising bipartite graphs between these classes essentially behave like random (bipartite) graphs. To state it precisely we, need some definitions.

Let $G = (V, E)$ be a graph and $A, B \subseteq V$ be pairwise disjoint nonempty vertex sets. The density $d(A, B)$ is defined by $d(A, B) = \frac{e(A,B)}{|A|\cdot|B|}$, where $e(A, B)$ denotes the number of edges between A and B. The pair (A, B) is called *ϵ-regular*, if all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \epsilon|A|$ and $|B'| \geq \epsilon|B|$ satisfy $|d(A', B') - d(A, B)| < \epsilon$. A pair (A, B) which is not ϵ-regular is called *ϵ-irregular*. A partition $V = A_0 \cup A_1 \cup \ldots \cup A_k$ is called *ϵ-uniform*, if $|A_0| < \epsilon|V|$, $|A_1| = \ldots = |A_k|$ and at most ϵk^2 of the tuples (A_i, A_j), $1 \leq i < j \leq k$, are ϵ-irregular.

Theorem 3.3.1 (Szemerédi's Uniformity Lemma) *For every $\epsilon > 0$ and $l \in \mathbb{N}$ there exist positive integers $n_0 = n_0(\epsilon, l)$ and $m = m(\epsilon, l)$ such that every graph on $n \geq n_0$ vertices has an ϵ-uniform partition $V = A_0 \cup A_1 \cup \ldots \cup A_k$, where $l \leq k \leq m$.*

For a direct proof of Theorem 3.3.1 we defer the reader to Szemerédi [81]. Here we will prove a generalization of the Uniformity Lemma to hypergraphs, which was independently proven by Steger (1990), Chung (1991), and Frankl and Rödl (1992). To state it we need some straightforward extensions of the notions introduced above.

Let $\mathcal{H} = (V, F)$ be an r-uniform hypergraph and let $A_1, \ldots, A_r \subseteq V$ be pairwise disjoint nonempty vertex sets. The density $d(A_1, \ldots, A_r)$ is defined by

$$d(A_1, \ldots, A_r) = \frac{e(A_1, \ldots, A_r)}{|A_1| \cdot \ldots \cdot |A_r|},$$

where

$$e(A_1, \ldots, A_r) = |\{f \in F \ : \ |f \cap A_i| = 1 \text{ for all } i = 1, \ldots, r\}|.$$

The tuple $(A_1, \ldots, A_r)$ is called *ϵ-regular*, if for all tuples $A_i' \subseteq A_i$, $i = 1, \ldots, r$ with $|A_i'| \geq \epsilon|A_i|$ one has

$$|d(A_1', \ldots, A_r') - d(A_1, \ldots, A_r)| < \epsilon.$$

A tuple $(A_1, \ldots, A_r)$ which is not ϵ-regular is called *ϵ-irregular*. A partition $V = A_0 \cup A_1 \cup \ldots \cup A_k$ is called *ϵ-uniform*, if $|A_0| < \epsilon|V|$, $|A_1| = \ldots = |A_k|$ and at most ϵk^r of the tuples $(A_{i_1}, \ldots, A_{i_r})$, $1 \leq i_1 < \ldots < i_r \leq k$, are ϵ-irregular. Note that in the case $r = 2$ these definitions correspond exactly to the graph-theoretic notions introduced above.

Theorem 3.3.2 *For every $\epsilon > 0$ and $l, r \in \mathbb{N}$, there exist positive integers $n_0 = n_0(\epsilon, l, r)$ and $m = m(\epsilon, l, r)$ such that every r-regular hypergraph $\mathcal{H} = (V_n, F)$ on $n \geq n_0$ vertices has an ϵ-uniform partition $V_n = A_0 \cup A_1 \cup ... \cup A_k$, where $l \leq k \leq m$.*

Define an *index* of a partition $V = A_0 \cup A_1 \cup ... \cup A_k$ into $k + 1$ classes with exceptional class A_0 as follows:

$$\operatorname{ind}((A_i)_{i=1,\ldots,k}) = \frac{1}{k^r} \sum_{1 \leq i_1 < \ldots < i_r \leq k} d(A_{i_1}, ..., A_{i_r})^2.$$

Note that we have $\mathrm{ind}((A_i)_{i=1,\dots,k}) < \frac{1}{r!}$.

The core of the proof of Theorem 3.3.2 is the following lemma, which shows that every partition with at least ϵk^r many ϵ-irregular tuples can be *refined* in such a way that the index increases by a fixed amount.

Lemma 3.3.3 *Let $\epsilon > 0$ and $k \in \mathbb{N}$ be given such that $2^{-k^{r-1}} \leq \frac{\epsilon^2}{10\,4^{r+1}}$. Let $\mathcal{H} = (V_n, F)$ be a r-regular hypergraph on n vertices with $n \geq \frac{400\,k\,4^{k^{r-1}+k+r}}{\epsilon^{r+3}(1-\epsilon)}$ and let $V_n = A_0 \cup A_1 \cup \dots \cup A_k$ be a partition with $|A_0| < \epsilon n$ and $|A_1| = \dots = |A_k|$, for which at least ϵk^r many tuples $(A_{i_1}, \dots, A_{i_r})$, $1 \leq i_1 < \dots < i_r \leq k$, are ϵ-irregular. Then there exists a partition $V = B_0 \cup B_1 \cup \dots \cup B_{k4^{k^{r-1}}}$ such that*

(1) $|B_0| \leq |A_0| + \epsilon^{r+3}\frac{n}{4^k}$,

(2) $|B_1| = \dots = |B_{k4^{k^{r-1}}}|$,

(3) *the partition $(B_0, \dots, B_{k4^{k^{r-1}}})$ is a refinement of the partition $(A_0, \dots, A_k)$, i.e., $A_0 \subseteq B_0$, and for each $1 \leq i \leq k4^{k^{r-1}}$ there exists a $1 \leq j \leq k$, such that $B_i \subseteq A_j$, and*

(4) *$ind((B_i)_{i=1,\dots,k4^{k^{r-1}}}) \geq ind((A_i)_{i=1,\dots,k}) + \frac{\epsilon^{r+3}}{20}$.*

Proof. For every ϵ–irregular tuple $(A_{i_1}, \dots, A_{i_r})$ with $1 \leq i_1 < \dots < i_r \leq k$ choose subsets $X_j \subseteq A_{i_j}, 1 \leq j \leq r$, which demonstrate that $(A_{i_1}, \dots, A_{i_r})$ is not ϵ-regular, i.e. which satisfy

(3.3.4) $|X_j| \geq \epsilon |A_{i_j}|$ for all $j = 1, \dots, r$ and $|d(X_1, \dots, X_r) - d(A_{i_1}, \dots, A_{i_r})| \geq \epsilon$.

These sets X_j decompose every set A_i into at most $2^{\binom{k-1}{r-1}} \leq 2^{k^{r-1}}$ many classes. Therefore we may find in every A_i sets $B_{(i-1)4^{k^{r-1}}+1}, \dots, B_{i4^{k^{r-1}}}$ in such a way that

(3.3.5) $|B_i| = \lfloor \frac{|A_1|}{4^{k^{r-1}}} \rfloor$ for all $i = 1, \dots, k4^{k^{r-1}}$ and

(3.3.6) every set X_j contains at least $\frac{|X_j|}{|B_1|} - 2^{k^{r-1}}$ many sets B_i.

With $B_0 = V \setminus \bigcup_{i=1}^{k4^{k^{r-1}}} B_i$, we have that $B_0, B_1, \dots, B_{k4^{k^{r-1}}}$ form a partition of V, which satisfies conditions (2) and (3). To see that (1) holds observe that

$$|B_0| \leq |A_0| + k \cdot 4^{k^{r-1}} \leq |A_0| + \epsilon^{r+3}\frac{n}{4^k}$$

by the assumption on n.

We adopt the following notation:

$$b := |B_1| = \Big\lfloor \frac{|A_1|}{4^{k^{r-1}}} \Big\rfloor, \quad q := 4^{k^{r-1}}$$

and

$$A_i^j := B_{(i-1)q+j} \text{ for } 1 \leq i \leq k \text{ and } 1 \leq j \leq q.$$

To verify (4) we show in a first step that all tuples $(A_{i_1}, \ldots, A_{i_r})$ satisfy:

$$\frac{1}{q^r} \cdot \sum_{j_1=1}^{q} \cdots \sum_{j_r=1}^{q} d(A_{i_1}^{j_1}, \ldots, A_{i_r}^{j_r})^2 \geq d(A_{i_1}, \ldots, A_{i_r})^2 - \frac{\epsilon^{r+3}}{50}. \tag{3.3.7}$$

Without loss of generality we consider only the tupel $(A_1, \ldots, A_r)$. Let $A_i^* := \bigcup_{j=1}^{q} A_i^j$. We then have

$$|A_i| - |A_i^*| \leq q$$

and therefore by choice of n also

$$|A_i^*| \geq |A_i|(1 - \frac{q}{|A_i|}) \geq |A_i|(1 - \frac{kq}{(1-\epsilon)n}) \geq |A_i|(1 - \frac{\epsilon^{r+3}}{100 \cdot 4^{r+1}}).$$

For $i = 1, \ldots, r$ choose $0 \leq d_i \leq \frac{\epsilon^{r+3}}{100 \cdot 4^{r+1}}$ such that $|A_i^*| = |A_i|(1 - d_i)$. With $C_i^0 = A_i^*$ and $C_i^1 = A_i \setminus A_i^*$ we observe:

$$\begin{aligned}
&|d(A_1, \ldots, A_r) - d(A_1^*, \ldots, A_r^*)| \\
&= \frac{|\prod_{i=1}^{r}(1-d_i) \cdot \sum_{\delta_i \in \{0,1\}} e(C_1^{\delta_1}, \ldots, C_r^{\delta_r}) - e(C_1^0, \ldots, C_r^0)|}{\prod_{i=1}^{r}(1-d_i)|A_i|} \\
&\leq \frac{\sum_{\substack{\delta_i \in \{0,1\} \\ \sum \delta_i > 0}} \left(\prod_{i=1}^{r} d_i^{\delta_i}\right) e(A_1, \ldots, A_r) + \sum_{\substack{\delta_i \in \{0,1\} \\ \sum \delta_i > 0}} e(C_1^{\delta_1}, \ldots, C_r^{\delta_r})}{\prod_{i=1}^{r}(1-d_i)|A_i|} \\
&\leq \frac{\sum_{\substack{\delta_i \in \{0,1\} \\ \sum \delta_i > 0}} \prod_{i=1}^{r} d_i^{\delta_i} + \sum_{\substack{\delta_i \in \{0,1\} \\ \sum \delta_i > 0}} \left(\prod_{i=1}^{r}(1-d_i)^{1-\delta_i} d_i^{\delta_i}\right)}{\prod_{i=1}^{r}(1-d_i)} \\
&\leq \frac{2^{r+1} \cdot \max_{1 \leq i \leq r} d_i}{\left(\frac{1}{2}\right)^r} \\
&\leq \frac{\epsilon^{r+3}}{100}
\end{aligned} \tag{3.3.8}$$

and hence

$$\begin{aligned}
&|d(A_1, \ldots, A_r)^2 - d(A_1^*, \ldots, A_r^*)^2| \\
&\quad = |d(A_1, \ldots, A_r) + d(A_1^*, \ldots, A_r^*)| \cdot |d(A_1, \ldots, A_r) - d(A_1^*, \ldots, A_r^*)|
\end{aligned} \tag{3.3.9}$$

$$\leq \frac{\epsilon^{r+3}}{50}.$$

Using the Cauchy-Schwarz inequality, i.e. that $\sum_{i=1}^n a_i^2 \geq \frac{1}{n}(\sum_{i=1}^n a_i)^2$ for real numbers a_i, this implies

$$\begin{aligned}
\frac{1}{q^r}\sum_{j_1=1}^{q}\cdots\sum_{j_r=1}^{q} d(A_1^{j_1},\ldots,A_r^{j_r})^2 &= \frac{1}{q^r\cdot b^{2r}}\sum_{j_1=1}^{q}\cdots\sum_{j_r=1}^{q} e(A_1^{j_1},\ldots,A_r^{j_r})^2\\
&\geq \frac{1}{q^{2r}b^{2r}}\Big(\sum_{j_1=1}^{q}\cdots\sum_{j_r=1}^{q} e(A_1^{j_1},\ldots,A_r^{j_r})\Big)^2\\
&= d(A_1^*,\ldots,A_r^*)^2\\
&\geq d(A_1,\ldots,A_r)^2-\frac{\epsilon^{r+3}}{50}
\end{aligned}$$

and hence (3.3.7).

In a second step we show that every ϵ-irregular tuple $(A_{i_1},\ldots,A_{i_r})$ satisfies:

$$(3.3.10)\quad \frac{1}{q^r}\sum_{j_1=1}^{q}\cdots\sum_{j_r=1}^{q} d(A_{i_1}^{j_1},\ldots,A_{i_r}^{j_r})^2 \quad\geq\quad d(A_{i_1},\ldots,A_{i_r})^2-\frac{\epsilon^{r+3}}{50}+\frac{\epsilon^{r+2}}{8}.$$

Again we assume without loss of generality that $(A_1,\ldots,A_r)$ is ϵ-irregular and show (3.3.10) for this tuple. Consider the sets $X_1,\ldots,X_r$ chosen for this tuple. By (3.3.6) we have for $X_i^* := \bigcup\{A_i^j | 1\leq j\leq q, A_i^j\subseteq X_i\}$

$$|X_i^*|\geq |X_i| - 2^{k^{r-1}}b \geq |X_i| - 2^{-k^{r-1}}|A_i|.$$

Since $|X_i|\geq \epsilon|A_i|$ (cf. (3.3.4)) this implies using the assumptions on ϵ and k :

$$|X_i^*|\geq |X_i|\Big(1-\frac{1}{\epsilon}2^{-k^{r-1}}\Big)\geq |X_i|\Big(1-\frac{\epsilon}{10\cdot 4^{r+1}}\Big).$$

As in the proof of (3.3.8) we can conclude that

$$|d(X_1,\ldots,X_r)-d(X_1^*,\ldots,X_r^*)|\leq \frac{\epsilon}{10}$$

and hence because of (3.3.4) and (3.3.8) also

$$|d(X_1^*,\ldots,X_r^*)-d(A_1^*,\ldots,A_r^*)| > \frac{\epsilon}{2}.$$

Let $x_i = \frac{|X_i^*|}{b}$ and $\delta = d(X_1^*,\ldots,X_r^*)-d(A_1^*,\ldots,A_r^*)$ and assume without loss of generality that $X_i^* = \bigcup_{j=1}^{x_i} A_i^j$ for all $i=1,\ldots,r$. This implies

$$\sum_{j_1=1}^{x_1}\cdots\sum_{j_r=1}^{x_r} e(A_1^{j_1},\ldots,A_r^{j_r}) = \Big(\prod_{i=1}^{r} x_i\Big)b^r d(X_1^*,\ldots,X_r^*)$$

$$= \Big(\prod_{i=1}^{r} x_i\Big) b^r (d(A_1^*, \dots, A_r^*) + \delta)$$
$$= \frac{\prod_{i=1}^{r} x_i}{q^r} \sum_{j_1=1}^{q} \cdots \sum_{j_r=1}^{q} e(A_1^{j_1}, \dots, A_r^{j_r}) + \delta b^r \cdot \prod_{i=1}^{r} x_i.$$

We now need the defect form of the Cauchy-Schwarz inequality (cf. Appendix A), i.e. if $m \leq n$ and $\sum_{i=1}^{m} a_i = \frac{m}{n}\sum_{i=1}^{n} a_i + \delta$ then $\sum_{i=1}^{m} a_i^2 \geq \frac{1}{n}(\sum_{i=1}^{n} a_i)^2 + \frac{\delta^2 n}{m(n-m)}$. From this we deduce

$$\frac{1}{q^r} \sum_{j_1=1}^{q} \cdots \sum_{j_r=1}^{q} d(A_1^{j_1}, ..., A_r^{j_r})^2$$
$$= \frac{1}{q^r b^{2r}} \sum_{j_1=1}^{q} \cdots \sum_{j_r=1}^{q} e(A_1^{j_1}, ..., A_r^{j_r})^2$$
$$\geq \frac{1}{q^r b^{2r}} \sum_{j_1=1}^{x_1} \cdots \sum_{j_r=1}^{x_r} e(A_1^{j_1}, ..., A_r^{j_r})^2$$
$$\geq \frac{1}{q^r b^{2r}} \Big[\frac{1}{q^r} \Big(\sum_{j_1=1}^{q} \cdots \sum_{j_r=1}^{q} e(A_1^{j_1}, ..., A_r^{j_r}) \Big)^2 + \frac{\delta^2 b^{2r} q^r \prod_{i=1}^{r} x_i}{q^r - \prod_{i=1}^{r} x_i} \Big]$$
$$= d(A_1^*, ..., A_r^*)^2 + \frac{\delta^2 \prod_{i=1}^{r} x_i}{q^r - \prod_{i=1}^{r} x_i}.$$

Using that $x_i = \frac{|X_i^*|}{b} \geq \frac{|X_i|}{b}\Big(1 - \frac{\epsilon}{10 \cdot 4^{r+1}}\Big) \geq \frac{\epsilon}{b}\Big(1 - \frac{1}{4^{r+1}}\Big)|A_1|$, that $|\delta| \geq \frac{\epsilon}{2}$ and that therefore

$$\frac{\delta^2 \prod_{i=1}^{r} x_i}{q^r - \prod_{i=1}^{r} x_i} \geq \frac{\delta^2}{q^r} \prod_{i=1}^{r} x_i \geq \frac{\epsilon^{r+2}}{4} \Big(1 - \frac{1}{4^{r+1}}\Big)^r \frac{|A_1|^r}{(qb)^r} \geq \frac{\epsilon^{r+2}}{8}$$

this together with (3.3.9) implies (3.3.10).

Finally, (3.3.7) and (3.3.10) and the assumption that at least ϵk^r many tuples are ϵ-irregular conclude the proof of the lemma as follows:

$$\mathrm{ind}\Big((B_i)_{i=1,\dots,kq}\Big) = \frac{1}{k^r q^r} \sum_{1 \leq i_1 < \cdots < i_r \leq kq} d(B_{i_1}, \dots, B_{i_r})^2$$
$$\geq \frac{1}{k^r q^r} \sum_{1 \leq i_1 < \cdots < i_r \leq k} \sum_{j_1=1}^{q} \cdots \sum_{j_r=1}^{q} d(A_{i_1}^{j_1}, \dots, A_{i_r}^{j_r})^2$$

$$\begin{aligned}
&\geq \frac{1}{k^r} \sum_{1 \leq i_1 < \cdots < i_r \leq k} d(A_{i_1}, \ldots, A_{i_r})^2 - \frac{1}{k^r} \binom{k}{r} \frac{\epsilon^{r+3}}{50} + \epsilon \frac{\epsilon^{r+2}}{8} \\
&\geq \operatorname{ind}\Big((A_i)_{i=1,\ldots,k}\Big) + \frac{\epsilon^{r+3}}{20}.
\end{aligned}$$

□

Proof of Theorem 3.3.2. Choose $l_0 \geq l$ in such a way that $2^{-l_0^{r-1}} \leq \frac{\epsilon^2}{10 \cdot 4^{r+1}}$. The index of every partition is less that $\frac{1}{r!}$. Lemma 3.3.3 therefore guarantees that starting from an arbitrary partition $V = A_0 \cup A_1 \cup \cdots \cup A_{l_0}$ in $l_0 + 1$ classes we reach a uniform partition after at most $20\epsilon^{-r-3}$ many refinements — if $n = |V|$ is sufficiently large. More precisely, consider the function f defined by $f(0) = l_0$ and $f(t+1) = f(t) \cdot 4^{f(t)^{r-1}}$ and set

$$n_0 = \frac{400 f(\lceil 20\epsilon^{-r-3} \rceil) \cdot 4^{f(\lceil 20\epsilon^{-r-3} \rceil - 1)} \cdot 4^r}{\epsilon^{r+3}(1-\epsilon)}$$

and

$$m = f(\lceil 20\epsilon^{-r-3} \rceil).$$

Starting from a sufficiently large partition $V = A_0 \cup A_1 \cup \cdots \cup A_{l_0}$ with $|A_1| = \cdots = |A_{l_0}| = \lfloor \frac{n}{l_0} \rfloor$ we reach an ϵ-uniform partition $V = B_0 \cup B_1 \cup \cdots \cup B_k$ with $l_0 \leq k \leq m$ and $|B_0| \leq |A_0| + \sum_{i=l_0}^{\infty} \epsilon^{r+3} \frac{n}{4^i} \leq l_0 + \frac{1}{3}\epsilon^{r+3} n \leq \epsilon^{r+3} n$. □

3.4 The Erdős, Frankl, Rödl Theorem

A consequence of the Erdős, Stone Theorem is that the number of edges of an extremal graph is essentially determined by the chromatic number of the forbidden graph (cf. Corollary 2.2.5). The aim of this section is to show that a similar result is true, if one considers the number of H-free graphs instead of the number of edges of an extremal graph.

Theorem 3.4.1 [Erdős, Frankl, Rödl, 1986] *Let H be a finite graph. Then*

$$|\mathcal{F}orb_n(H)| = 2^{(1 - \frac{1}{\chi(H)-1})\frac{n^2}{2} + o(n^2)}.$$

The main idea of the proof of Theorem 3.4.1 is to show that every H-free graph can be made $K_{\chi(H)}$-free by deleting at most $o(n^2)$ edges. The key tools for achieving this are the Uniformity Lemma of the previous section and the following result.

Lemma 3.4.2 *For all $r, t \in \mathbb{N}$ there exists $0 < \epsilon_0 < \frac{1}{2}$ such that for all $\epsilon \leq \epsilon_0$ there exists $n_0 = n_0(\epsilon)$ so that for all $n \geq n_0$ the following is true. If $G = (\bigcup_{i=1}^r A_i, E)$ is an r-partite graph with parts $|A_1| = \cdots = |A_r| = n$ such that for all $1 \leq i < j \leq r$ the pair (A_i, A_j) is ϵ^{rt}-regular and has density at least 2ϵ, then G contains a $K_r(t)$.*

Proof. Let $\epsilon_0 = 1/r$ and $n_0(\epsilon) = \epsilon^{-rt}$ for all $\epsilon \leq \epsilon_0$, and assume that G is an r-partite graph satisfying the assumptions of the lemma with respect to some $\epsilon \leq \epsilon_0$ and $n \geq n_0(\epsilon)$. First observe that the assumption on the regularity and the density of the pairs (A_i, A_j) implies:

($*$) If $1 \leq s < rt$ is an integer and $X \subseteq A_i$ and $Y \subseteq A_j$ are two sets such that $|X| \geq \epsilon^s n$ and $|Y| \geq \epsilon^s n$, then the set $X' := \{v \in X \mid |\Gamma(v) \cap Y| < \epsilon |Y|\}$ satisfies $|X'| \leq \epsilon |X|$.

Let $v_1, \ldots, v_k$ be the longest sequence of at most rt pairwise different vertices such that the following properties are satisfied simultaneoulsy

(i) $v_i \in A_{\lceil i/t \rceil}$ for all $1 \leq i \leq k$,

(ii) $\{v_i, v_j\} \in E(G)$ for all $1 \leq i < j \leq k$ with $\lceil i/t \rceil < \lceil j/t \rceil$, and

(iii) for all $2 \leq j \leq r$:
$$|\bigcap_{\substack{1 \leq i \leq k \\ \lceil i/t \rceil < j}} \Gamma(v_i) \cap A_j| \;\geq\; \epsilon^k |A_j|$$

Clearly, if $k = r \cdot t$, then the vertices v_i induce the desired $K_r(t)$. So assume $k < r \cdot t$ and let $j_0 = \lceil (k+1)/j \rceil$ and
$$A'_j := \bigcap_{\substack{1 \leq i \leq k \\ \lceil i/t \rceil < j}} \Gamma(v_i) \cap A_j$$
for all $1 \leq j \leq r$. Then, by assumption (iii), all pairs (A'_{j_0}, A'_j), $j_0 < j \leq r$, are ϵ^{rt-k}-regular. By ($*$) and choice of ϵ_0 and $n_0(\epsilon)$ there therefore exists a vertex $v_{k+1} \in A'_{j_0}$ such that $|\Gamma(v_{k+1}) \cap A'_j| \geq \epsilon |A'_j|$ for all $j_0 < j \leq r$, contradicting the maximality of k. $\square$

Proof of Theorem 3.4.1. Let $0 < \delta < \frac{1}{2}$ be an arbitrary but fixed constant, let ϵ_0 be the constant from Lemma 3.4.2 with respect to $r = \chi(H)$ and $t = |V(H)|$, and let $0 < \epsilon \leq \min\{\delta, \epsilon_0\}$ be a constant such that $n\binom{n}{2\epsilon n} < 2^{2\delta n}$ for all sufficiently large n. Let $\epsilon' = \epsilon^{\chi(H)\cdot t}$, $l = \lceil \delta^{-1} \rceil$ and $m = m(\epsilon', l)$ denote the constant from Theorem 3.3.1. By Theorem 3.3.1 we know that for n sufficiently large all graphs on n vertices have an ϵ'-uniform partition. We can therefore construct all graphs in $\mathcal{F}orb_n(H)$ as follows. First we choose a partition $V_n = A_0 \cup A_1 \cup \ldots \cup A_k$ of V_n with $|A_0| < \epsilon' n$, $|A_1| = \ldots = |A_k|$ and $l \leq k \leq m$. This can be done in at most n^n different ways. Then we choose the edges which are completely contained in some class $A_i, 1 \leq i \leq k$, and connect the vertices in A_0 to all other vertices. This can be done in at most $2^{k \cdot \binom{|A_1|}{2}} \cdot 2^{|A_0| \cdot n} \leq 2^{\frac{n^2}{2k} + \epsilon' n^2}$ ways. Next we decide for every pair $(A_i, A_j), 1 \leq i < j \leq k$ whether it is ϵ'-irregular, ϵ'-regular with density less than 2ϵ, or ϵ'-regular with density at least 2ϵ. There are $3^{\binom{k}{2}} < 2^{k^2}$ ways to do this.

By Theorem 3.3.1 we know that there are at most $\epsilon' k^2$ many ϵ'-irregular pairs. Therefore all edges which are contained in ϵ'-irregular pairs can be inserted in at most

$2^{|A_1|^2\epsilon' k^2} \le 2^{\epsilon' n^2}$ different ways. For a sparse ϵ'-regular pair there are (by choice of ϵ), at most $\sum_{i=1}^{2\epsilon|A_1|^2} \binom{|A_1|^2}{i} < 2^{2\delta \frac{n^2}{k^2}}$ ways to choose the edges contained in this pair. Therefore there are at most $2^{\delta n^2}$ ways to insert all edges contained in some sparse ϵ'-regular pair. Finally, we conclude from Lemma 3.4.2 that for any $\chi(H)$ sets A_i the remaining ϵ'-regular pairs cannot contain all $\binom{\chi(H)}{2}$ pairs between these sets. In particular this shows that the edges of G contained in dense ϵ'-regular pairs may not contain a $K_{\chi(H)}$. That is, there are at most $Forb_n(K_{\chi(H)})$ ways to insert these edges.
Therefore, together with Theorem 3.2.2, for sufficiently large n

$$Forb_n(H) \le n^n \cdot 2^{\frac{n^2}{2k}+\epsilon' n^2} \cdot 2^{k^2} \cdot 2^{\epsilon' n^2} \cdot 2^{\delta n^2} \cdot Forb_n(K_{\chi(H)}) \le 2^{(1-\frac{1}{\chi(H)-1})\frac{n^2}{2}+4\delta n^2}.$$

Since δ was chosen arbitrarily this implies that

$$Forb_n(H) \le 2^{(1-\frac{1}{\chi(H)-1})\frac{n^2}{2}+o(n^2)}.$$

As on the other hand the definition of an extremal graph implies that $Forb_n(H) \ge 2^{ex(n,H)}$ this together with Corollary 2.2.5 concludes the proof of Theorem 3.4.1. □

Corollary 3.4.3 *Let H be a finite graph with chromatic number $\chi(H) \ge 3$. Then*

$$\log |\mathcal{F}orb_n(H)| = (1 + o(1)) \cdot ex_n(H).$$ □

3.5 Degenerate graphs

While Theorem 3.4.1 nicely determinates the order of magnitude of the class $\mathcal{F}orb_H(n)$ whenever $\chi(H) \ge 3$, it degenerates to $|\mathcal{F}orb_n(H)| = 2^{o(n^2)}$, whenever H is bipartite. One also easily checks, that in fact Corollary 3.4.3 does not hold for bipartite graphs in general.

Lemma 3.5.1 *Let $H = T_r$ be a tree on r vertices. Then there exists a constant $c > 0$ such that for all sufficiently large n:*

$$|\mathcal{F}orb_n(T_r)| > 2^{ex_n(T_r)\cdot c \log n}.$$

Proof. From Lemma 2.3.4 we know that $ex_n(T_r) \le (r-2)n$. On the other hand every partition of V_n into pairwise disjoint cliques of size at most $r-1$ forms a T_r-free graph. Therefore, (with $r' = n - \lfloor \frac{n}{r-1} \rfloor \cdot (r-1)$)

$$|\mathcal{F}orb_n(T_r)| \ \ge \ \binom{n}{r-1, \ldots, r-1, r'} \ = \ 2^{\Omega(n \log n)}.$$

□

Nevertheless it is believed that also for bipartite graphs (different from trees) the subgraphs of the extremal graph(s) already represent "most" H-free graphs. More precisely, Erdős (cf. e.g. [29]) conjectured:

Conjecture 3.5.2 (Erdős) *If H is bipartite and not a tree then*

$$|\mathcal{F}orb_n(H)| = 2^{ex_n(H)\cdot(1+o(1))}.$$

At present this conjecture seems to be rather intractable. It is not even known for special cases. Even for the simplest case, a cycle of length four, the best known upper bound is $2^{cn^{3/2}}$ with $c \approx 1.08$ due to Kleitman and Winston (1982).

Theorem 3.5.3 *The number of C_4-free graphs on n vertices satisfies:*

$$2^{\frac{1}{2}n^{3/2}+o(n^{3/2})} \leq |\mathcal{F}orb_n(C_4)| \leq 2^{1.08n^{3/2}}. \qquad \square$$

PART II

Special Topics

Chapter 4

Graphs with a Color-Critical Vertex

Given a graph H, what is the structure of a typical, i.e., random, H-free graph? For triangle-free graphs this question was answered by Erdős, Kleitman and Rothschild (1976), cf. Theorem 3.1.1. In the following years, a sequence of results shed more and more light on the general picture: among these are Lamken, Rothschild (1984), who solved the problem for odd circuits, Kolaitis, Prömel and Rothschild (1987), who answered the question for all cliques K_l of a fixed size, and Prömel (1990) and Prömel, Steger (1993), who gave a complete answer for all graphs containing a color-critical edge. A consequence common to these results was the following: in all these cases the structure of a typical H-free graph is essentially given by the extremal H-free graph! While the exact statement of the results differ from case to case (cf. Section 4.2), one can summarize these results as follows: in all the considered cases the structure of a typical H-free graph is essentially given by the extremal H-free graph.

In this chapter we completely solve the problem for all graphs H containing a color-critical vertex. (A vertex v of a graph H is called *color-critical* if the deletion of v from H decreases the chromatic number.) While this result subsumes all the results mentioned above, its relatively short proof is self-contained. Before we present it, in Section 4.2, we turn our attention to the related question about extremal graphs.

4.1 Extremal graphs

A strengthening of Turán's theorem is Simonovits's inverse extremal theorem [77] which gives a complete characterization of those graphs H sharing the property that $T_n(l)$ is the extremal graph in $\mathcal{F}orb_n(H)$. To formulate it, we need the notion of a color-critical edge. An edge e of H is called *color-critical* if the deletion of e from H decreases the chromatic number. Obviously, if H contains a critical edge then it also contains a color-critical vertex. Moreover, observe that trivially each clique K_l contains a color-critical edge.

Theorem 4.1.1 *Let $l > 1$ and let H be a graph of chromatic number $l+1$. Then the Turán graph $T_n(l)$ is an extremal graph in $\mathcal{F}orb_n(H)$ if and only if H contains a color-critical edge.*

□

Another extension of Turán's theorem, also due to Simonovits (1968), gives a characterization of the extremal graphs for certain graphs containing a color-critical vertex.

Theorem 4.1.2 *Let $l > 1, 1 \leq r_1 \leq \ldots \leq r_l$ and n be sufficiently large. If G is an extremal graph in $\mathcal{F}orb_n(K_{l+1}(1, r_1, \ldots, r_l))$ then G admits a representation as $\bigtimes_{i=1}^{l} G_i$ where*

(1) $|V(G_i)| = \frac{n}{l} + o(n)$, *for each $i = 1, \ldots, l$, and*

(2) *G_i is an extremal graph in $\mathcal{F}orb_n(K_2(1, r_1))$ for each $i = 1, \ldots, l$.*

□

This result was later generalized by Erdős and Simonovits (1971) to excluded complete graphs $K_{l+1}(r_0, r_1, \ldots, r_l)$ with $\min_{0 \leq i \leq l} r_i \leq 3$.

Here we will prove a different generalization of Theorem 4.1.2. Let H be a vertex-critical graph and let v be a critical vertex in H. We say that v has *criticallity* d, iff there exist d edges $e_1, \ldots, e_d$ incident to v such that $H \setminus (e_1 + \cdots + e_d)$ has chromatic number l, and d is minimal with respect to this property. Let $t_n(p)$ denote the number of edges of a complete p-partite graph on n vertices with parts as equal as possible.

Theorem 4.1.3 *Let d, k, p be integers such that $1 \leq d \leq k$ and $p \geq 2$, and let H be a $(p+1)$-partite graph with parts $H_0, \ldots, H_p$ such that $H_0 = \{v_0\}$, $|H_1| = \cdots = |H_p| = k$, $|\Gamma(v_0) \cap H_1| = d$, $H_1 \cup \cdots \cup H_p$ form a complete p-partite graph, and v_0 is completely connected to $H_2 \cup \cdots \cup H_p$. Then for n sufficiently large*

$$ex_n(H) \leq t_n(p) + \frac{1}{2}n(d-1)$$

and $ex_n(H) = t_n(p) + \frac{1}{2}n(d-1)$ implies that every extremal H-free graph is the union of the Turán-graph $T_n(p)$ and a graph with maximal degree $d-1$.

Proof. Let $t = (3d+4)pk$ and recall that for sufficiently large n, say for $n \geq n_0$, by the Erdős-Stone Theorem 2.2.4 every graph on n vertices and $t_n(p)$ edges contains a $K_p(t)$, i.e. a complete p-partite graph with t vertices in every class, as a subgraph.

Define $\Delta_n(H)$ as $\Delta_n(H) = ex_n(H) - (t_n(p) + \frac{1}{2}n(d-1))$. We need to show that $\Delta_n(H) \leq 0$ whenever n is sufficiently large. To proof this it obviously suffices to show that for all $n \geq n_0$ either $\Delta_n(H) \leq 0$ or $\Delta_n(H) \leq \Delta_{n-pt}(H) - ptk$.

Fix $n \geq n_0$ and let G be an extremal H-free graph, that is let G be an H-free graph with $|E(G)| = ex_n(H)$. If $|E(G)| \leq t_n(p)$ there is nothing to be shown. So assume not. Then G contains a $K_p(t)$ as a subgraph. Let $K_1, \dots, K_p$ denote its parts and set $K = K_1 \cup \dots \cup K_p$. We partition $V(G) \setminus K$ as follows:

$$X = \{v \in V(G) \setminus K \quad | \quad \exists\, i < j \text{ s.t. } |\Gamma(v) \cap K_i| \leq k-1 \text{ and } |\Gamma(v) \cap K_j| \leq k-1\},$$

$$Y = \{v \in V(G) \setminus (X \cup K) \quad | \quad \exists\, i \neq j \text{ s.t. } |\Gamma(v) \cap K_i| \leq t - 3pk \text{ and } |\Gamma(v) \cap K_j| \leq d-1\},$$

and for $i = 1, \dots, p$ let

$$V_i = \{v \in V(G) \setminus (X \cup Y \cup K) \quad | \quad |\Gamma(v) \cap K_i| \leq d-1 \text{ but } |\Gamma(v) \cap K_j| \geq t - 3pk \text{ for all } j \neq i\}.$$

As G is H-free, one easily observes that indeed

$$V(G) = K \cup X \cup Y \cup \bigcup_{i=1}^{p} V_i.$$

Observe that the definition of these sets and the fact that G is H-free implies that

$$|\Gamma(v) \cap (K_i \cup V_i)| < d \qquad \text{for all } v \in K_i \cup V_i \text{ and all } i = 1, \dots, p.$$

So we have the following inequalities:

$$\begin{aligned}
e(X, K) &\leq [(p-2)t + 2(k-1)] \cdot |X| \leq [(p-1)t - 2pk] \cdot |X| \\
e(Y, K) &\leq [(p-1)t - (3pk - d + 1)] \cdot |Y| \leq [(p-1)t - 2pk] \cdot |Y| \\
e(G[K_i]) + e(K_i, V_i) &\leq t(d-1) \qquad \text{for all } i = 1, \dots, p \\
e(G[K_i \cup V_i]) &\leq \frac{1}{2}(t + |V_i|)(d-1) \qquad \text{for all } i = 1, \dots, p \\
\sum_{i=1}^{p} e(K_i, \bigcup_{\substack{1 \leq j \leq p \\ j \neq i}} V_j) &\leq t(p-1)(n - pt - |X| - Y|).
\end{aligned}$$

Hence,

$$\begin{aligned}
ex_n(H) \leq\ & ex_{n-pt}(H) + \binom{p}{2} t^2 + e(X, K) + e(Y, K) + \\
& \sum_{i=1}^{p} \left(e(G[K_i]) + e(K_i, V_i) + e(K_i, \bigcup_{\substack{1 \leq j \leq p \\ j \neq i}} V_j) \right),
\end{aligned}$$

that is

$$\Delta_n(H) \leq \Delta_{n-pt}(H) - t_n(p) - \frac{1}{2} n(d-1) + t_{n-pt}(p) + \frac{1}{2}(n - pt)(d-1)$$

$$+\binom{p}{2}t^2+[(p-1)t-2pk](|X|+|Y|)+pt(d-1)$$
$$+t(p-1)(n-pt-|X|-|Y|).$$

After some straightforward calculations this yields

$$\Delta_n(H) \quad \leq \quad \Delta_{n-pt}(H)-2pk(|X|+|Y|)+\frac{1}{2}pt(d-1).$$

If $|X|+|Y|\geq t$ we are home. We therefore assume that $|X|+|Y|<t$. To simplify the subsequent calculations we would also like to assume that the cardinality of $Z=X\cup Y$ is a multiple of p. We can achieve this by moving at most $p-1$ vertices from $\bigcup V_i$ to Z and replacing the inequality $e(X\cup Y,K)\leq[(p-1)t-2pk]\cdot|X\cup Y|$ by $e(Z,K)\leq[(p-1)t-k]\cdot|Z|$. We therefore assume without loss of generality that $|Z|=pz$, where $z\leq t$. Then

$$\begin{aligned}
ex_n(H) &= \binom{p}{2}t^2+e(Z,K)+e(Z,\bigcup_{i=1}^{p}V_i)+e(G[Z]) \\
&\quad +\sum_{i=1}^{p}\left(e(G[K_i\cup V_i])+e(K_i,\bigcup_{\substack{1\leq j\leq p\\ j\neq i}}V_j)+\frac{1}{2}e(V_i,\bigcup_{\substack{1\leq j\leq p\\ j\neq i}}V_j)\right) \\
&\leq \binom{p}{2}t^2+[(p-1)t-k]\cdot pz+e(Z,\bigcup_{i=1}^{p}V_i)+e(G[Z]) \\
&\quad +\frac{1}{2}(n-pz)(d-1)+(p-1)t(n-p(t+z))+\frac{1}{2}\sum_{i=1}^{p}e(V_i,\bigcup_{\substack{1\leq j\leq p\\ j\neq i}}V_j) \\
&\leq \frac{1}{2}n(d-1)+(p-1)tn-\binom{p}{2}t^2-[k+\frac{1}{2}(d-1)]\cdot pz \\
&\quad +e(G[Z])+e(Z,\bigcup_{i=1}^{p}V_i)+\frac{1}{2}\sum_{i=1}^{p}e(V_i,\bigcup_{\substack{1\leq j\leq p\\ j\neq i}}V_j)
\end{aligned}$$

Using $t_n(p)=t_{n-p(t+z)}(p)+(p-1)(t+z)n-\binom{p}{2}(t+z)^2$ this implies

$$(*)\qquad \begin{aligned} ex_n(H) \leq\ & t_n(p)+\frac{1}{2}n(d-1)-t_{n-p(t+z)}(p)-(p-1)z(n-pt)+\binom{p}{2}z^2 \\ & -[k+\frac{1}{2}(d-1)]\cdot pz+e(G[Z])+e(Z,\bigcup_{i=1}^{p}V_i)+\frac{1}{2}\sum_{i=1}^{p}e(V_i,\bigcup_{\substack{1\leq j\leq p\\ j\neq i}}V_j). \end{aligned}$$

Recall now that if $x_1,\ldots,x_p$ are integers such that $\sum_{i=1}^{p}x_i=n$ and $x_1=\lceil\frac{n}{p}\rceil+\delta$ for some $\delta\geq 0$ that then $\sum_{i<j}x_ix_j\leq t_n(p)-\frac{1}{2}\delta^2$. If we let $\delta=\max_{1\leq i\leq p}|V_i|-\lceil\frac{n-p(t+z)}{p}\rceil$

we therefore obtain

$$(**) \qquad ex_n(H) \le t_n(p) + \frac{1}{2}n(d-1) - \frac{1}{2}\delta^2 - (p-1)z(n-pt) + \binom{p}{2}z^2 - [k + \frac{1}{2}(d-1)] \cdot pz + e(G[Z]) + e(Z, \bigcup_{i=1}^{p} V_i).$$

This implies in particular that

$$ex_n(H) \le t_n(p) + \frac{1}{2}n(d-1) + zn - \frac{1}{2}\delta^2,$$

from which we immediately deduce that we may assume without loss of generality that $\delta \le \sqrt{2zn}$. Next we further partition the sets V_i. Let $c_0 = 8t^2p^4$ and set

$$V_i' = \{v \in V_i \mid |\Gamma(v) \cap V_j| \ge |V_j| - c_0 \text{ for all } j \ne i\}$$

and let $V_i'' = V_i \setminus V_i'$. Then $(**)$ implies

$$ex_n(H) \le t_n(p) + \frac{1}{2}n(d-1) + zn - \frac{1}{2}\sum_{i=1}^{p} c_0|V_i''|$$

and we may therefore assume without loss of generality that $\sum_{i=1}^{p}|V_i''| \le \frac{2z}{c_0}n \le \frac{1}{4tp^4}n$.

Observe that if a vertex $z \in Z$ satisfies $|\Gamma(z) \cap V_i'| \ge d$ for some $1 \le i \le p$, then there has to exist a $j \ne i$ such that $|\Gamma(z) \cap V_j'| \le c_0(d + (p-2)k) + k - 1$, as otherwise G would contain a copy of the excluded subgraph H. Therefore, assuming without loss of generality that $|V_1'| \le \cdots \le |V_p'|$ and using that $|V_1'| = |V_1| - |V_1''| \ge \lfloor\frac{n-p(t+z)}{p}\rfloor - \delta - |V_1''|$, we obtain

$$\begin{aligned} |\Gamma(z) \cap \bigcup_{i=1}^{p} V_i'| &\le \max\{n - p(t+z) - |V_1'| + d - 1, \\ &\qquad n - p(t+z) - |V_1'| - |V_2'| + 2c_0(d + pk)\} \\ &\le \frac{p-1}{p}(n - p(t+z)) + \delta + |V_1''| + d, \end{aligned}$$

for n sufficiently large. Using $(**)$ once more, this implies

$$\begin{aligned} ex_n(H) &\le t_n(p) + \frac{1}{2}n(d-1) - (p-1)z(n-pt) + \binom{p}{2}z^2 - [k + \frac{1}{2}(d-1)]pz \\ &\quad + \binom{pz}{2} + pz \cdot [\frac{p-1}{p}(n - p(t+z)) + \delta + |V_1''| + d] - \frac{1}{2}\delta^2 - \frac{1}{2}c_0\sum_{i=1}^{p}|V_i''| \\ &\le t_n(p) + \frac{1}{2}n(d-1) + \frac{1}{2}pz(z-1) - [k - \frac{1}{2}(d+1)]pz \\ &\quad + pz\delta - \frac{1}{2}\delta^2 - (\frac{1}{2}c_0 - pz)\sum_{i=1}^{p}|V_i''| \end{aligned}$$

$$\leq \quad t_n(p) + \frac{1}{2}n(d-1) + \frac{1}{2}pt^2 + pt\delta - \frac{1}{2}\delta^2 - (\frac{1}{2}c_0 - pt)\sum_{i=1}^{p}|V_i''|,$$

from which we conclude (recall that $c_0 = 8t^2p^4$) that we may assume that $\sum_{i=1}^p |V_i''| = 0$. In a last step we partition the set Z. Let

$$Z_i = \{v \in Z \setminus \bigcup_{i'<i} Z_{i'} \mid |\Gamma(v) \cap V_j| \geq |V_j| - c_0 \text{ for all } j \neq i\}$$

and set $Z_0 = Z \setminus \bigcup_{i=1}^p Z_i$. As before we deduce that for each $1 \leq i \leq p$ the graph induced by $Z_i \cup V_i$ has maximal degree $d-1$. Furthermore, every vertex $v \in Z_0$ either has at most $d-1$ neigbours in some set V_i and at most $|V_j| - c_0$ neighbours in some other set V_j, $j \neq i$, or there exits $i \neq j$ such that v has at most $c_0(d+pt)$ neighbors in V_i as well as in V_j. This implies in particular that the following inequality is valid:

$$e(Z_0, \bigcup_{i=1}^{p}(Z_i \cup V_i)) + \frac{1}{2}\sum_{i=1}^{p} e(Z_i \cup V_i, \bigcup_{\substack{1\leq j\leq p\\ j\neq i}}(Z_j \cup V_j)) \leq t_{n-pt}(p) - (c_0 - d + 1)|Z_0|.$$

Using $(*)$ we obtain (recall that $|\Gamma(v) \cap V_i| \leq d-1$ for all $v \in Z_i$)

$$\begin{aligned}
ex_n(H) \quad &\leq \quad t_n(p) + \frac{1}{2}n(d-1) - t_{n-p(t+z)}(p) - (p-1)z(n-pt) + \binom{p}{2}z^2 \\
&\quad - [k + \frac{1}{2}(d-1)]pz + e(G[Z_0]) + e(Z_0, \bigcup_{i=1}^{p}(Z_i \cup V_i)) \\
&\quad + \sum_{i=1}^{p}\left(e(G[Z_i]) + e(Z_i, V_i) + \frac{1}{2}e(Z_i \cup V_i, \bigcup_{\substack{1\leq j\leq p\\ j\neq i}}(Z_j \cup V_j))\right) \\
&\leq \quad t_n(p) + \frac{1}{2}n(d-1) + t_{n-pt}(p) - t_{n-p(t+z)}(p) - (p-1)z(n-pt) + \binom{p}{2}z^2 \\
&\quad - [k + \frac{1}{2}(d-1)]pz + |Z_0|^2 + (pz - |Z_0|)(d-1) - (c_0 - d + 1)|Z_0| \\
&\leq \quad t_n(p) + \frac{1}{2}n(d-1) - \frac{1}{2}kpz.
\end{aligned}$$

Hence, either $ex_n(H) < t_n(p) + \frac{1}{2}n(d-1)$ or $Z = \emptyset$ and G is a union of the Turán-graph $T_n(p)$ and a graph with maximal degree $d-1$. $\square$

Remark. One easily observes that Theorem 4.1.3 in fact implies that all graphs H which contain a color-critical vertex satisfy

$$t_n(\chi(H) - 1) \leq ex_n(H) \leq t_n(\chi(H) - 1) + \mathcal{O}(n).$$

For the graphs H of Theorem 4.1.3 one can also quite easily show that all graphs obtained as a union of the Turán-graph $T_n(p)$ and a graph with maximal degree $d-1$ not containg a K_3 or C_4 are in fact H-free – implying slightly stronger bounds on $ex_n(H)$ for those graphs.

4.2 Asymptotic structure

In Section 3.1 we have seen that a typical triangle-free graph is bipartite, i.e., is two-colorable. Answering a question of Paul Erdős, a counterpart to Simonovits' inverse extremal theorem (Theorem 4.1.1) was proved in Prömel, Steger (1992c).

Theorem 4.2.1 *Let $l > 1$ and let H be a graph of chromatic number $l+1$. Then*

$$|\mathcal{F}orb_n(H)| \;=\; (1+o(1))\cdot|\mathcal{C}ol_l(n)|$$

if and only if H contains a color-critical edge. □

In particular, choosing $H = K_3$, one gets as a corollary the result of Erdős, Kleitman and Rothschild that almost every triangle-free graph is two-colorable. Another special case is the result of Kolaitis, Prömel and Rothschild (1987) stating that almost every K_{l+1}-free graph is l-colorable.

The aim of this section is to prove a generalization of Theorem 4.2.1 to all graphs H containing a color-critical vertex due to Hundack, Prömel, Steger (1993). Recall that a color-critical vertex v has *criticallity* d, if there exist d edges $e_1,\ldots,e_d$ incident to v such that $H\setminus(e_1+\cdots+e_d)$ has chromatic number l, and d is minimal with respect to this property. Furthermore we denote by $\mathcal{P}_n(l,d)$ the class of graphs on n vertices which can be represented as a product of l graphs each of which has maximal degree at most d, i.e.,

$$\mathcal{P}_n(l,d) := \{G=(V_n,E)\mid G\subseteq \mathop{\times}_{i=1}^{l} G_i,\ \sum_{i=1}^{l}|V(G_i)| = n,\ \Delta(G_i)\le d\ \forall\ i=1,\ldots,l\}.$$

Theorem 4.2.2 *If H has chromatic number $l+1$ and contains a color-critical vertex v_0 with criticallity d, then*

$$|\mathcal{F}orb_n(H)| = (1+o(2^{-cn}))\cdot|\mathcal{F}orb_n(H)\cap\mathcal{P}_n(l,d-1)|,$$

for some constant $c>0$.

The key idea of the proof of Theorem 4.2.2 is to partition the class $\mathcal{F}orb_n(H)$ into a finite number of subclasses and to show that all but one of these subclasses are asymptotically negligible — a technique which we have already used in the proof of Theorem 3.1.1. In fact a similar approach was first used by Kleitman and Rothschild [53] to count asymptotically partially ordered sets. The following Theorem gives this method a formal framework (cf.Steger (1990)).

Theorem 4.2.3 (Kleitman-Rothschild Method) *Let k be a positive integer, let $\mathcal{E}(n)\subseteq\mathcal{F}(n)$ and $\mathcal{Z}_i(n)$, $i=1,\ldots,k$ be $k+2$ families of graphs on n vertices and let $\alpha,\beta>0$ be reals such that for all sufficiently large n:*

$$\log\frac{|\mathcal{E}(n)|}{|\mathcal{E}(n+1)|} \le -\alpha\cdot n+\beta\sqrt{n}, \quad \textit{and} \tag{1}$$

$$\mathcal{F}(n) \subseteq \mathcal{E}(n) \cup \bigcup_{i=1}^{k} \mathcal{Z}_i(n). \tag{2}$$

Moreover, let $z_i = z_i(n)$ be a positive integral functions such that $z_i = o(n)$ and let $\epsilon_i > 0$ be constants such that for all sufficiently large n we have that

$$\log \frac{|\mathcal{Z}_i(n)|}{|\mathcal{F}(n-z_i)|} \leq (\alpha - \epsilon_i) \cdot z_i n \quad \textit{for every } i = 1, \ldots, k. \tag{3}$$

Then there exist $c \geq 1$ and $\gamma > 1$ such that the following inequality holds for every $n \in \mathbb{N}$:

$$|\mathcal{F}(n)| \leq (1 + c \cdot \gamma^{-n}) \cdot |\mathcal{E}(n)|.$$

Proof. Let $\epsilon = \min_{1 \leq i \leq k} \epsilon_i$, $\gamma = 2^{\epsilon/2}$ and choose n_0 sufficiently large such that (1) - (3) of the theorem and $\frac{\epsilon}{2}n - \alpha z_i - \beta\sqrt{n} \geq 1$ is fulfilled for all $n \geq n_0$. Finally, choose $c \geq 1$ sufficiently large such that the claim of Theorem 4.2.3 hold for all $n \leq n_0$. We proceed by induction on n. Using (2) we conclude that it suffices to show that

$$\frac{|\mathcal{Z}_i(n)|}{|\mathcal{E}(n)|} \leq \frac{c}{k} \cdot \gamma^{-n} \qquad \text{for every } i = 1, \ldots, k. \tag{4.2.1}$$

Using (3), the induction hypothesis, and (1) we obtain that

$$\begin{aligned} \frac{|\mathcal{Z}_i(n)|}{|\mathcal{E}(n)|} &\leq \frac{|\mathcal{Z}_i(n)|}{|\mathcal{F}(n-z_i)|} \cdot \frac{|\mathcal{F}(n-z_i)|}{|\mathcal{E}(n-z_i)|} \cdot \prod_{j=1}^{z_i} \frac{|\mathcal{E}(n-j)|}{|\mathcal{E}(n+1-j)|} \\ &\leq 2^{(\alpha-\epsilon)\cdot z_i n} \cdot (1 + c\gamma^{-(n-z_i)}) \cdot 2^{\sum_{j=1}^{z_i}(-\alpha\cdot(n-j)+\beta\sqrt{n-j})}. \end{aligned}$$

(4.2.1) follows now by straightforward calculations, similar to the one in the proof of Theorem 3.1.1. ☐

The proof of Theorem 4.2.2 will be an application of Theorem 4.2.3. In particular, we let $\mathcal{E}(n) = \mathcal{F}orb_n(H) \cap \mathcal{P}_n(l, d-1)$ and $\mathcal{F}(n) = \mathcal{F}orb_n(H)$. First we determine the constants α and β such that (1) of Theorem 4.2.3 is satisfied.

Lemma 4.2.4 *There exists a constant $D > 0$ such that for all sufficiently large n:*

$$\log \frac{|\mathcal{F}orb_n(H) \cap \mathcal{P}_n(l, d-1)|}{|\mathcal{F}orb_{n+1}(H) \cap \mathcal{P}_{n+1}(l, d-1)|} \leq -\frac{l-1}{l} \cdot n + D\sqrt{n}.$$

Proof. Let $\mathcal{E}(n) = \mathcal{F}orb_n(H) \cap \mathcal{P}_n(l, d-1)$, let $\mathcal{C}ol_n(l)$ denote the set of all l-colorable graphs on n vertices, and let $\mathcal{X}_n(d)$ denote the set of all graphs with maximal degree at most d. Then obviously

$$|\mathcal{C}ol_n(l)| \leq |\mathcal{E}(n)| \leq |\mathcal{C}ol_n(l)| \cdot |\mathcal{X}_n(d-1)|.$$

It is well-known (compare also Section 5.3, where a much more general result is proven) that the number of l-colorable graphs satisfies

$$|\mathcal{C}ol_n(l)| = \Theta\left(2^{\frac{l-1}{2l}\cdot n^2 + n\cdot\log l - \frac{l-1}{2}\cdot\log n}\right).$$

Furthermore, we easily obtain the following crude bound for $\mathcal{X}_n(d)$:

$$|\mathcal{X}_n(d)| \leq 2^{dn\log n}.$$

We need some more notations. Let $\mathcal{Q}$ be a property which a l-colorable graph may have. (For example the existence of a color-critical vertex is such a property.) Let

$$\mathcal{C}_n(\mathcal{Q}) = \{G \in \mathcal{C}ol_n(l) \mid G_2 \text{ has property } \mathcal{Q}\}$$

and

$$\mathcal{E}_n(\mathcal{Q}) = \{G \in \mathcal{F}orb_n(H) \mid G = G_1 \dot\cup G_2, G_1 \in \mathcal{X}_n(d-1), G_2 \in \mathcal{C}ol_n(l), G_2 \text{ has property } \mathcal{Q}\}.$$

Obviously for all such properties $\mathcal{Q}$:

$$|\mathcal{E}_n(\mathcal{Q})| \leq |\mathcal{C}_n(\mathcal{Q})| \cdot |\mathcal{X}_n(d-1)| \leq |\mathcal{C}_n(\mathcal{Q})| \cdot 2^{dn\log n}.$$

That is, if

$$(*) \qquad |\mathcal{C}_n(\mathcal{Q})| = o(2^{-dn\log n}) \cdot |\mathcal{C}ol_n(l)|$$

then $|\mathcal{E}_n(\mathcal{Q})| = o(|\mathcal{E}(n)|)$.

We will consider the following two properties:

$\mathcal{Q}_1 =$ there exists a coloring with color classes $V_1, \ldots, V_l$ such that $\min_{1\leq i\leq l}|V_i| \leq \frac{n}{2l}$, and

$\mathcal{Q}_2 =$ for all colorings with color classes $V_1, \ldots, V_l$ such that $|V_1| = \min_{1\leq i\leq l}|V_i| \geq \frac{n}{2l}$ there exist sets A_{ij}, $1 \leq i \leq \sqrt{n}$, $2 \leq j \leq l$ such that

- $A_{ij} \subseteq V_j$, $|A_{ij}| = |V(H)|$,
- $A_{i_1j} \cap A_{i_2j} = \emptyset$ for all $i_1 \neq i_2$, and
- $|\{v \in V_1 \mid \bigcup_{2\leq j\leq l} A_{ij} \subseteq \Gamma(v)\}| \leq |V(H)|$ for all $1 \leq i \leq \sqrt{n}$.

One easily checks that $\mathcal{Q}_1$ as well as $\mathcal{Q}_2$ satisfy inequality $(*)$. (Overestimate the number of graphs contained in $\mathcal{C}_n(\mathcal{Q}_i)$ by first choosing appropriate color classes (less than l^n ways), the sets A_{ij} (less than $\binom{n}{|V(H)|}^{(l-1)\sqrt{n}}$ ways), and then choosing the edges between the color classes.) More precisely,

$$|\mathcal{C}_n(\mathcal{Q}_1)| \leq l^n \cdot 2^{\frac{n}{2l}(n-\frac{n}{2l})+\binom{l-1}{2}(\frac{n-\frac{n}{2l}}{l-1})^2} = o(2^{-dn\log n}) \cdot |\mathcal{C}ol_n(l)|$$

and (here we let $h = |V(H)|$)

$$\begin{aligned} |\mathcal{C}_n(\mathcal{Q}_2)| &\leq l^n \cdot \binom{n}{h}^{(l-1)\sqrt{n}} \cdot 2^{\binom{l-1}{2}(\frac{n-|V_1|}{l-1})^2+|V_1|\cdot(n-|V_1|-(l-1)h\sqrt{n})} \cdot \\ &\quad \left[\sum_{i=0}^{h}\binom{|V_1|}{i}\right]^{\sqrt{n}} \cdot (2^{(l-1)h}-1)^{|V_1|\sqrt{n}} \\ &\leq 2^{\binom{l}{2}n^2+n\log l-\frac{1}{2^{l\cdot h}\cdot l}n\sqrt{n}} \end{aligned}$$

$$= \quad o(2^{-dn\log n}) \cdot |\mathcal{C}ol_n(l)|.$$

The cardinality of $\mathcal{E}(n+1)$ can now be bounded as follows. Fix an arbitrary vertex $v_0 \in V_{n+1}$. We claim that are at least $2^{\frac{l-1}{l}(n-l|V(H)|\sqrt{n})}$ possibilities to connect v_0 to any graph $G \in \mathcal{E}_n(\bar{\mathcal{Q}}_1 \wedge \bar{\mathcal{Q}}_2)$ induced on $V_{n+1} \setminus v_0$. Indeed, by the definition of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ every graph $G \in \mathcal{E}_n(\bar{\mathcal{Q}}_1 \wedge \bar{\mathcal{Q}}_2)$ may be written as $G = G_1 \cup G_2$ with $G_2 \in \mathcal{C}_n(\bar{\mathcal{Q}}_1 \wedge \bar{\mathcal{Q}}_2)$. Let $|V_1| \leq \cdots \leq |V_l|$ be the color classes of an appropriate coloring of G. Then $|V_1| \geq \frac{n}{2l}$ and for $i = 2, \ldots, l$ there exist sets $S_i \subseteq V_i$ of size $|S_i| = \sqrt{n} \cdot |V(H)|$ such that any tuple $(X_2, \ldots, X_l)$ with $X_i \subseteq V_i \setminus S_i$ and $|X_i| = |V(H)|$ has the property that $\bigcup_{2 \leq i \leq l} X_i$ has at least $|V(H)$ many neighbors in V_1. In particular, this implies that v_0 may be connected to $\bigcup_{i=2}^{l}(V_i \setminus S_i)$ in an arbitrary way without generating a copy of the forbidden subgraph H. Therefore

$$\begin{aligned}
\mathcal{E}(n+1) &\geq |\mathcal{E}_n(\bar{\mathcal{Q}}_1 \wedge \bar{\mathcal{Q}}_2)| \cdot 2^{\frac{l-1}{l}(n-l|V(H)|\sqrt{n})} \\
&\geq (|\mathcal{E}(n)| - |\mathcal{E}_n(\mathcal{Q}_1)| - |\mathcal{E}_n(\mathcal{Q}_2)|) \cdot 2^{\frac{l-1}{l}(n-l|V(H)|\sqrt{n})} \\
&\geq (1 - o(1)) \cdot |\mathcal{E}(n)| \cdot 2^{\frac{l-1}{l}(n-l|V(H)|\sqrt{n})},
\end{aligned}$$

from which the desired inequality follows easily. □

For the rest of this section fix integers $l \geq 2$ and $1 \leq d \leq k$ and let H be an arbitrary but fixed $(l+1)$-partite graph containing a color-critical vertex of criticality d. The main part of the proof is devoted to defining appropriate sets $\mathcal{Z}_i(n)$.

We proceed with some definitions. Throughout the rest of this section we fix $\epsilon_l := 1/(2^{3d+8} \cdot l^3)$ and an $\epsilon_0 > 0$ such that

$$-(\epsilon_0 \log \epsilon_0 + (1-\epsilon_0)\log(1-\epsilon_0)) = \frac{1}{5}\epsilon_l.$$

Observe that this implies $n^2 \cdot \binom{n}{\epsilon_0 n} < 2^{\frac{1}{4}\epsilon_l n}$ for n sufficiently large.

For $0 \leq k < l$ let f_k denote the integral function

$$f_k(n) = \lceil \log^{(k^2-k+2)} n \rceil.$$

A *p_k-set* is an ordered set $P = \{v_1, \ldots, v_{kf_k(n)}\} \subseteq V_n$ such that for the sets $P_i = \{v_{(i-1)f_k(n)+1}, \ldots, v_{if_k(n)}\}$, $i = 1, \ldots, k$, the following conditions are satisfied:

(1) $P_i \cap P_j = \emptyset$ for every $1 \leq i < j \leq k$,

(2) $|P_i| = f_k(n)$ for every $1 \leq i \leq k$ and

(3) P_i and P_j are completely connected for every pair $1 \leq i < j \leq k$.

For a p_k-set $P = \bigcup_{i=1}^{k} P_i$ we denote by $T_k(P) = T_k(P_1, \ldots, P_k)$ the set

$$T_k(P) = \{v \in V_n \setminus \bigcup_{i=1}^{k} P_i \mid |\Gamma(v) \cap P_i| \geq \epsilon_0 |P_i| \text{ for every } i = 1, \ldots, k\}.$$

Observe that, by definition, a p_0-set is empty and $T_0(\emptyset) = V_n$. For simplicity a p_{l-1}-set Q is also called a *q-set*. The corresponding set $T_{l-1}(Q)$ is also denoted by $R(Q)$ and called the *r-set* of Q. In addition we let q denote the size of a q-set , i.e. $q = (l-1)f_{l-1}$. A q-set *of a vertex* v is a q-set contained in the neighborhood $\Gamma(v)$ of v.

Lemma 4.2.5 *Let $1 \leq k \leq l-1$ be an integer and let $\mathcal{A}_k(n)$ denote the set of all graphs $G \in \mathcal{F}orb_n(H)$ that contain a p_k-set P such that the set*

$$|T_k(P)| \leq (\frac{l-k}{l} - \epsilon_l)n.$$

Then

$$\log \frac{|\mathcal{A}_k(n)|}{|\mathcal{F}orb_{n-kf_k}(H)|} \leq (\frac{l-1}{l} - \frac{1}{2k}\epsilon_l)kf_k n$$

for all sufficiently large n.

Proof. Construct all graphs in $\mathcal{A}_k(n)$ as follows. First choose the p_k-set $P = \bigcup_{i=1}^k P_i$ and an H-free graph on $V_n \setminus P$ (at most $n^{kf_k} \cdot |\mathcal{F}orb_{n-kf_k}(H)|$ ways). Then choose edges inside P (less than $2^{\frac{1}{2}(kf_k)^2}$ ways), choose the set $T_k(P)$ (less than 2^n ways) and connect P to $T_k(P)$ (at most $2^{kf_k|T_k(P)|}$ ways). Finally, connect P to $V_n \setminus (P \cup T_k(P))$. Do this by first choosing for each $v \in V_n \setminus (P \cup T_k(P))$ an index i, $1 \leq i \leq k$, so that v has less than $\epsilon_0 f_k$ neighbors in P_i. In this way the total number of ways to connect P to $V_n \setminus (P \cup T_k(P))$ can, for n sufficiently large, obviously be bounded by

$$\left(k \cdot \sum_{j=0}^{\epsilon_0 f_k} \binom{f_k}{j} \cdot 2^{(k-1)f_k} \right)^{n-|T_k(P)|} \leq 2^{(\frac{1}{4}\epsilon_l f_k + (k-1)f_k)\cdot(n-|T_k(P)|)}.$$

Together this gives

$$\begin{aligned}
\log \frac{|\mathcal{A}_k(n)|}{|\mathcal{F}orb_{n-kf_k}(H)|} &\leq kf_k \log n + \frac{1}{2}(kf_k)^2 + n + kf_k|T_k(P)| \\
&\quad + (\frac{1}{4}\epsilon_l f_k + (k-1)f_k)\cdot(n - |T_k(P)|) \\
&\leq (k - 1 + \frac{l-k}{l} - \epsilon_l + \frac{1}{2}\epsilon_l) \cdot f_k n \\
&\leq (\frac{l-1}{l} - \frac{1}{2k}\epsilon_l) \cdot kf_k n
\end{aligned}$$

for n sufficiently large. □

Lemma 4.2.6 *Let $0 \leq k \leq l-2$ be an integer and let $\mathcal{B}_k(n)$ denote the set of all graphs $G \in \mathcal{F}orb_n(H)$ that contain a vertex v and a p_k-set P contained in $\Gamma(v)$ such that*

$$|T_k(P)| \geq (\frac{2}{l} - \epsilon_l)n \qquad \textit{and} \qquad |\Gamma(v) \cap T_k(P)| \leq f_k(n).$$

Then

$$\log \frac{|\mathcal{B}_k(n)|}{|\mathcal{F}orb_{n-1}(H)|} \leq (\frac{l-1}{l} - \frac{1}{2l})n$$

for all sufficiently large n.

Proof. Construct all graphs in $\mathcal{B}_k(n)$ as follows. First choose the vertex v and an H-free graph on $V_n \setminus v$ (at most $n \cdot |\mathcal{F}orb_{n-1}(H)|$ ways). Then choose the p_k-set P (less than n^{kf_k} ways) — observe that this implicitly defines the set $T_k(P)$. Next connect v with $T_k(P)$ (less than $\sum_{i=0}^{f_k} \binom{n}{i} \leq 2^{f_k \log n}$ ways) and with $V_n \setminus (P \cup T_k(P))$ (less than $2^{n-|T_k(P)|}$ ways).
Together this gives

$$\begin{aligned}
\log \frac{|\mathcal{B}_k(n)|}{|\mathcal{F}orb_{n-1}(H)|} &\leq \log n + kf_k \log n + f_k \log n + n - |T_k(P)| \\
&\leq (1 - \frac{2}{l} + 2\epsilon_l)n \\
&\leq (\frac{l-1}{l} - \frac{1}{2l})n
\end{aligned}$$

for n sufficiently large. □

Lemma 4.2.7 *For n sufficiently large, every graph $G = (V_n, E)$ in $\mathcal{F}orb_n(H) \setminus (\bigcup_{k=1}^{l-2} \mathcal{A}_k(n) \cup \bigcup_{k=0}^{l-2} \mathcal{B}_k(n))$ has the following property: the neighbourhood $\Gamma(v)$ of every vertex $v \in V_n$ contains a q-set Q.*

The proof of this lemma is analogous to the proof of a similar result in Kolaitis, Prömel, Rothschild [55]. In particular we need the following lemma from [55], which generalizes a result of Bollobás and Erdős [11].

Lemma 4.2.8 [Kolaitis, Prömel, Rothschild, 1987] *Let $0 < \epsilon < 1$ and $k \in \mathbb{N}$ be given. Then there exists $N_0 = N_0(\epsilon, k)$ such that the following is true. Let G be a graph with vertex set $A_0 \cup \cdots \cup A_k$, where the A_i are pairwise disjoint sets each of size $N \geq N_0$, and suppose that $|\Gamma(v) \cap A_i| \geq \epsilon N$ for all $v \in A_0$ and all $1 \leq i \leq k$. Then there exist sets $A_i' \subseteq A_i$, $0 \leq i \leq k$, such that $|A_i'| = \lceil \log^{(2k)} N \rceil$ for each $i = 0, \ldots, k$ and such that $\bigcup_{i=1}^{k} A_i' \subseteq \Gamma(v)$ for all $v \in A_0'$.* □

Proof of Lemma 4.2.7. Let v be an arbitrary vertex of G. By definition of the set $\mathcal{B}_0(n)$ we observe that $\Gamma(v)$ contains a set P_{11} of size $f_1(n)$. We conclude the proof by induction on k. Suppose for some k, $1 \leq k \leq l-2$, there exist sets $P_{k1}, \ldots, P_{kk}$ satisfying the following two conditions:

i) the P_{ki}, $1 \leq i \leq k$ are pairwise disjoint, and

ii) $P_{k1} \cup \cdots \cup P_{kk}$ form a p_k-set contained in $\Gamma(v)$.

As G is neither contained in $\mathcal{A}_k(n)$ nor in $\mathcal{B}_k(n)$ we know that

$$|\Gamma(v) \cap T_k(P_{k1} \cup \cdots \cup P_{kk})| > f_k(n).$$

Let $\tilde{P}$ be an arbitrary subset of $\Gamma(v) \cap T_k(P_{k1} \cup \cdots \cup P_{kk})$ of size $f_k(n)$. Apply Lemma 4.2.8 to $A_0 = \tilde{P}$, and $A_i = P_{ki}$, $1 \leq i \leq k$ to obtain sets $P_{k+1,1}, \ldots P_{k+1,k+1}$ satisfying the conditions i) and ii) for $k+1$. Inductively we thus obtain sets $P_{l-1,1}, \ldots, P_{l-1,l-1}$ which form a q-set contained in $\Gamma(v)$. □

Observe also that another immediate consequence of Lemma 4.2.8 is that a vertex may not have many neighbours in the r-set of a q-set contained in its neighbourhood.

Corollary 4.2.9 *For all sufficiently large n every graph $G = (V_n, E)$ in $\mathcal{F}orb_n(H)$ has the following property: if $v \in V_n$ is a vertex with q-set Q, then*

$$|\Gamma(v) \cap R(Q)| \leq q.$$

□

Lemma 4.2.10 *Let $\mathcal{C}(n)$ denote the set of all graphs $G \in \mathcal{F}orb_n(H)$ that contain a vertex v with q-set Q such that*

$$|R(Q)| \geq (\frac{1}{l} + \epsilon_l)n.$$

Then

$$\log \frac{|\mathcal{C}(n)|}{|\mathcal{F}orb_{n-1}(H)|} \leq (\frac{l-1}{l} - \frac{1}{2}\epsilon_l)n$$

for all sufficiently large n.

Proof. Construct all graphs in $\mathcal{C}(n)$ as follows. First choose the vertex v and an H-free graph on $V_n \setminus v$ (at most $n \cdot |\mathcal{F}orb_{n-1}(H)|$ ways). Then choose the q-set Q in $V_n \setminus v$ (less than n^q ways) and connect v to $V_n \setminus R(Q)$ (less than $2^{n-|R(Q)|}$ ways). Finally connect v to $R(Q)$. Observe that by Corollary 4.2.9 there are at most

$$\sum_{i=0}^{q} \binom{n}{i} \leq 2^{q \log n}$$

ways to do this.
Together this gives

$$\begin{aligned} \log \frac{|\mathcal{C}(n)|}{|\mathcal{F}orb_{n-1}(H)|} &\leq \log n + q \log n + n - |R(Q)| + q \log n \\ &\leq (\frac{l-1}{l} - \frac{1}{2}\epsilon_l)n \end{aligned}$$

for n sufficiently large. □

Corollary 4.2.11 *For n sufficiently large, every graph $G = (V_n, E)$ in $\mathcal{F}orb_n(H) \setminus (\mathcal{A}_{l-1}(n) \cup \mathcal{C}(n))$ has the following property: if v is a vertex with q-set Q then*

$$(\frac{1}{l} - \epsilon_l)n \leq |R(Q)| \leq (\frac{1}{l} + \epsilon_l)n.$$

□

Lemma 4.2.12 *Let $0 \leq k \leq l-2$ and $m \in \{2, d+1\}$ be a integers, and let $\mathcal{D}_{k,m}(n)$ denote the set of all graphs $G \in \mathcal{F}orb_n(H)$ that contain distinct vertices $v_1, \ldots, v_m$ with*

q-sets $Q_1, \ldots, Q_m$, respectively, and a p_k-set P contained in $\bigcap_{i=1}^m \Gamma(v_i)$ such that the following properties are satisfied simultaneously:

$$|T_k(P)| \geq (\frac{2}{l} - \epsilon_l)n, \qquad |\bigcap_{i=1}^m \Gamma(v_i) \cap T_k(P)| \leq f_k(n),$$

$$|R(Q_1) \cap \bigcap_{i=2}^m R(Q_i)| \geq 2^{3d+6}\epsilon_l n, \qquad |\bigcap_{i=2}^m R(Q_i)| \geq (\frac{1}{l} - 2^{d+4}\epsilon_l)n,$$

and

$$(\frac{1}{l} - \epsilon_l)n \leq |R(Q_i)| \leq (\frac{1}{l} + \epsilon_l)n \quad \textit{for} \quad i = 1, \ldots, m.$$

Then

$$\log \frac{|\mathcal{D}_{k,m}(n)|}{|\mathcal{F}orb_{n-m}(H)|} \leq (\frac{l-1}{l} - \frac{\epsilon_l}{m})mn$$

for all sufficiently large n.

Proof. Similarly as in Lemma 4.2.6 we construct all graphs in $\mathcal{D}_{k,m}(n)$ as follows. We first choose the vertices $v_1, \ldots, v_m$, an H-free graph on $V_n \setminus \{v_1, \ldots, v_m\}$, choose the q-sets $Q_1, \ldots, Q_m$, and the p_k-set P (less than $n^m \cdot |\mathcal{F}orb_{n-m}(H)| \cdot n^{mq} \cdot n^{kf_k}$ ways). Then we choose the edges among the v_i (less than 2^{m^2} ways), connect v_1 to $R_1 := R(Q_1)$ and the v_i, $i \geq 2$, to $\tilde{R} := \bigcap_{i=2}^m R(Q_i)$ (less than $2^{m \cdot q \log n}$ ways, cf. Corollary 4.2.9), connect v_i, $i \geq 2$, to $R_1 \setminus \tilde{R}$ and v_1 to $\tilde{R} \setminus R_1$ (less than $2^{(m-1)(|R_1| - |\tilde{R} \cap R_1|)} \cdot 2^{|\tilde{R}| - |\tilde{R} \cap R_1|}$ ways), and connect v_i, $i \geq 1$, to $V_n \setminus (\tilde{R} \cup R_1 \cup T_k(P))$ (less than $2^{m(n - |T_k(P) \setminus (\tilde{R} \cup R_1)| - |\tilde{R}| - |R_1| + |\tilde{R} \cap R_1|)}$ ways). Finally, we connect the v_i, $i \geq 1$, to $T_k(P) \setminus (\tilde{R} \cup R_1)$. This can be done by choosing first the at most $f_k(n)$ many vertices of $T_k(P) \setminus (\tilde{R} \cup R_1)$ connected to all the v_i and then connecting the remaining vertices to at most $m-1$ of the v_i. That is, there are at most

$$\sum_{j=0}^{f_k} \binom{n}{j} \cdot (2^m - 1)^{|T_k(P) \setminus (\tilde{R} \cup R_1)|} \leq 2^{f_k \log n + (m - \frac{1}{2^{m+1}})|T_k(P) \setminus (\tilde{R} \cup R_1)|}$$

ways to connect the v_i to $T_k(P) \setminus (\tilde{R} \cup R_1)$.
Together this gives

$$\begin{aligned}
\log \frac{|\mathcal{D}_{k,m}(n)|}{|\mathcal{F}orb_{n-m}(H)|} &\leq m \log n + mq \log n + kf_k \log n + m^2 + mq \log n + (m-1)|R_1| \\
&\quad + |\tilde{R}| + m \cdot (n - |T_k(P) \setminus (\tilde{R} \cup R_1)| - |\tilde{R}| - |R_1|) \\
&\quad + f_k \log n + (m - \frac{1}{2^{m+1}})|T_k(P) \setminus (\tilde{R} \cup R_1)| \\
&\leq mn - \frac{1}{2^{m+1}}|T_k(P) \setminus (\tilde{R} \cup R_1)| - (m-1)|\tilde{R}| - |R_1| + \frac{1}{2^{m+1}}\epsilon_l n \\
&\leq \frac{l-1}{l} \cdot mn - \underbrace{(\frac{2^{3d+6} - 3}{2^{m+1}}\epsilon_l - d2^{d+4}\epsilon_l - \epsilon_l - \frac{1}{2^{m+1}}\epsilon_l)}_{\geq \epsilon_l} \cdot n
\end{aligned}$$

for n sufficiently large. $\square$

The following corollaries can be proved in exactly the same way as Corollary 4.2.7, if one uses the $\mathcal{D}_{k,m}(n)$-sets instead of the $\mathcal{B}_k(n)$-sets.

Corollary 4.2.13 *For n sufficiently large, every graph $G = (V_n, E)$ in $\mathcal{F}orb_n(H) \setminus (\bigcup_{k=1}^{l-1} \mathcal{A}_k(n) \cup \bigcup_{k=0}^{l-2} \mathcal{B}_k(n) \cup \mathcal{C}(n) \cup \bigcup_{k=0}^{l-2} \mathcal{D}_{k,2}(n))$ has the following property: for every two vertices $v_1, v_2 \in V_n$ which have q-sets Q_1 and Q_2, such that $|R(Q_1) \cap R(Q_2)| \geq 2^{3d+6}\epsilon_l n$ there exists a q-set Q contained in $\Gamma(v_1) \cap \Gamma(v_2)$.* $\square$

Corollary 4.2.14 *For n sufficiently large, every graph $G = (V_n, E)$ in $\mathcal{F}orb_n(H) \setminus \left(\bigcup_{k=1}^{l-1} \mathcal{A}_k(n) \cup \bigcup_{k=0}^{l-2} \mathcal{B}_k(n) \cup \mathcal{C}(n) \cup \bigcup_{k=0}^{l-2} \mathcal{D}_{k,d+1}(n)\right)$ has the following property: for every $d+1$ vertices $v_1, \ldots, v_{d+1}$ which have q-sets $Q_1, \ldots, Q_{d+1}$, respectively, such that $|\bigcap_{i=1}^{d+1} R(Q_i)| \geq (\frac{1}{l} - 2^{d+4}\epsilon_l)n$ there exists a q-set Q contained in $\bigcap_{i=1}^{d+1} \Gamma(v_i)$.* $\square$

Lemma 4.2.15 *Let $\mathcal{E}(n)$ denote the set of all graphs $G \in \mathcal{F}orb_n(H)$ that contain a vertex v with two q-sets Q_1 and Q_2 such that the following two properties are satisfied simultaneously:*

$$|R(Q_1)| \geq (\frac{1}{l} - \epsilon_l)n \qquad \text{and} \qquad |R(Q_2) \setminus R(Q_1)| \geq 4\epsilon_l n.$$

Then

$$\log \frac{|\mathcal{E}(n)|}{|\mathcal{F}orb_{n-1}(H)|} \leq (\frac{l-1}{l} - 24\epsilon_l)n$$

for all sufficiently large n.

Proof. Construct all graphs in $\mathcal{E}(n)$ as follows. First choose the vertex v and an H-free graph on $V_n \setminus v$ (at most $n \cdot |\mathcal{F}orb_{n-1}(H)|$ ways). Then choose the q-sets Q_1 and Q_2 (less than n^{2q} ways) and connect v with $V_n \setminus (R(Q_1) \cup R(Q_2))$ (less than $2^{n-|R(Q_1)|-|R(Q_2)\setminus R(Q_1)|}$ ways). Finally connect v to $R(Q_1) \cup R(Q_2)$. Observe that by Corollary 4.2.9 there are at most $2^{2q \log n}$ ways to do this.
Together this gives

$$\begin{aligned}
\log \frac{|\mathcal{E}(n)|}{|\mathcal{F}orb_{n-1}(H)|} &\leq \log n + 2q \log n + n - |R(Q_1)| - |R(Q_2) \setminus R(Q_1)| + 2q \log n \\
&\leq (\frac{l-1}{l} - 2\epsilon_l)n
\end{aligned}$$

for n sufficiently large. $\square$

Corollary 4.2.16 *For n sufficiently large, every graph $G = (V_n, E)$ in $\mathcal{F}orb_n(H) \setminus \left(\bigcup_{k=1}^{l-1} \mathcal{A}_k(n) \cup \bigcup_{k=0}^{l-2} \mathcal{B}_k(n) \cup \mathcal{C}(n) \cup \bigcup_{k=0}^{l-2} \mathcal{D}_{k,2}(n) \cup \mathcal{E}(n)\right)$ has the following property: If v_1, v_2 are two vertices with q-sets Q_1 and Q_2, resp., then the corresponding r-sets either "coincide" or are "disjoint", that is either*

$$|R(Q_1) \cap R(Q_2)| \geq (\frac{1}{l} - 9\epsilon_l)n \qquad \text{or} \qquad |R(Q_1) \cap R(Q_2)| \leq 2^{3d+6}\epsilon_l n.$$

Proof. Let v_1 and v_2 be two vertices with q-sets Q_1 and Q_2, respectively. Assume that $|R(Q_1) \cap R(Q_2)| \geq 2^{3d+6}\epsilon_l n$. According to Corollary 4.2.13 there exists a q-set Q_{12}, which is a q-set of v_1 as well as of v_2. As G is not in $\mathcal{E}(n)$ this implies that

$$|R(Q_1) \setminus R(Q_{12})| \leq 4\epsilon_l n \qquad \text{and} \qquad |R(Q_{12}) \setminus R(Q_2)| \leq 4\epsilon_l n.$$

Therefore,

$$\begin{aligned} |R(Q_1) \cap R(Q_2)| &\geq |R(Q_1) \cap R(Q_{12}) \cap R(Q_2)| \\ &\geq |R(Q_1)| - |R(Q_1) \setminus R(Q_{12})| - |R(Q_{12}) \setminus R(Q_2))| \geq \\ &\geq (\frac{1}{l} - 9\epsilon_l)n. \end{aligned}$$

□

Corollary 4.2.17 *For n sufficiently large,*

$$\mathcal{F}orb_n(H) \setminus \left(\bigcup_{k=1}^{l-1} \mathcal{A}_k(n) \cup \bigcup_{k=0}^{l-2} \mathcal{B}_k(n) \cup \mathcal{C}(n) \cup \bigcup_{k=0}^{l-2} (\mathcal{D}_{k,2}(n) \cup \mathcal{D}_{k,d+1}(n)) \cup \mathcal{E}(n) \right) \subseteq \mathcal{F}orb_n(H) \cap \mathcal{P}_n(l, d-1).$$

Proof. Let $G = (V_n, E)$ be a graph contained in the set on the left hand side and assume n is so large that all of the above lemmas and corollaries hold. We now construct a partition $V_n = \bigcup_i X_i$ and show that it has the desired properties.

By Lemma 4.2.7 the neighbourhood of every vertex $v \in V_n$ contains at least one q-set. For every vertex v fix one such q-set, lets call it Q_v, and denote the corresponding r-set by $R_v = R(Q_v)$. Choose a maximum number of vertices $v_1, \ldots, v_s \in V_n$ such that

$$|R_{v_i} \cap R_{v_j}| \leq 2^{3d+6}\epsilon_l n \qquad \text{for all pairs } 1 \leq i < j \leq s$$

and let

$$X_i := \{x \in V_n \mid |R_x \cap R_{v_i}| \geq \frac{2}{3}\frac{n}{l}\}.$$

Claim 1: The sets X_i partition the set V_n.
Choose $x \in V_n \setminus \{v_1, \ldots, v_s\}$ arbitrarily. By the maximality of the set $\{v_1, \ldots, v_s\}$ there has to exist at least one v_i such that $|R_x \cap R_{v_i}| > 2^{3d+6}\epsilon_l n$. By Corollary 4.2.16 this implies $|R_x \cap R_{v_i}| \geq \frac{2}{3}\frac{n}{l}$, that is $x \in X_i$. Assume now there also exists an $1 \leq j \leq s$, $j \neq i$, such that $x \in X_j$. Then

$$\frac{4}{3}\frac{n}{l} \leq |R_x \cap R_{v_i}| + |R_x \cap R_{v_j}| \leq |R_x| + |R_{v_i} \cap R_{v_j}| \leq (\frac{1}{l} + \epsilon_l + 2^{3d+6}\epsilon_l)n.$$

Contradiction.
Claim 2: $G[X_i]$ has maximal degree at most $d-1$, for all $i = 1, \ldots, s$.
Assume there exist $1 \leq i_0 \leq s$ and vertices $x_0, x_1, \ldots, x_d \in X_{i_0}$ such that x_0 is connected to all vertices $x_1, \ldots, x_d$. Then the definition of X_{i_0} and Corollaries 4.2.11 and 4.2.16 imply

$$|R_{x_i} \cap R_{x_j}| \geq (\frac{1}{l} - 9\epsilon_l)n \qquad \text{for all } 0 \leq i < j \leq d.$$

Therefore

$$|R_{x_i} \setminus R_{x_j}| \leq 10\epsilon_l n \qquad \text{for all } 0 \leq i < j \leq d$$

and

$$|\bigcap_{i=0}^{d} R_{x_i}| \;\geq\; |R_{x_0}| - \sum_{i=1}^{d} |R_{x_0} \setminus R_{x_i}| \;\geq\; (\frac{1}{l} - \epsilon_l - 10d\epsilon_l)n \;\geq\; (\frac{1}{l} - 2^{d+4}\epsilon_l)n.$$

By Corollary 4.2.14 we know that therefore there exists a q-set $\tilde{Q} = \bigcup_{i=1}^{l-1} \tilde{Q}_i$ contained in $\bigcap_{i=0}^{d} \Gamma(x_i)$. Recall now that the definition of the r-set together with Lemma 4.2.8 implies that there exist sets $H_1 \subseteq R(\tilde{Q})$ and $H_{i+1} \subseteq \tilde{Q}_i$, $i = 1, \ldots, l-1$ so that $\{x_0\}, H_1 \cup \{x_1, \ldots, x_d\}, H_2, \ldots, H_l$ form the parts of the forbidden subgraph H. Contradiction.

Claim 3: $s = l$.

By Claim 2 at most $d-1$ vertices of Q_{v_1} can belong to X_1. Similarly, we observe that the $l-1$ parts of Q_{v_1} must belong to different sets X_i. This shows $s \geq l$. As on the other hand

$$n \geq |\bigcup_{i=1}^{s} R_{v_i}| \geq \sum_{i=1}^{s} |R_{v_i}| - \sum_{i<j} |R_{v_i} \cap R_{v_j}| \geq s(\frac{1}{l} - \epsilon_l)n - \binom{s}{2} 2^{3d+6}\epsilon_l n,$$

we immediately observe that also $s \leq l$, that is $s = l$. □

This completes the proof of Theorem 4.2.2.

Chapter 5

Graph Coloring Algorithms

One of the most intrinsic problems in graph theory is the graph-coloring problem: given a graph $G = (V, E)$, decide its chromatic number $\chi(G)$. Unfortunately, even the question of whether $\chi(G) \leq l$ for some fixed $l \geq 3$ is NP-complete. The importance of the problem has led many researchers to find algorithms which are - though exponential in the worst case - at least fast "on average".

Considering average-case complexity, Wilf (1984) observed that there exists an algorithm which decides the question whether $\chi(G) \leq l$ in constant expected time, assuming equal distribution on all graphs. We describe a simplified version of his algorithm in Section 5.1.

If we restrict consideration to the "small" class of K_{l+1}-free graphs, the problem becomes much more difficult. Though it is known that almost all K_{l+1}-free graphs are l-colorable, it is not difficult to see that the fraction of those K_{l+1}-free graphs which are not l-colorable grows at least as $2^{-\Theta(n)}$. Since there is no coloring algorithm of subexponential complexity known for this class, one needs more structural properties to design an algorithm which colors every K_{l+1}-free graph with a minimal number of colors in polynomial expected time. In Section 5.2 we prove such a structural result and exhibit an algorithm which colors every K_{l+1}-free graph G with $\chi(G)$ colors in $\mathcal{O}(n^2)$ expected time.

Restricting the class under consideration even further, Turner (1988) had earlier been able to exhibit a polynomial time algorithm which finds a proper coloring for almost every l-colorable graph. His result was strengthened by Dyer and Frieze (1989) (for constant l) and by Kučera (1989) (for all $l \leq \sqrt{n/(28 \log n)}$) who proposed an algorithm which colors every l-colorable graph on n vertices properly in $\mathcal{O}(n^2)$ expected time. All these algorithms are based on the fact that for $l \leq \sqrt{n/(28 \log n)}$ almost all l-colorable graphs are uniquely l-colorable. In Section 5.3 we generalize this result considerably by showing that not only does there exist a threshold function $l = l(n)$ for the property of being uniquely l-colorable, but that this threshold corresponds to the chromatic number of a random graph.

5.1 Wilf's algorithm

In 1984 H. Wilf published an algorithm which decides in constant expected time, whether a given graph (randomly drawn from the set of all graphs) has chromatic number at most l. The reason for this at first sight very surprising result is simply that almost every graph contains many cliques of size $l+1$, thus is in particular not l-colorable. Even more is true: with high probability a graph contains a clique of size $l+1$ already on an initial piece of constant length. In these cases the algorithm just says "no". For the K_{l+1}-free graphs on the other hand the algorithm decides the l-colorability simply by complete enumeration. Since the fraction of these graphs is sufficiently small, this does not destroy the constant expected time.

In the following we state a simplified version (cf. [48]) of Wilf's algorithm. Assume that Exact-Color(G) is an algorithm which determines the chromatic number of a graph G exactly. We denote its worst-case running time on graphs with n vertices by $\mathrm{run}_C(n)$. For any integer $k \leq |V(G)|$ let $G[1,k]$ denote the subgraph of G induced by the first k vertices.

Algorithm Test-Colorability$_l(G)$
Input: Graph G with n vertices;
Output: True if G has chromatic number $\chi(G) \leq l$, False otherwise.
begin
  $k := 1$;
  while $k \leq n$ **do begin**
    if $G[1,k]$ contains a K_{l+1}-subgraph **then** **return** False
      else $k := k+1$;
  end while
  return (Exact-Color(G) $\leq l$);
end.

The correctness of the algorithm is immediate. To bound the average running time we first state two observations which are easily seen to be true.

- *If* Test-Colorability *stops with $k = t$, then $\mathcal{O}(t^{l+1})$ is a crude upper bound for the running time of* Test-Colorability *in this case.*
- *The number of graphs on n vertices for which* Test-Colorability *completes at least t passes of the while-loop is exactly $|\mathcal{F}orb_t(K_{l+1})| \cdot 2^{\binom{n}{2}-\binom{t}{2}}$.*

Let $\mathrm{ave}_T(n)$ denote the average running time of the algorithm Test-Colorability on graphs with n vertices, assuming that every graph is equally likely. An upper bound for $\mathrm{ave}_T(n)$ is then easily derived:

$$\mathrm{ave}_T(n) \quad \leq \quad 2^{-\binom{n}{2}} \cdot \left(\sum_{t=1}^{n} \mathcal{O}(t^{l+1}) \cdot |\mathcal{F}orb_t(K_{l+1})| \cdot 2^{\binom{n}{2}-\binom{t}{2}} + \mathrm{run}_C(n) \cdot |\mathcal{F}orb_n(K_{l+1})| \right).$$

As, trivially, $\mathrm{run}_C(n) \leq n^n$ and, according to Theorem 3.2.2, we also know that $|\mathcal{F}orb_n(K_{l+1})| = 2^{(1-\frac{1}{l})\binom{n}{2}+o(n^2)}$, this immediately implies the following theorem.

Theorem 5.1.1 *For every fixed $l \geq 3$ the algorithm* TEST-COLORABILTY$_l(G)$ *decides in $\mathcal{O}(1)$ expected time, whether $\chi(G) \leq l$.*

□

Observe, that one could even demand that the algorithm should output a proper coloring with $\chi(G)$ colors, whenever $\chi(G) \leq l$, without spoiling the constant expected running time!

5.2 Coloring clique-free graphs

In 1989 Dyer and Frieze described an algorithm which colors every l-colorable graph (here l is a fixed constant!) in $\mathcal{O}(n^2)$ expected time. Following Prömel, Steger (1992) we extend this considerably by exhibiting an algorithm which colors every K_{l+1}-free graph G with $\chi(G)$ colors in $O(n^2)$ expected time, regardless what the value of $\chi(G)$ is. Observing that almost every K_{l+1}-free graph has $\Omega(n^2)$ edges, this means that the complexity of the algorithm presented is in fact linear. Moreover, since the class of l-colorable graphs is dense in the class of K_{l+1}-free graphs, the algorithm in particular colors l-colorable graphs in linear expected time. This result heavily relies on a structural result, which allows to estimate precisely the number of graphs with certain properties. As a consequence we obtain, for example, that the portion of K_{l+1}-free graphs on n vertices with chromatic number larger than $h(n)$, where $h(n) \geq l$, is at most $2^{-\Omega(n \cdot h(n))}$, hence, in particular, the fraction of K_{l+1}-free graphs which are not l-colorable, is logarithmically small.

5.2.1 The algorithm

The key idea of the algorithm is to partition the class $\mathcal{F}orb_n(K_{l+1})$ of $K_{\ell+1}$-free graphs into subclasses. The first and largest subclass $\mathcal{H}_n$ consists of l-colorable graphs which satisfy some additional structural requirements so that they can be l-colored using an $O(n^2)$ algorithm. Unfortunately, there are too many graphs left in $\mathcal{F}orb_n(K_{l+1}) \setminus \mathcal{H}_n$, to color all of them by complete enumeration without spoiling the linear expected running time. Therefore, in a next step, we therefore split off a "big" subclass $\mathcal{I}_n$ of $K_{\ell+1}$-free graphs from these remaining graphs which still satisfy some structural requirements. These properties of the graphs in $\mathcal{I}_n$ will enable us to color every graph in this class in subexponential time, provided its chromatic number is sufficiently small. The remaining step, then, is to show that the number of graphs in $\mathcal{I}_n$ having a big chromatic number and the number of graphs outside of $\mathcal{H}_n \cup \mathcal{I}_n$ is sufficiently small such that an optimal coloring of them can be determined by complete enumeration.

In order to make this procedure more precise, we start with some notations and definitions. In all probabilistic statements in this section we assume uniform distribution on the class $\mathcal{F}orb_n(K_{l+1})$. In particular this implies that $Prob[A(n)] = |A(n)|/|\mathcal{F}orb_n(K_{l+1})|$ for every set $A(n) \subseteq \mathcal{F}orb_n(K_{l+1})$.

We say that a graph $G = (V_n, E)$ has the *clique property*, if for every k, $1 \leq k \leq l-1$ every k-clique S_k satisfies that $\{v \in V_n \mid S_k \subseteq \Gamma(v)\} \neq \emptyset$.

Two vertices v and x of G are connected via a *spindle*, if there are vertices $w_1, \ldots, w_{l-1}$ such that v and x are different from the w_i's, the w_i's form a clique of size $l-1$ and they are as well in the neighborhood of v as in the neighborhood of x. Observe that in any l-coloring of G the vertices v and x have to be colored the same.

If $S_l = \{v_1, \ldots, v_l\}$ is an l-clique in G then we denote by $SP(v_i)$ the set of all vertices $x \in V_n$ which are connected to v_i via a spindle. G is said to be *forcible l-colorable* if for every l-clique $S_l = \{v_1, \ldots, v_l\}$ we have that $\bigcup_{i=1}^{l} SP(v_i) = V_n$, that $SP(v_i) \cap SP(v_j) = \emptyset$ for every pair $1 \leq i < j \leq l$, and that $SP(v_i)$ is stable for every $i = 1, \ldots, l$. Thus, in a forcible l-colorable graph G the spindle sets $SP(v_i)$ are exactly the color–classes of the uniquely l-colorable graph G.

Observe that a graph which has the clique-property and which is forcible l-colorable can be l-colored in polynomial time: simply choose any vertex, extend it to an l-clique, and determine the spindle sets of this l-clique by enumerating all subsets of size $l-1$. While, according to our assumption that l is fixed, this procedure requires only polynomially many steps, we would like to speed it up to an $O(n^2)$ algorithm. For this purpose we need more structural information of the graph.

We say that G is *linearly forcible l-colorable* if it is forcible l-colorable and, in addition, for every pair of vertices x, y which are contained in the same color–class every k-clique S_k, for $0 \leq k \leq l-2$, which is contained in $\Gamma(x) \cap \Gamma(y)$ can be extended to a $k+1$-clique in $\Gamma(x) \cap \Gamma(y)$, i.e. if $\{v \in V_n \mid v \in \Gamma(x) \cap \Gamma(y) \text{ and } S_k \subseteq \Gamma(v)\} \neq \emptyset$. Observe that a linearly forcible l-colorable graph which has the clique property can really be colored in $O(n^2)$ many steps.

Let $g(n)$ be a positive integral function and $G = (V_n, E)$ be a uniquely l- colorable graph. We say that G has the *$g(n)$-intersection property* if every pair of sets S and T, sharing the properties that each set is completely contained in one color–class and each set is of size at least $g(n)$, satisfies $\Gamma(S) \cap T \neq \emptyset$. Observe that the smaller $g(n)$ the stronger the $g(n)$-intersection property.

The following structure theorem for K_{l+1}-free graphs exhibits the connections between the properties discussed above and the fraction of K_{l+1}-free graphs which do not have these properties.

Theorem 5.2.1 *Let $f(n)$ and $g(n)$ be integral functions such that for all n which are sufficiently large we have that $1 \leq f(n) \leq n/(\log n)^3$ and that $n/log^{(l^2+1)}n \leq g(n) \leq n/(l^4 2^l)$. Then there exists $\mathcal{X}_n(f,g) \subseteq \mathcal{F}orb_n(K_{l+1})$ such that*

$$Prob[\mathcal{X}_n(f,g)] = 2^{-\Omega(f(n)g(n))}$$

and every $G = (V_n, E)$ in $\mathcal{F}orb_n(K_{l+1}) \setminus \mathcal{X}_n(f,g)$ has the following property:

There exists a set $X \subseteq V_n$ of size $|X| \leq f(n) - 1$ such that

- *$G[V_n \setminus X]$ has the clique property,*
- *$G[V_n \setminus X]$ is linearly forcible l-colorable, and*
- *$G[V_n \setminus X]$ has the $g(n)$-intersection property.*

Note that choosing $g(n) = \frac{1}{l^4 2^l} \cdot n$ and $f(n) \equiv 1$ we obtain in particular that almost every K_{l+1}-free graph is uniquely l-colorable. In fact we get even more, viz. that the fraction of K_{l+1}-free graphs which are not l-colorable is logarithmically small.

While Theorem 5.2.1 is essential for proving the linear running time of our algorithm, we defer its proof to the next section. In the remaining part of this section we will describe the algorithm and deduce that its expected running time is linear.

By $\mathcal{H}_n$ we denote the set of all graphs in $\mathcal{F}orb_n(K_{l+1})$ which have the clique property and which are linearly forcible l-colorable. As already outlined above, it is not difficult to design a procedure $COLOR\mathcal{H}$ which colors every graph in $\mathcal{H}_n$ with l colors and whose running time is bounded by $O(n^2)$. Observe that from Theorem 5.2.1 we obtain immediately the following fact.

Corollary 5.2.2 $$\text{Prob}\,[G \notin \mathcal{H}_n] = 2^{-\Omega(n)}.$$ □

By $\mathcal{I}_n$ we denote the set of all those graphs $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \mathcal{H}_n$ for which there exists a set $X(G) \subseteq V_n$ of "bad" vertices of size at most $\log n - 1$ such that $G[V_n \setminus X(G)]$ has the clique property, is linearly forcible l-colorable and has the $\lceil \frac{n}{\log^{(l^2+1)} n} \rceil$-intersection property. We will design a procedure $STRUCTURE$ which decides for every $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \mathcal{H}_n$ whether it has these structural properties and which has a running time bounded by $2^{o(n)}$. Again using Theorem 5.2.1, we obtain the following bound on the size of $\mathcal{F}orb_n(K_{l+1}) \setminus (\mathcal{H}_n \cup \mathcal{I}_n)$. For notational convenience we put from now on $q = l^2 + 1$.

Corollary 5.2.3 $$\text{Prob}\,[G \notin (\mathcal{H}_n \cup \mathcal{I}_n)] = 2^{-\Omega\left(\frac{n \cdot \log n}{\log^{(q)} n}\right)}.$$ □

It is not difficult to design an algorithm $COLOR\mathcal{F}$ which colors every graph in $\mathcal{F}orb_n(K_{l+1})$ with its chromatic number of colors - in fact even every graph on n vertices - and which has a running time that is exponentially bounded. For example, Lawler (1976) described such an algorithm having running time $O(n^3 \cdot 2.445^n)$. Hence, every graph in $\mathcal{F}orb_n(K_{l+1}) \setminus (\mathcal{H}_n \cup \mathcal{I}_n)$ can be colored by $COLOR\mathcal{F}$ without spoiling the linear expected running time.

In order to be able to color the graphs in $\mathcal{I}_n$ sufficiently fast, we have to split this class into two subclasses, say $\mathcal{J}_n$ and $\mathcal{K}_n$, where

$$\begin{aligned} \mathcal{J}_n &= \{G \in \mathcal{I}_n \mid \chi(G) \leq \log^{(q+2)} n\} \\ \mathcal{K}_n &= \mathcal{I}_n \setminus \mathcal{J}_n. \end{aligned}$$

We will exhibit a procedure $COLOR\mathcal{J}$ which colors every graph in $\mathcal{J}_n$ with its chromatic number of colors and which has running time $2^{o(n)}$. Recalling Corollary 5.2.2, this is obviously sufficiently fast. Then, concerning the class $\mathcal{K}_n$, it will be shown that this class is subexponentially small, i.e. the following corollary:

Corollary 5.2.4 $$\text{Prob}\,[G \in \mathcal{K}_n] = 2^{-\Omega(n \cdot \log^{(q+2)} n)}.$$ □

Thus, every graph in $\mathcal{K}_n$ can be colored by $COLOR\mathcal{F}$ without destroying the linear expected running time. In order to verify Corollary 5.2.4 we prove the following more general consequence of Theorem 5.2.1.

Lemma 5.2.5 *Let $h(n)$ be a function such that $h(n) \geq l$ for every n and let $G \in \mathcal{F}orb_n(K_{l+1})$. Then we have that*

$$Prob[\chi(G) > h(n)] = 2^{-\Omega(n \cdot h(n))}.$$

Proof. Let $g(n) = n/(l^4 2^l)$ and $f(n) = h(n) - l + 1$. Then by Theorem 5.2.1 there exists a subset $\mathcal{X}_n(f,g) \subseteq \mathcal{F}orb_n(K_{l+1})$ such that $Prob[\mathcal{X}_n(f,g)] = 2^{-\Omega(n \cdot f(n))}$ and such that every graph in $\mathcal{F}orb_n(K_{l+1}) \setminus \mathcal{X}_n(f,g)$ has chromatic number at most $l + f(n) - 1 = h(n)$. Obviously, this proves Lemma 2.5 and hence Corollary 5.2.4. □

Before we will start discussing the procedures $COLOR\mathcal{H}$, $STRUCTURE$ and $COLOR\mathcal{J}$ in detail, we briefly summarize the algorithm $COLOR$ which finds an optimal coloring for every $K_{\ell+1}$-free graph.

ALGORITHM COLOR
Input: a graph $G \in \mathcal{F}orb_n(K_{l+1})$;
Output: a coloring of G with $\chi(G)$ colors;
begin
  $COLOR\mathcal{H}(G)$;
  if $COLOR\mathcal{H}(G)$ fails **then**
    $STRUCTURE(G)$;
    if $STRUCTURE(G)$ successful **then**
      $COLOR\mathcal{J}(G)$;
      if $COLOR\mathcal{J}(G)$ fails **then**
        $COLOR\mathcal{F}(G)$;
      end if
    else $COLOR\mathcal{F}(G)$;
    end if
  end if
end.

The key for the linear expected running time of the algorithm $COLOR$ is that either the whole graph (in $\mathcal{H}_n$) or at least a big portion (in $\mathcal{J}_n$) is linearly forcible l-colorable. Let $G \in \mathcal{F}orb_n(K_{l+1})$, $Y \subseteq V_n$ be a set of vertices and $S_l = \{x_1, \ldots, x_l\}$ be an l-clique in $V_n \setminus Y$. A family $SP(x_i) \subseteq V_n \setminus Y$, $i = 1, \ldots, l$, such that for every i every $y \in SP(x_i)$ is connected to x_i via a spindle through $V_n \setminus Y$, that $\bigcup_{i=1}^{l} SP(x_i) = V_n \setminus Y$, that each $SP(x_i)$ is stable, and that for each pair $1 \leq i < j \leq l$ we have that $SP(x_i) \cap SP(x_j) = \emptyset$, is called a *$S_l$-spindle family* for $G[V_n \setminus Y]$.

Of course, every spindle family for $G[V_n \setminus Y]$ yields a proper l-coloring of $G[V_n \setminus Y]$. The following procedure $SPINDLE$ is designed to find spindle families.

PROCEDURE SPINDLE

Input: a graph $G \in \mathcal{F}orb_n(K_{l+1})$, a subset $Y \subseteq V_n$, and an l-clique $S_l = \{x_1, \ldots, x_l\}$ in $V_n \setminus Y$;
Output: either an S_l-spindle family for $G[V_n \setminus Y]$ or $SPINDLE$ stops with failure;
begin
    for every vertex $x \in S_l$ **do**
        $SP(x) := \emptyset$;
        for every vertex $y \in V_n \setminus (Y \cup \Gamma(x))$ **do**
            if there exists $z \in (\Gamma(x) \cap \Gamma(y)) \setminus Y$ **then**
                $Cl := \{z\}$;
                for every $w \in V_n$ **do**
                    if w is connected to all $v \in Cl \cup \{x, y\}$ **then**
                        $Cl := Cl \cup \{w\}$;
                    end if
                end for
            end if
            if Cl is an $(l-1)$-clique **then**
                $SP(x) := SP(x) \cup \{y\}$;
            end if
        end for
    end for
    if each $SP(x_i), i = 1, \ldots, l$, is stable and $SP(x_i) \cap SP(x_j) = \emptyset$ for every pair $1 \leq i < j \leq l$, and $\bigcup_{i=1}^{l} SP(x_i) = V_n \setminus Y$ **then**
        output $SP(x_i), i = 1, \ldots, l$, **stop**
    else stop with failure;
    end if
end.

The following observation is straightforward.

Lemma 5.2.6 *The procedure $SPINDLE$ terminates successfully whenever $G[V_n \setminus Y]$ is linearly forcible l-colorable. The complexity of the procedure $SPINDLE$ is bounded by $O(n^2)$.*

□

We will use procedure $SPINDLE$ as a subroutine to l-color every graph in $\mathcal{H}_n$.

PROCEDURE COLOR$\mathcal{H}$

Input: a graph $G \in \mathcal{F}orb_n(K_{l+1})$;
Output: either an l-coloring of G or $COLOR\mathcal{H}$ stops with failure;
begin
    choose $v \in V_n$, $Cl := \{v\}$;
    for every vertex $w \in V_n$ **do**
        if w is connected to all $x \in Cl$ **then**
            $Cl := Cl \cup \{w\}$;
        end if
    end for

if Cl is an l-clique **then**
$SPINDLE(G, \emptyset, Cl)$;
if $SPINDLE(G, \emptyset, Cl)$ terminates successfully **then**
output spindle family as l-coloring of G, **stop**;
else stop with failure;
else stop with failure;
end if
end.

The following lemma is obvious by definition of $\mathcal{H}_n$ and by Lemma 5.2.6.

Lemma 5.2.7 *The procedure $COLOR\mathcal{H}$ terminates successfully for every graph G in $\mathcal{H}_n$. The complexity of $COLOR\mathcal{H}$ is bounded by $O(n^2)$.*

□

Let $G \in \mathcal{I}_n$. Then there exists a set $X(G) \subseteq V_n$ of "bad" vertices of size at most $\log n - 1$ such that the graph $G[V_n \setminus X(G)]$ has the clique property, is linearly forcible l-colorable and has the $\lceil \frac{n}{\log^{(q)} n} \rceil$-intersection property. The procedure $STRUCTURE$ simply checks for all potentially "bad" sets $X \subseteq V_n$ whether $G \setminus X$ has the desired properties.

PROCEDURE STRUCTURE

Input: a graph $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \mathcal{H}_n$;
Output: either a set $X \subseteq V_n$ of size at most $\log n - 1$,
a clique $S_l = \{x_1, \ldots, x_l\}$ in $V_n \setminus X$, and
an S_l-spindle family $SP(x_i)$, $i = 1, \ldots, l$ for $G[V_n \setminus X]$
with $\lceil \frac{n}{\log^{(q)} n} \rceil$-intersection property
or $STRUCTURE$ stops with failure;
begin
for every $X \subseteq V_n$ of size at most $\log n - 1$ **do**
choose $w \in V_n \setminus X$; $Cl := \{w\}$;
for every $x \in V_n \setminus X$ **do**
if x is connected to all $y \in Cl$ **then**
$Cl := Cl \cup \{x\}$;
end if
end for
if Cl is an l-clique **then**
$SPINDLE(G, X, Cl)$;
if $SPINDLE(G, X, Cl)$ terminates successfully with
Cl-spindle family $SP(x_i)$, $i = 1, \ldots, l$ **then**
if for every pair $1 \leq i < j \leq l$ and for every pair $S \subseteq SP(x_i)$
and $T \subseteq SP(x_j)$ with $|S|, |T| = \lceil \frac{n}{\log^{(q)} n} \rceil$ it is true that
$\Gamma(S) \cap T \neq \emptyset$ **then**
output set X, l-clique $Cl = \{x_i, \ldots, x_l\}$,
Cl-spindle family $SP(x_i)$, $i = 1, \ldots, l$, **stop**;
end if
end if

end for
stop with failure;
end.

The following lemma is obvious by the definition of the class $\mathcal{I}_n$ and the construction of algorithm $STRUCTURE$.

Lemma 5.2.8 *The procedure $STRUCTURE$ terminates successfully for every graph in $\mathcal{I}_n$.*

□

In the next lemma, which bounds the complexity of the procedure $STRUCTURE$, we will use the following elementary inequality. Let k be a positive integer and n be sufficiently large, then

$$\frac{n}{\log^{(k)} n} \cdot \binom{n}{\frac{n}{\log^{(k)} n}} \; < \; 2^{n/\log^{(k+1)} n} \; .$$

This inequality will tacitly be used several times in this paper.

Lemma 5.2.9 *The complexity of the procedure $STRUCTURE$ is bounded by $2^{o(n)}$.*

Proof. There are at most

$$\log n \cdot \binom{n}{\log n} \leq 2^{(\log n)^2}$$

many sets X to consider. For each such set the number of steps for finding a clique is at most $O(n)$. Then procedure $SPINDLE$ is performed once, which takes at most $O(n^2)$ many steps. Finally, it is checked whether the spindle family obtained by procedure $SPINDLE$ has the $\lceil \frac{n}{\log^{(q)} n} \rceil$- intersection property. This requires at most

$$\binom{n}{\lceil \frac{n}{\log^{(q)} n} \rceil}^2 \cdot n \cdot \left\lceil \frac{n}{\log^{(q)} n} \right\rceil = 2^{O(\frac{n}{\log^{(q+1)} n})}$$

many steps. Together this shows that the complexity of $STRUCTURE$ is bounded by $2^{o(n)}$. □

The next step is to color every graph in $\mathcal{J}_n$ properly, i.e. to color every graph for which the procedure $STRUCTURE$ has terminated successfully and which in addition has chromatic number at most $\log^{(q+2)} n$. The procedure $COLOR\mathcal{J}$ is straightforward and its efficiency is based on the fact that every graph $G \in \mathcal{J}_n$ has the $\lceil \frac{n}{\log^{(q)} n} \rceil$-intersection property. This implies that in every proper coloring of G every color cannot be used more than $\lceil \frac{n}{\log^{(q)} n} \rceil$ many times in more than one of the spindle sets. So $COLOR\mathcal{J}$ just enumerates all such potential colorings by testing all possible assignments of "major" colors to spindle sets, all colorings of vertices by "minor" colors, and all colorings of the vertices in X. We omit a detailed description of this procedure.

Lemma 5.2.10 *The procedure $COLOR\mathcal{J}$ terminates successfully for every graph in $\mathcal{J}_n$.*

□

Lemma 5.2.11 *The complexity of procedure $COLOR\mathcal{J}$ is bounded by $2^{o(n)}$.*

Proof. There are at most $(l+1)^{\log^{(q+2)} n}$ possibilities to associate "major" colors to the spindle sets. Moreover, the number of possible families of "minor" colors is bounded from above by

$$\left(\left\lceil \frac{n}{\log^{(q)} n} \right\rceil \cdot \binom{n}{\lceil \frac{n}{\log^{(q)} n} \rceil}\right)^{l\cdot\log^{(q+2)} n} = 2^{O(\frac{n\cdot\log^{(q+2)} n}{\log^{(q+1)} n})}.$$

Finally, the number of possible colorings of X is at most

$$(\log^{(q+2)} n)^{\log n - 1}.$$

Having chosen such a partial coloring the number of steps to check whether it can be extended to a proper coloring is easily seen to be bounded from above by $O(n^2)$. Together we get that the complexity of $COLOR\mathcal{J}$ is bounded by $2^{o(n)}$. □

Theorem 5.2.12 *The algorithm COLOR determines for every $G \in \mathcal{F}orb_n(K_{l+1})$ a coloring with $\chi(G)$ colors. The expected running time of the algorithm COLOR is $O(n^2)$.*

Proof. The first part of Theorem 5.2.12 follows immediately from Lemma 5.2.7, Lemma 5.2.10 and the definition of the algorithm $COLOR$.
In order to verify the second part, let $R_{exp}(COLOR)$ denote the expected running time of $COLOR$ and for an algorithm A let $R_{max}(A)$ denote the worst case running time of A. Then $R_{exp}(COLOR)$ can obviously be bounded from above by

$$\begin{aligned} R_{exp}(COLOR) &\le R_{max}(COLOR\mathcal{H}) \\ &+(R_{max}(STRUCTURE) + R_{max}(COLOR\mathcal{J})) \cdot Prob[G \notin \mathcal{H}_n] \\ &+R_{max}(COLOR\mathcal{F})(Prob[G \in \mathcal{K}_n] + Prob[G \notin \mathcal{H}_n \cup \mathcal{I}_n]). \end{aligned}$$

By Lemma 2.7 we know that the first summand is bounded by $O(n^2)$. From Lemma 5.2.9, Lemma 5.2.11 and Corollary 5.2.2 we derive that the second summand tends to 0. Finally, from Corollary 5.2.3, Corollary 5.2.4 together with the observation that the running time of $COLOR\mathcal{F}$ can be bounded by $O(n^3 \cdot 2.445^n)$ it follows that the third summand vanishes as well. Hence, $R_{exp}(COLOR) = O(n^2)$ completing the proof of Theorem 5.2.12. □

It is worthwhile to observe that the algorithm $COLOR$ can easily be converted to a parallel algorithm which colors every K_{l+1}-free graph with a minimum number of colors, which uses only polynomially many processors, and has an expected running time which is bounded by a polynomial in the logarithm of the size of the graph.

5.2.2 Proof of the structure theorem

The proof of the Structure Theorem 5.2.1 is based on the Kleitman-Rothschild method (cf. Theorem 4.2.3). Here we need a slight generalization.

Theorem 5.2.13 (THE KLEITMAN-ROTHSCHILD METHOD – VERSION II) *Let k be a positive integer, let $\mathcal{E}(n)$ and $\mathcal{F}(n)$ and $\mathcal{Z}_i(n)$, $i = 1, \dots, k$ be k+2 families of graphs on n vertices and let $\alpha, \beta > 0$ be reals such that for all n sufficiently large we have that*

$$\log \frac{|\mathcal{F}(n)|}{|\mathcal{F}(n+1)|} \leq -\alpha \cdot n + \beta, \quad \textit{and} \tag{1}$$

$$\mathcal{F}(n) \subseteq \mathcal{E}(n) \cup \bigcup_{i=1}^{k} \mathcal{Z}_i(n). \tag{2}$$

Moreover, let $x_i(n)$, $y_i(n)$ for $i = 1, \dots, k$ be positive monotone ascending functions satisfying $1/x_i(n) = o(1)$ such that for n sufficiently large we have that

$$1 \ \leq \ y_i(n) \leq x_i(n)/2\alpha \ \leq \ n \tag{3}$$

$$[1ex] \log \frac{|\mathcal{Z}_i(n)|}{|\mathcal{F}(n - y_i(n))|} \leq (\alpha \cdot n - x_i(n)) \cdot y_i(n) \quad \textit{for every}\, i = 1, \dots, k. \tag{4}$$

Then there exist $c \geq 1$ and $\gamma > 1$ such that the following inequality holds for every $n \in \mathbb{N}$:

$$\left(1 - c \cdot \gamma^{-x(n) \cdot y(n)}\right) \cdot |\mathcal{F}(n)| \ \leq \ |\mathcal{E}(n)| \,,$$

where $x(n) = min\{\, x_i(n) |\, i = 1, \dots, k\}$ and $y(n) = min\{\, y_i(n) |\, i = 1, \dots, k\}$.

Before showing how this result can be used to derive Theorem 5.2.1 we prove Theorem 5.2.13.

Proof of Theorem 5.2.13. Put $\gamma = \sqrt[3]{2} > 1$ and choose n_0 sufficiently large such that (1) - (4) are fulfilled for all $n \geq n_0$. Let $n_1 \geq n_0$ be such that

$$-\frac{x_i(n)}{4} + \frac{\alpha}{2} + \beta \ \leq \ 0 \qquad \text{for every} \quad i = 1, \dots, k \quad \text{and for every} \quad n \geq n_1$$

and

$$2^{-x(n) \cdot y(n)/2} \ \leq \ \frac{1}{k} \cdot 2^{-x(n) \cdot y(n)/3} \qquad \text{for every} \quad n \geq n_1.$$

Finally, choose $c \geq 1$ sufficiently large such that

$$\left(1 - c \cdot \gamma^{-x(n) \cdot y(n)}\right) \cdot |\mathcal{F}(n)| \ \leq \ |\mathcal{E}(n)| \qquad \text{for every} \quad n \leq n_1.$$

By (2) we know that

$$|\mathcal{F}(n)| - \sum_{i=1}^{k} |\mathcal{Z}_i(n)| \ \leq \ |\mathcal{E}(n)|.$$

Thus, it suffices to show that

$$\frac{|\mathcal{Z}_i(n)|}{|\mathcal{F}(n)|} \le \frac{c}{k} \cdot \gamma^{-x(n)\cdot y(n)} \qquad \text{for every } i.$$

Using (4) and (1) we obtain that

$$\begin{aligned} \frac{|\mathcal{Z}_i(n)|}{|\mathcal{F}(n)|} &= \frac{|\mathcal{Z}_i(n)|}{|\mathcal{F}(n-y_i(n))|} \cdot \prod_{j=1}^{y_i(n)} \frac{|\mathcal{F}(n-j)|}{|\mathcal{F}(n+1-j)|} \\ &\le 2^{(\alpha\cdot n - x_i(n))\cdot y_i(n)} \cdot 2^{\sum_{j=1}^{y_i(n)}(-\alpha\cdot(n-j)+\beta)}. \end{aligned}$$

Using (3) we obtain that

$$2^{\sum_{j=1}^{y_i(n)}(-\alpha\cdot(n-j)+\beta)} \le 2^{-\alpha\cdot n\cdot y_i(n)+(x_i(n)/4+\alpha/2+\beta)\cdot y_i(n)}.$$

Hence, we get for every $n \ge n_1$:

$$\begin{aligned} \frac{|\mathcal{Z}_i(n)|}{|\mathcal{F}(n)|} &\le 2^{-x_i(n)\cdot y_i(n)/2} \cdot 2^{(-x_i(n)/4+\alpha/2+\beta)\cdot y_i(n)} \\ &\le 2^{-x(n)\cdot y(n)/2} \le \frac{1}{k} \cdot \gamma^{-x(n)\cdot y(n)} \\ &\le \frac{c}{k} \cdot \gamma^{-x(n)\cdot y(n)}. \end{aligned}$$

□

Proof of Theorem 5.2.1. Fix two functions $f(n)$ and $g(n)$ which satisfy the conditions of Theorem 5.2.1 and let $h(n)$ be the largest integral function such that $2^{6\cdot l}\cdot(h(n)-1)+1 \le f(n)$. We will define sets $\mathcal{Z}_i(n) = \mathcal{Z}_i(n,g,h) \subseteq \mathcal{F}orb_n(K_{l+1})$ for $i = 1,\ldots,4l-2$, functions $x_i(n)$ and $y_i(n)$, $i = 1,\ldots,4l-2$, and reals $\alpha,\beta > 0$ satisfying the assumptions of Theorem 5.2.13, provided that the set $\mathcal{F}orb_n(K_{l+1})$ of K_{l+1}-free graphs plays the role of $\mathcal{F}(n)$ and $\mathcal{E}(n) \subseteq \mathcal{F}orb_n(K_{l+1})$ denotes the set of those K_{l+1}-free graphs $G = (V_n, E)$ which satisfy the conditions of Theorem 5.2.1. Then observing that $x(n) = \Theta(g(n))$ and $y(n) = \Theta(h(n)) = \Theta(f(n))$ obviously suffices to prove Theorem 5.2.1.
As a first step we determine the constants α and β in Theorem 5.2.13.

Lemma 5.2.14 *There exists a constant $D > 0$ such that for all sufficiently large n we have that*

$$\log \frac{|\mathcal{F}orb_n(K_{l+1})|}{|\mathcal{F}orb_{n+1}(K_{l+1})|} \le -\frac{l-1}{l}\cdot n + D.$$

Proof. The number of l-colorable graphs satisfies (cf. e.g. Section 5.3)

$$|\mathcal{C}ol_l(n)| = \Theta\left(2^{\frac{l-1}{2l}\cdot n^2 + n\cdot\log l - \frac{l-1}{2}\cdot\log n}\right).$$

Moreover from the Kolaitis, Prömel and Rothschild result Theorem 3.2.1 (cf. also Theorem 4.2.2) we know that

$$\lim_{n\to\infty} \frac{|\mathcal{C}ol_l(n)|}{|\mathcal{F}orb_n(K_{l+1})|} = 1.$$

Hence, combining these results we obtain the desired inequality. $\square$

We proceed with some definitions. For $0 \le k < l$ let f_k denote the integral function

$$f_k(n) = \lceil \log^{(k^2-k+2)} n \rceil.$$

A Q_k*-set* is an ordered set $Q = \{v_1, \ldots, v_{kf_k(n)}\} \subseteq V_n$ such that for the sets $P_i = \{v_{(i-1)f_k(n)+1}, \cdots, v_{if_k(n)}\}$, $i = 1, \ldots, k$, the following conditions are satisfied:

(1) $\quad P_i \cap P_j = \emptyset \quad$ for every $1 \le i < j \le k$,

(2) $\quad |P_i| = f_k(n) \quad$ for every $1 \le i \le k$ and

(3) $\quad P_i$ and P_j are completely connected $\quad$ for every pair $1 \le i < j \le k$.

For a Q_k-set $Q = \bigcup_{i=1}^k P_i$ we denote by $T_k(Q) = T_k(P_1, \ldots, P_k)$ the set

$$T_k(Q) = \{v \in V_n \setminus \bigcup_{i=1}^k P_i \mid |\Gamma(v) \cap P_i| \ge \left\lceil \frac{f_k(n)}{\log f_k(n)} \right\rceil \text{ for every } i = 1, \ldots, k\}.$$

Observe that, by definition, a Q_0-set is empty and $T_0(\emptyset) = V_n$. For simplicity a Q_{l-1}-set Q is also called a Q*-set*. The corresponding set $T_{l-1}(Q)$ is also denoted by $R(Q)$ and called the R*-set* of Q.
A Q-set *of a vertex* v is a Q-set contained in the neighborhood $\Gamma(v)$ of v.
Now we start defining, step by step, the sets $\mathcal{Z}_i$ and the functions $x_i(n)$, $y_i(n)$ resp., and prove that they have the required properties.

Lemma 5.2.15 *Let* $1 \le k \le l-1$ *be an integer, let* $x(n) = x_{\mathcal{A}}(n)$ *be a function satisfying*

$$n = o(x(n) \cdot \log^{(2)} f_k(n))$$

and let $y(n) = y_{\mathcal{A}}(n)$ *be an integral function such that for all sufficiently large* n *the following inequality is valid:*

$$1 \le y(n) \le n/\log n.$$

Let $\mathcal{A} = \mathcal{A}(n,k,x,y)$ *denote the set of all graphs* $G \in \mathcal{F}orb_n(K_{l+1})$ *that contain* $y(n)$ *pairwise disjoint* Q_k*-sets* $Q(i)$, $i = 1, \ldots, y(n)$, *such that for all* $i = 1, \ldots, y(n)$ *the sets* $T_k(Q(i))$ *satisfy*

$$|T_k(Q(i))| \le \frac{l-k}{l} n - x(n).$$

Then the following inequality is valid for all sufficiently large n*:*

$$\log \frac{|\mathcal{A}|}{|\mathcal{F}orb_{n-ky(n)f_k(n)}(K_{l+1})|} \quad \le \quad \left(\frac{l-1}{l} \cdot n - \frac{x(n)}{2k}\right) \cdot k \cdot y(n) \cdot f_k(n).$$

Proof. Construct all graphs $G \in \mathcal{A}$ as follows. First choose $y(n)$ pairwise disjoint sets $Q(i) = \bigcup_{j=1}^{k} P_j(i) \subseteq V_n$ and edges therein. For this one has at most

$$\binom{n}{f_k(n)}^{k \cdot y(n)} \cdot 2^{\binom{f_k(n)}{2} \cdot k \cdot y(n)} < 2^{k \cdot y(n) \cdot f_k(n) \log n + k \cdot y(n) \cdot f_k(n)^2}$$

possibilities. Then choose an appropriate K_{l+1}-free graph on the remaining vertices (in at most $|\mathcal{F}orb_{n-ky(n)f_k(n)}(K_{l+1})|$ ways). Next we connect every $Q(i)$, $i = 1, \ldots, y(n)$, with the remaining vertices. First choose the set $T_k(Q(i))$ and connect $Q(i)$ with $T_k(Q(i))$ (in at most $2^n \cdot 2^{k \cdot f_k(n) \cdot |T_k(Q(i))|}$ many ways). Finally, to connect $Q(i)$ with $V_n \setminus (Q(i) \cup T_k(Q(i)))$ we choose for every $x \in V_n \setminus (Q(i) \cup T_k(Q(i))$ a set $P_{j_0}(i)$ which contains less than $\left\lceil \frac{f_k(n)}{\log f_k(n)} \right\rceil$ neighbors of x. Then connect x to the remaining $P_j(i)$ arbitrarily. Hence, there are at most

$$\left[k \cdot \frac{f_k(n)}{\log f_k(n)} \cdot \binom{f_k(n)}{\frac{f_k(n)}{\log f_k(n)}} \cdot 2^{(k-1) \cdot f_k(n)} \right]^{(n - |T_k(Q(i))| - k \cdot f_k(n))}$$

many possibilities to connect $Q(i)$ to $V_n \setminus (Q(i) \cup T_k(Q(i)))$. Putting things together and using the abbreviation $|T| = \max\{|T_k(Q(i))| \mid i = 1, \ldots, y(n)\}$ we get for all sufficiently large n:

$$\begin{aligned}
&\log \frac{|\mathcal{A}|}{|\mathcal{F}orb_{n-ky(n)f_k(n)}(K_{l+1})|} \\
&\leq \; k \cdot y(n) \cdot f_k(n) \cdot \log n + k \cdot y(n) \cdot f_k(n)^2 + y(n) \cdot \Big[n + k \cdot f_k(n) \cdot |T| \\
&\quad + (\log k + \frac{f_k(n)}{\log^{(2)} f_k(n)} + (k-1) \cdot f_k(n)) \cdot (n - |T| - k \cdot f_k(n)) \Big] \\
&\leq \; y(n) \cdot \left[n \cdot (k-1) \cdot f_k(n) + |T| \cdot f_k(n) + \frac{1}{2} \cdot x(n) \cdot f_k(n) \right] \\
&\leq \; y(n) \cdot f_k(n) \cdot \left[n \cdot (k-1) + \frac{l-k}{l} \cdot n - \frac{1}{2} \cdot x(n) \right] \\
&\leq \; \left(\frac{l-1}{l} \cdot k \cdot n - \frac{1}{2} \cdot x(n) \right) \cdot y(n) \cdot f_k(n).
\end{aligned}$$

□

Put $x_{\mathcal{A}}(n) = 1/3 \cdot g(n)$ and $y_{\mathcal{A}}(n) = \lceil h(n)/f_k(n) \rceil$. Then by choice of $g(n)$ and $h(n)$ these functions fulfill the requirements of Lemma 5.2.15. Now define $\mathcal{Z}_{2k}(n)$, $x_{2k}(n)$ and $y_{2k}(n)$ for $k = 1, \ldots, l-1$ as follows:

$$\begin{aligned}
\mathcal{Z}_{2k}(n) &= \mathcal{A}(n, k, x_{\mathcal{A}}(n), y_{\mathcal{A}}(n)) \\
x_{2k}(n) &= \frac{1}{2k} \cdot x_{\mathcal{A}}(n)
\end{aligned}$$

$$y_{2k}(n) \quad = \quad k \cdot f_k(n) \cdot y_{\mathcal{A}}(n)$$

Obviously, $x_{2k}(n) = \Theta(g(n))$, $y_{2k}(n) = \Theta(h(n))$ for every $k = 1, \ldots, l-1$, and condition (3) and condition (4) of Theorem 5.2.13 are fulfilled.

Lemma 5.2.16 *Let $0 \le k \le l-1$ be an integer, let $x = x_{\mathcal{B}}$ be a function and let $y = y_{\mathcal{B}}$ be a positive integral function satisfying*

$$y(n) = o\left(\frac{x(n)}{f_k(n) \cdot \log n}\right).$$

Let $\mathcal{B} = \mathcal{B}(n,k,x,y)$ denote the set of all graphs $G \in \mathcal{F}orb_n(K_{l+1})$ that contain $y(n)$ pairwise distinct vertices $v_1, \ldots, v_{y(n)}$ such that there exist Q_k-sets $Q(i), i = 1, \ldots, y(n)$, so that for every $i = 1, \ldots, y(n)$ the set $T_k(Q(i))$ satisfies

$$|T_k(Q(i))| \;\ge\; \frac{1}{l} \cdot n + x(n)$$

and

$$|T_k(Q(i)) \cap \Gamma(v_i)| \;\le\; f_k(n) \cdot (1 + y(n)).$$

Then the following inequality is valid for all sufficiently large n:

$$\log \frac{|\mathcal{B}|}{|\mathcal{F}orb_{n-y(n)}(K_{l+1})|} \quad \le \quad \left(\frac{l-1}{l} \cdot n - \frac{x(n)}{2}\right) \cdot y(n).$$

Proof. Construct all graphs $G \in \mathcal{B}$ as follows. Choose vertices $v_1, \ldots, v_{y(n)}$ and a suitable graph on $V_n \setminus \{v_1, \ldots, v_{y(n)}\}$ (in at most $n^{y(n)} \cdot |\mathcal{F}orb_{n-y(n)}(K_{l+1})|$ ways). For each $i = 1, \ldots, y(n)$ then choose the set $Q(i)$ (in at most $n^{kf_k(n)}$ ways) and connect v_i with $T_k(Q(i))$. For this one has at most

$$f_k(n) \cdot (1 + y(n)) \cdot \binom{n}{f_k(n)(1 + y(n))} \;\le\; n^{f_k(n) \cdot (1+y(n))}$$

possibilities. Finally one has to connect v_i with $V_n \setminus T_k(Q(i))$. This can be done in at most $2^{n-|T_k(Q(i))|}$ ways.
Putting things together and abbreviating $\min\{|T_k(Q(i))| \mid i = 1, \ldots, y(n)\}$ by $|T|$ we get for all n sufficiently large:

$$\begin{aligned}
&\log \frac{|\mathcal{B}|}{|\mathcal{F}orb_{n-y(n)}(K_{l+1})|} \\
&\le \quad y(n) \cdot \log n + y(n) \cdot \Big[k \cdot f_k(n) \cdot \log n + f_k(n) \cdot (1 + y(n)) \cdot \log n \\
&\qquad\quad + n - |T|\Big] \\
&\le \quad y(n) \cdot \left[\frac{l-1}{l} \cdot n + \log n + (k+1) \cdot f_k(n) \cdot \log n\right.
\end{aligned}$$

$$\left. +y(n) \cdot f_k(n) \cdot \log n - x(n) \right]$$

$$\leq \left(\frac{l-1}{l} \cdot n - \frac{x(n)}{2} \right) \cdot y(n).$$

□

Put $x_{\mathcal{B}}(n) = g(n)/3$ and $y_{\mathcal{B}}(n) = h(n)$. Then by choice of $g(n)$ and $h(n)$ these functions fulfill the requirements of Lemma 5.2.16. We define $\mathcal{Z}_{2k+1}$, x_{2k+1} and y_{2k+1} for $k = 0, \ldots, l-1$ as follows:

$$\begin{aligned} \mathcal{Z}_{2k+1}(n) &= \mathcal{B}(n, k, x_{\mathcal{B}}(n), y_{\mathcal{B}}(n)) \\ x_{2k+1}(n) &= \frac{1}{2} \cdot x_{\mathcal{B}}(n) \\ y_{2k+1}(n) &= y_{\mathcal{B}}(n) \end{aligned}$$

Obviously, $x_{2k+1}(n) = \Theta(g(n))$, $y_{2k+1} = \Theta(h(n))$ for every $k = 0, \ldots, l-1$, and condition (3) and condition (4) of Theorem 5.2.13 are fulfilled.
Next we derive some properties of graphs not contained in $\bigcup_{i=1}^{2l-1} \mathcal{Z}_i(n)$. So let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{2l-1} \mathcal{Z}_i(n)$. Then by definition of $\mathcal{Z}_1(n)$ there are at most $h(n) - 1$ vertices which satisfy the definition of the corresponding $\mathcal{B}$-set, i.e. which have at most $f_0(n) \cdot (1 + h(n))$ many neighbors. We denote these "bad" vertices by $B_1(G) \subseteq V(G)$. Similarly, by the definition of $\mathcal{Z}_2(n)$ there are at most $\left\lceil \frac{h(n)}{f_1(n)} \right\rceil - 1$ many pairwise disjoint Q_1-sets in G which satisfy the conditions of the corresponding $\mathcal{A}$-set. We denote the vertices which are contained in these "bad" Q_1-sets by $A_2(G) \subseteq V(G)$. Then $|A_2(G)| \leq \left(\left\lceil \frac{h(n)}{f_1(n)} \right\rceil - 1 \right) \cdot f_1(n) < h(n)$ and hence, recalling that $h(n)$ is an integral function, $|A_2(G)| \leq h(n) - 1$. Continuing this in this way gives sets $B_1, B_3, \ldots, B_{2l-1}$ and $A_2, A_4, \ldots, A_{2l-2}$ each of size at most $h(n) - 1$. Now let $A(G) = \bigcup_{i=1}^{l-1} A_{2i}(G)$ and $B(G) = \bigcup_{i=0}^{l-1} B_{2i+1}(G)$ and put $X_{\mathcal{B}}(G) = A(G) \cup B(G)$. Then obviously $|X_{\mathcal{B}}(G)| \leq (2l-1) \cdot (h(n) - 1)$. Note that $X_{\mathcal{B}}(G)$ is not necessarily uniquely defined.
Let $Q = \bigcup_{i=1}^{l-1} P_i$ be a Q-set. Recall that the R-set of Q is defined as $R(Q) = T_{l-1}(Q)$ and that $Q \subseteq \Gamma(v)$, if Q is a Q-set of a vertex v.

Corollary 5.2.17 *Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{2l-1} \mathcal{Z}_i(n)$ and let $v \in V_n \setminus X_{\mathcal{B}}(G)$ be a vertex with Q-set $Q \subseteq V_n \setminus X_{\mathcal{B}}(G)$. Then we have for all sufficiently large n the following inequality:*

$$\frac{n}{l} - \frac{g(n)}{3} \leq |R(Q)| \leq \frac{n}{l} + \frac{g(n)}{3}.$$

Proof. The assertion follows immediately from the facts that $G \notin \mathcal{Z}_{2l-2}(n) \cup \mathcal{Z}_{2l-1}(n)$, that Q is disjoint from A_{2l-2} and from B_{2l-1}, and that v cannot be connected to any vertex in $R(Q)$. □

Let $z(n, t)$ denote the maximal number of edges a bipartite graph on $n \times n$ vertices can have without containing a complete bipartite graph on $t \times t$ vertices as a subgraph.

Znám [1963] proved that for every $t = 2, \ldots, n-1$

$$z(n,t) \;\leq\; (t-1)^{1/t} \cdot n^{2-1/t} + \frac{t-1}{2} \cdot n.$$

For a proof and for more background on these Zarankiewicz–numbers see, e.g., Bollobás [1978].

Lemma 5.2.18 *Let h be given. Then there exists a positive integer n such that the following is true: let G be a graph with vertex–set $P_0 \cup P_1 \cup \ldots \cup P_h$ where the P_i are mutually disjoint sets and $|P_0| = \ldots = |P_h| = n$. Suppose that for $i = 1, \ldots, h$ and each $v \in P_0$ we have*

$$|\Gamma(v) \cap P_i| \;\geq\; \left\lceil \frac{n}{\log n} \right\rceil.$$

Then there are subsets $P_i^{'} \subseteq P_i$, for every $i = 0, \ldots, h$, each of size $\left\lceil \log^{(2h)} n \right\rceil$ such that every vertex $v \in P_0^{'}$ is connected to every vertex in $\bigcup_{i=1}^{h} P_i^{'}$.

The proof relies on the following two facts.

Proposition 5.2.19

$$z\left(n, 2\ \log^{(2)} n\right) < n^{2-\frac{1}{4\log^{(2)} n}},$$

for n sufficiently large.

□

Proposition 5.2.20 *Let $j \geq 0$ be an integer and $\epsilon > 0$ be sufficiently small. Let B be a set of size n and A be a set of size $n^{1-j\epsilon}$ such that every $a \in A$ has at least $\frac{n}{\log n}$ many neighbors in B. Then there exist sets $A' \subset A$ of size $|A'| \geq n^{1-(j+1)\epsilon}$ and $B' \subset B$ of size $|B'| = \log\log n$ such that for every $a \in A'$ we have that*

$$|\Gamma(a) \cap B'| \geq \frac{\log\log n}{\log^{(3)} n}.$$

Proof. In fact we will show that subsets $B^{'} \subseteq B$ of size $\log\log n$ have on average $n^{1-(j+1)\epsilon}$ many vertices $a \in A$ such that $|\Gamma(a) \cap B^{'}| \geq \log\log n / \log^{(3)} n$. Abbreviating put $x := \log\log n / \log^{(3)} n$ and let $a \in A$. Then there are at least

$$\binom{\frac{n}{\log n}}{x} \cdot \binom{n-x}{\log\log n - x} \cdot \binom{\log\log n}{x}^{-1}$$

many sets $B^{'} \subseteq B$ of size $\log\log n$ which intersect the neighborhood of a in at least x vertices. Hence, on average a set $B^{'} \subseteq B$ of size $\log\log n$ intersects at least

$$n^{1-j\epsilon} \cdot \binom{\frac{n}{\log n}}{x} \cdot \binom{n-x}{\log\log n - x} \cdot \binom{\log\log n}{x}^{-1} \cdot \binom{n}{\log\log n}^{-1}$$

many neighborhoods of vertices $a \in A$ each in at least x vertices.
A straightforward calculation shows that this expression is at least $n^{1-(j+1)\epsilon}$, provided that n was chosen sufficiently large. This proves Proposition 5.2.20. $\square$

Using these two facts we prove Lemma 5.2.18 as follows.
Proof of Lemma 5.2.18. For some $j < h$ let $P_0^{'} \subseteq P_0, \dots, P_j^{'} \subseteq P_j, \dots, P_h^{'} \subseteq P_h$ be sets each of size $\log^{(2j)} n$ such that firstly every vertex $v \in P_0^{'}$ is connected to every vertex in $\bigcup_{i=1}^{j} P_i^{'}$ and secondly for every $i = j+1, \dots, h$ every vertex $v \in P_0^{'}$ is connected to at least $\frac{\log^{(2j)} n}{\log^{(2j+1)} n}$ vertices in $P_i^{'}$.
Observe that for $j = 0$ this assumption is fulfilled by the requirements of the lemma. So assume it to be true for some $0 \leq j < h$.
Applying Proposition 5.2.20 successively to $A = P_0^{'}$ and $B = P_{j+2}^{'}, \dots, B = P_h^{'}$ gives some $P_0^{''} \subseteq P_0^{'}$ of size

$$s := \left(\log^{(2j)} n\right)^{1-(h-j-1)\epsilon}$$

and $P_{j+2}^{''} \subseteq P_{j+2}^{'}, \dots, P_h^{''} \subseteq P_h^{'}$ each of size $\log^{(2j+2)} n$ such that for every $v \in P_0^{''}$ and every $i = j+2, \dots, h$ we have that

$$\left|\Gamma(v) \cap P_i^{''}\right| \geq \frac{\log^{(2j+2)} n}{\log^{(2j+3)} n}.$$

Next observe that by averaging one easily obtains a subset $P_{j+1}^{''} \subseteq P_{j+1}^{'}$ of size s such that there are at least $\frac{s^2}{\log^{(2j+1)} n}$ many edges between $P_0^{''}$ and $P_{j+1}^{''}$.
Since

$$\frac{s^2}{\log^{(2j+1)} n} \geq s^{2-1/(4\log^{(2)} s)},$$

we can use Proposition 5.2.19 to find subsets $P_{j+1}^{'''} \subseteq P_{j+1}^{''}$ and $P_0^{'''} \subseteq P_0^{''}$ satisfying $|P_{j+1}^{'''}| = |P_0^{'''}| = 2 \log^{(2)} s$ such that $P_{j+1}^{'''}$ and $P_0^{'''}$ induce a complete bipartite graph.
Observing that

$$2 \log^{(2)} s \geq \log^{(2j+2)} n$$

for n sufficiently large and induction on j completes the proof of Lemma 5.2.18. $\square$

Corollary 5.2.21 *Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{2l-1} \mathcal{Z}_i(n)$, let $Y \subseteq V_n \setminus X_{\mathcal{B}}(G)$ be an arbitrary set of size at most $(2^{6\cdot l} - (2l-1)) \cdot (h(n) - 1)$, and let n be sufficiently large. Then for every $v \in V_n \setminus (X_{\mathcal{B}}(G) \cup Y)$ the set $\Gamma(v) \setminus (X_{\mathcal{B}}(G) \cup Y)$ contains a Q-set Q.*

Proof. Let $v \in V_n \setminus X_{\mathcal{B}}(G)$. Since $G \notin \mathcal{Z}_1(n)$ and recalling that $|X_{\mathcal{B}}(G) \cup Y| \leq 2^{6\cdot l} \cdot (h(n) - 1)$ we know that $\Gamma(v) \setminus (X_{\mathcal{B}}(G) \cup Y)$ contains a set P_{11} of size $f_1(n) = \left\lceil \log^{(2)} n \right\rceil$. We continue by induction on k. Assume that for some positive integer $k \leq l-2$ there exist sets $P_{k1}, \dots, P_{kk}$ which satisfy the following four conditions:

(1) $P_{ki} \cap P_{kj} = \emptyset$ for every $1 \leq i < j \leq k$,

(2) $\bigcup_{i=1}^{k} P_{ki} \subseteq \Gamma(v) \setminus (X_{\mathcal{B}}(G) \cup Y)$,

(3) $|P_{ki}| = f_k(n)$ for every $1 \leq i \leq k$ and

(4) P_i and P_j are completely connected for every pair $1 \leq i < j \leq k$.

Since $G \notin \mathcal{Z}_{2k}(n)$ and $\bigcup_{i=1}^{k} P_{ki} \cap (X_{\mathcal{B}}(G) \cup Y) = \emptyset$ we have that

$$|T_k(P_{k1}, ..., P_{kk})| > \frac{l-k}{l} \cdot n - \frac{1}{3} \cdot g(n).$$

Since $G \notin \mathcal{Z}_{2k+1}(n)$ and $v \notin X_{\mathcal{B}}(G)$ we get from this, provided that n was chosen sufficiently large,

$$|T_k(P_{k1}, ..., P_{kk}) \cap \Gamma(v)| > f_k(n) \cdot (1 + h(n)).$$

Thus, let $P \subseteq T_k(P_{k1}, ..., P_{kk}) \cap \Gamma(v) \setminus (X_{\mathcal{B}}(G) \cup Y)$ be a set of size $f_k(n)$. Applying Lemma 5.2.18 to $P_0 = P$ and $P_1 = P_{k1}, \ldots, P_k = P_{kk}$ gives sets $P_{k+1\,1}, ..., P_{k+1\,k+1}$, each of size

$$\begin{aligned} \left\lceil \log^{(2k)}(f_k(n)) \right\rceil &= \left\lceil \log^{(2k)}\left(\left\lceil \log^{(k^2-k+2)} n \right\rceil \right) \right\rceil \geq \left\lceil \log^{((k+1)^2-(k+1)+2)} n \right\rceil \\ &= f_{k+1}(n), \end{aligned}$$

obviously satisfying conditions (1) to (4). In this way we get finally $l-1$ sets $P_{l-1\,1}, ...,$ $P_{l-1\,l-1}$, each of size $f_{l-1}(n)$ which form a Q–set in $\Gamma(v) \setminus (X_{\mathcal{B}}(G) \cup Y)$. $\square$

Lemma 5.2.22 *Let $1 \leq k \leq l$ be an integer, let $x = x_{\mathcal{C}}$ be a function having the property that for all sufficiently large n*

$$\frac{n}{\log f_{l-1}(n)} \leq x(n)$$

and let $y = y_{\mathcal{C}}$ be a positive integral function satisfying for all sufficiently large n

$$y(n) \leq \frac{n}{(\log n)^2} \quad .$$

Let $\mathcal{C} = \mathcal{C}(n, k, x, y)$ denote the set of all graphs $G \in \mathcal{F}orb_n(K_{l+1})$ that contain $y(n)$ pairwise disjoint k-cliques $S_i = \{v_{i1}, \ldots, v_{ik}\}$, such that for every $i = 1, \ldots, y(n)$

(1) S_i is contained in at most $2^{6 \cdot l} \cdot l \cdot y(n)$ many $k+1$-cliques,

(2) for every vertex v_{ij} there exists a Q-set Q_{ij} in the neighborhood of v_{ij} such that $R(Q_{ij})$ satisfies

$$|R(Q_{ij})| \geq \frac{n}{l} - x(n)$$

(3) and the set

$$T_i = \{v \in V_n \backslash S_i \mid v \notin \bigcup_{j=1}^{k} R(Q_{ij})\}$$

satisfies

$$|T_i| \geq 3k \cdot 2^k \cdot x(n).$$

Then the following inequality is valid for all sufficiently large n:

$$\log \frac{|\mathcal{C}|}{|\mathcal{F}orb_{n-ky(n)}(K_{l+1})|} \quad \leq \quad \left(\frac{l-1}{l} \cdot n - \frac{x(n)}{k}\right) \cdot k \cdot y(n).$$

Proof. Construct all graphs $G \in \mathcal{C}$ as follows. First choose $y(n)$ cliques $S_i = \{v_{i1}, \ldots, v_{ik}\}$ (in at most $n^{ky(n)}$ ways). Then impose a suitable graph on the remaining vertices (in at most $|\mathcal{F}orb_{n-ky(n)}(K_{l+1})|$ ways).
For every $i = 1, \ldots, y(n)$ we now have to connect $S_i = \{v_{i1}, \ldots, v_{ik}\}$ with $V_n \setminus \{v_{i1}, \ldots, v_{ik}\}$. First choose Q-sets $Q_{i1}, \ldots, Q_{ik}$ (in at most $\binom{n}{f_{l-1}(n)}^{(l-1)k} \leq n^{(l-1)f_{l-1}(n)k}$) ways). Next we connect S_i with $\bigcup_{h=1}^{k} R(Q_{ih})$. Observe that v_{ij} cannot be connected to $R(Q_{ij})$ because this would give an $(l+1)$-clique. For each $h = 1, \ldots, k$ and each choice of $1 \leq j_1 < \ldots < j_h \leq k$ let

$$T_{j_1,\cdots,j_h}(i) \;=\; \bigcap_{j \in \{j_1,\cdots,j_h\}} R(Q_{ij}) \cap \bigcap_{j \notin \{j_1,\cdots,j_h\}} (V_n \setminus R(Q_{ij})).$$

Note that

$$T_i \;=\; V_n \backslash \left(\bigcup_h \bigcup_{j_1,\cdots,j_h} T_{j_1,\cdots,j_h}(i) \right).$$

Now the number of ways to connect S_i to $\bigcup_{h=1}^{k} R(Q_{ih})$ is at most

$$2^{\sum_{h=1}^{k} \sum_{j_1,\cdots,j_h} |T_{j_1,\cdots,j_h}(i)|(k-h)}.$$

The sum in the exponent can be bounded as follows:

$$\begin{aligned}
&\sum_{h=1}^{k} \sum_{j_1,\cdots,j_h} |T_{j_1,\cdots,j_h}(i)| \cdot (k-h) \\
&= \quad k \cdot (n - |T_i|) - \sum_{j=1}^{k} |R(Q_{ij})| \\
&\leq \quad k \cdot n - k \cdot |T_i| - k \cdot \left(\frac{n}{l} - x(n)\right) \\
&= \quad \frac{l-1}{l} \cdot n \cdot k - k \cdot (|T_i| - x(n)).
\end{aligned}$$

Finally, we have to connect S_i with T_i. This can be done by first choosing a set of at most $2^{6\cdot l} \cdot l \cdot y(n)$ many vertices from T_i which we connect completely with S_i and then connecting the remaining vertices of T_i to S_i arbitrarily. For this one has at most

$$2^{6\cdot l} \cdot l \cdot y(n) \cdot \binom{n}{2^{6\cdot l} \cdot l \cdot y(n)} \cdot (2^k - 1)^{|T_i|}$$

many possibilities.
Putting things together and abbreviating $\min\{|T_i| \mid i = 1, \ldots, y(n)\}$ by $|T|$ we get for all sufficiently large n:

$$\begin{aligned} & \log \frac{|\mathcal{C}|}{|\mathcal{F}orb_{n-ky(n)}(K_{l+1})|} \\ \leq\ & k \cdot y(n) \cdot \log n + y(n) \cdot \Big[(l-1) \cdot k \cdot f_{l-1}(n) \cdot \log n + \frac{l-1}{l} \cdot n \cdot k \\ & -k \cdot |T| + k \cdot x(n) + 2^{6\cdot l} \cdot l \cdot y(n) \cdot \log n + \left(k - \frac{1}{2^k}\right) \cdot |T|\Big] \\ \leq\ & \left(\frac{l-1}{l} \cdot n - \frac{x(n)}{k}\right) \cdot k \cdot y(n). \end{aligned}$$

□

Put $x_{\mathcal{C}}(n) = g(n)/3$ and $y_{\mathcal{C}}(n) = \left\lceil \frac{h(n)}{k} \right\rceil$. Then by choice of $g(n)$ and $h(n)$ these functions fulfill the requirements of Lemma 5.2.22. We define $\mathcal{Z}_{2l-1+k}$, x_{2l-1+k} and y_{2l-1+k} for $k = 1, \ldots, l$ as follows:

$$\begin{aligned} \mathcal{Z}_{2l-1+k}(n) &= \mathcal{C}(n, k, x_{\mathcal{C}}(n), y_{\mathcal{C}}(n)) \\ x_{2l-1+k}(n) &= \frac{1}{k} \cdot x_{\mathcal{C}}(n) \\ y_{2l-1+k}(n) &= k \cdot y_{\mathcal{C}}(n) \end{aligned}$$

Obviously $x_{2l-1+k}(n) = \Theta(g(n))$, $y_{2l-1+k}(n) = \Theta(h(n))$ for every $k = 1, \ldots, l$ and condition (3) and condition (4) of Theorem 5.2.13 are fulfilled.
Next we exhibit properties of graphs which are not contained in $\bigcup_{i=1}^{3l-1} \mathcal{Z}_i(n)$. So let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{3l-1} \mathcal{Z}_i(n)$. Then by definition of $\mathcal{Z}_{2l-1+k}$ there are at most $\left\lceil \frac{h(n)}{k} \right\rceil - 1$ many pairwise disjoint k-cliques in G which satisfy conditions (1) to (3) of Lemma 5.2.22. We denote the vertices contained in these "bad" k-cliques by $C_k(G) \subseteq V_n$. Obviously, $|C_k(G)| \leq \left(\left\lceil \frac{h(n)}{k} \right\rceil - 1\right) \cdot k \leq h(n) - 1$ for every $k = 1, \ldots, l$. Now put $X_{\mathcal{C}}(G) = X_{\mathcal{B}}(G) \cup \bigcup_{k=1}^{l} C_k(G)$, then $|X_{\mathcal{C}}(G)| \leq (3l-1) \cdot (h(n) - 1)$.

Corollary 5.2.23 *Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{3l-1} \mathcal{Z}_i(n)$, let $Y \subseteq V_n \setminus X_{\mathcal{C}}(G)$ be an arbitrary set of size at most $(2^{6\cdot l} - (3l-1)) \cdot (h(n) - 1)$, and let S_k be a k–clique in $V_n \setminus (X_{\mathcal{C}}(G) \cup Y)$ for some $1 \leq k \leq l-1$. Then for all n sufficiently large we have that*

$$\{v \in V_n \setminus (X_{\mathcal{C}}(G) \cup Y) \mid S_k \subseteq \Gamma(v)\} \neq \emptyset.$$

Proof. Let k be an integer between 1 and $l-1$ and let $S_k = \{v_1, \ldots, v_k\}$ be a k–clique in $V_n \setminus (X_{\mathcal{C}}(G) \cup Y)$. From Corollary 5.2.21 we know that for every $i = 1, \ldots, k$ the set $\Gamma(v_i) \setminus (X_{\mathcal{C}}(G) \cup Y)$ contains a Q-set Q_i. From Corollary 5.2.17 we can derive that

$$|R(Q_i)| \geq \frac{n}{l} - \frac{g(n)}{3} \qquad \text{and}$$

$$\begin{aligned} \left|\bigcup_{i=1}^{k} R(Q_i)\right| &\leq \frac{k}{l} \cdot n + \frac{k}{3} \cdot g(n) \\ &< n - 3k \cdot 2^k \cdot \frac{g(n)}{3} - k \end{aligned}$$

provided that n was chosen sufficiently large.
Since $G \notin \mathcal{Z}_{2l-1+k}(n)$ and $S_k \subseteq V_n \setminus X_{\mathcal{C}}(G)$ this implies that S_k is contained in at least $2^{6 \cdot l} \cdot l \cdot \left\lceil \frac{h(n)}{k} \right\rceil$ many $(k+1)$–cliques. Since $|X_{\mathcal{C}}(G) \cup Y| \leq 2^{6 \cdot l} \cdot (h(n) - 1)$ the assertion follows provided n was chosen sufficiently large. □

Corollary 5.2.24 *Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{3l-1} \mathcal{Z}_i(n)$, $x, y \in V_n \setminus X_{\mathcal{C}}(G)$ connected by an edge with Q-sets $Q_x, Q_y \subseteq V_n \setminus X_{\mathcal{C}}(G)$. Then*

$$|R(Q_x) \cap R(Q_y)| \leq 2l \cdot 2^l \cdot g(n).$$

Proof. Corollary 5.2.23 implies that x, y are contained in some l-clique $S = \{v_1, \ldots, v_l\} \subseteq V_n \setminus X_{\mathcal{C}}(G)$. Fix a Q-set in $V_n \setminus X_{\mathcal{C}}(G)$ for each of these vertices and let $R(v_1), \ldots, R(v_l)$ be the corresponding R-sets. Such Q-sets exist by Corollary 5.2.21. Since S is not contained in an $(l+1)$-clique and since Corollary 5.2.17 implies that

$$|R(v_i)| \geq \frac{n}{l} - \frac{1}{3} \cdot g(n) \quad \text{for every} \quad i = 1, \ldots, l,$$

it follows from $G \notin \mathcal{Z}_{3l-1}(n)$ that

$$\left|\{v \in V_n \setminus S \mid v \notin \bigcup_{i=1}^{l} R(v_i)\}\right| < 3l \cdot 2^l \cdot \frac{g(n)}{3},$$

i.e., $|\bigcup_{i=1}^{l} R(v_i)| \geq n - l \cdot 2^l \cdot g(n)$. Hence, since we have by Corollary 5.2.17 that $|R(v_i)| \leq n/l + g(n)/3$ for every $i = 1, \ldots, l$, we get for every pair $1 \leq i < j \leq l$ that

$$|R(v_i) \cap R(v_j)| \leq \left(\frac{l}{3} + l \cdot 2^l\right) \cdot g(n) \leq 2l \cdot 2^l \cdot g(n).$$

□

Corollary 5.2.25 *Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{3l-1} \mathcal{Z}_i(n)$, $x \in V_n \setminus X_{\mathcal{C}}(G)$ with Q-set $Q_x \subseteq V_n \setminus X_{\mathcal{C}}(G)$ and R-set $R(x)$. Moreover, let $y \in R(x) \setminus X_{\mathcal{C}}(G)$ with Q-set $Q_y \subseteq V_n \setminus X_{\mathcal{C}}(G)$ and R-set $R(y)$. Then*

$$|R(x) \cap R(y)| \geq \frac{n}{l} - 4l^3 \cdot 2^l \cdot g(n).$$

Proof. By definition, there exist vertices $v_1, \ldots, v_{l-1} \in Q_x$ such that as well $v_1, \ldots, v_{l-1}, x$ as $v_1, \ldots, v_{l-1}, y$ form an l-clique. By Corollary 5.2.21 each of the vertices $v_1, \ldots, v_{l-1}$ has a Q-set in $V_n \setminus X_{\mathcal{C}}(G)$. We denote the corresponding R-sets by $R(v_1), \ldots, R(v_{l-1})$. Then it follows from Corollary 5.2.24 that

$$\begin{aligned} \left| R(x) \cap \bigcup_{i=1}^{l-1} R(v_i) \right| &\leq 2l^2 \cdot 2^l \cdot g(n) \quad \text{and} \\ \left| R(y) \cap \bigcup_{i=1}^{l-1} R(v_i) \right| &\leq 2l^2 \cdot 2^l \cdot g(n). \end{aligned}$$

From Corollary 5.2.17 together with Corollary 5.2.24 we get that

$$\begin{aligned} \left| \bigcup_{i=1}^{l-1} R(v_i) \right| &\geq (l-1) \cdot \left(\frac{n}{l} - \frac{1}{3} \cdot g(n) \right) - \binom{l-1}{2} \cdot 2l \cdot 2^l \cdot g(n) \\ &\geq \frac{l-1}{l} \cdot n - l^3 \cdot 2^l \cdot g(n). \end{aligned}$$

Thus, using once more Corollary 5.2.17 we get

$$\begin{aligned} |R(x) \cap R(y)| &\geq 2 \cdot \left(\frac{n}{l} - \frac{g(n)}{3} - 2l^2 \cdot 2^l \cdot g(n) \right) - \left(\frac{n}{l} + l^3 \cdot 2^l \cdot g(n) \right) \\ &\geq \frac{n}{l} - 4l^3 \cdot 2^l \cdot g(n), \end{aligned}$$

which completes the proof. □

We say that two vertices v and z of G are connected via a *double spindle*, if there are vertices $w_1, \ldots, w_{l-1}, x, y_1, \ldots, y_{l-1}$ such that x is different from the w_i's as well as from the y_i's, the w_i's as well as the y_i's form a clique of size $l-1$ and the w_i's are in the neighborhood of v and x and the y_i's are in the neighborhood of of x and z. The degenerate case of a double spindle where $x = z$ and $w_i = y_i$ for every $i = 1, \ldots, l-1$ is just a spindle. Observe that for every $x \in V_n$ with Q-set Q_x and R-set $R(x)$ every vertex in $R(x)$ is connected to x via a spindle.

Corollary 5.2.26 *Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{3l-1} \mathcal{Z}_i(n)$, let $Y \subseteq V_n \setminus X_{\mathcal{C}}(G)$ be an arbitrary set of size at most $(2^{6 \cdot l} - (3l-1)) \cdot (h(n) - 1)$, and let $S_l = \{v_1, \ldots, v_l\}$ be an l-clique in $V_n \setminus (X_{\mathcal{C}}(G) \cup Y)$. Moreover, for $i = 1, \ldots, l$ let*

$$\begin{aligned} S(v_i) = \; & \{x \in V_n \setminus (X_{\mathcal{C}}(G) \cup Y) |\, x \text{ is connected to } v_i \text{ via a double} \\ & \text{spindle through vertices from } V_n \setminus (X_{\mathcal{C}}(G) \cup Y)\}. \end{aligned}$$

Then the following is true for all sufficiently large n:

(1) $\frac{n}{l} - \frac{g(n)}{2} \leq |S(v_i)| \leq \frac{n}{l} + \frac{l-1}{2} \cdot g(n)$, *for every $i = 1, \ldots, l$*

(2) $\bigcup_{i=1}^{l} S(v_i) = V_n \setminus (X_{\mathcal{C}}(G) \cup Y)$

(3) $S(v_i) \cap S(v_j) = \emptyset$, *for every pair $1 \leq i < j \leq l$*

(4) $S(v_i)$ is stable, for every $i = 1, \ldots, l$.

Proof. By Corollary 5.2.21 each vertex $x \in V_n \setminus (X_{\mathcal{C}}(G) \cup Y)$ and in particular each of the vertices $v_1, \ldots, v_l$ has a Q-set in $V_n \setminus (X_{\mathcal{C}}(G) \cup Y)$. We again denote the corresponding R-sets by $R(x)$ and $R(v_1), \ldots, R(v_l)$, respectively. First observe that obviously $R(v_i) \subseteq S(v_i) \cup X_{\mathcal{C}}(G) \cup Y$ for every $i = 1, \ldots, l$. This shows the left hand side of (1) recalling Corollary 5.2.17. Next observe that from $(R(x) \cap R(v_j)) \setminus (X_{\mathcal{C}}(G) \cup Y) \neq \emptyset$ for some vertex $x \in V_n \setminus (X_{\mathcal{C}}(G) \cup Y)$ it follows that $x \in S(v_j)$. Hence, combining Corollary 5.2.17 and Corollary 5.2.24 proves (2).
For every $i = 1, \ldots, l$ let

$$\overline{S}(v_i) = \{x \in V_n \setminus (X_{\mathcal{C}}(G) \cup Y) | \, |R(x) \cap R(v_i)| \geq \frac{2}{3} \cdot \frac{n}{l}\}.$$

Obviously, $\overline{S}(v_i) \subseteq S(v_i)$ and from Corollary 5.2.24 it follows that $\overline{S}(v_i)$ is stable. Moreover, Corollary 5.2.25 (together with Corollary 5.2.17) implies that whenever $x \in V_n \setminus X_{\mathcal{C}}(G)$ has an R-set satisfying $(R(x) \cap R(v_i)) \setminus (X_{\mathcal{C}}(G) \cup Y) \neq \emptyset$ then $|R(x) \cap R(v_i)| \geq \frac{n}{l} - (8 \cdot l^3 \cdot 2^l + \frac{1}{3}) \cdot g(n)$. Hence, by Corollary 5.2.17 and Corollary 5.2.24, $\bigcup_{i=1}^{l} \overline{S}(v_i) = V_n \setminus (X_{\mathcal{C}}(G) \cup Y)$. Thus, $G[V_n \setminus (X_{\mathcal{C}}(G) \cup Y)]$ is l-colorable. From this it follows that the $S(v_i)$ are mutually disjoint and so each $S(v_i)$ is stable which proves (3), (4) and, with (2), the right hand side of (1). □

Note that in particular for every graph $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{3l-1} \mathcal{Z}_i(n)$ and every set $Y \subseteq V_n \setminus X_{\mathcal{C}}(G)$ of size at most $(2^{6 \cdot l} - (3l-1)) \cdot (h(n) - 1)$ the graph $G[V_n \setminus (X_{\mathcal{C}} \cup Y)]$ is uniquely l-colorable, provided n was chosen sufficiently large.

Corollary 5.2.27 *Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{3l-1} \mathcal{Z}_i(n)$, let $Y \subseteq V_n \setminus X_{\mathcal{C}}(G)$ be an arbitrary set of size at most $(2^{6 \cdot l} - (3l-1)) \cdot (h(n) - 1)$, and let $S_l = \{v_1, \ldots, v_l\}$ be an l-clique in $V_n \setminus (X_{\mathcal{C}}(G) \cup Y))$. Moreover for $i = 1, \ldots, l$ let*

$$\begin{aligned} S(v_i) = \quad & \{x \in V_n \setminus (X_{\mathcal{C}}(G) \cup Y) | \, x \text{ is connected to } v_i \text{ via a double} \\ & \text{spindle through vertices from } V_n \setminus (X_{\mathcal{C}}(G) \cup Y)\}. \end{aligned}$$

Then the following is true for all sufficiently large n: If S and T are sets of size $g(n)$ such that $S \subseteq S(v_i)$ and $T \subseteq S(v_j)$ for some pair $1 \leq i < j \leq l$ then $\Gamma(S) \cap T \neq \emptyset$.

Proof. Let $S \subseteq S(v_i)$ be of size $f_1(n) < g(n)$. Since $G \notin \mathcal{Z}_2(n)$ and $S(v_i) \cap X_{\mathcal{B}}(G) = \emptyset$ it follows that

$$|\Gamma(S)| \geq \frac{l-1}{l} \cdot n - \frac{g(n)}{3}.$$

By (4) of Corollary 5.2.26, $S(v_i)$ is stable and hence $\Gamma(S) \cap S(v_i) = \emptyset$. By (1) of Corollary 5.2.26, $|S_i| \geq n/l - g(n)/2$ and so $\Gamma(S)$ has to intersect every set of size $g(n)$ outside of $S(v_i)$. □

Lemma 5.2.28 *Let $0 \leq k \leq l-2$ be an integer, let $x = x_{\mathcal{D}}$ be a function having the property that for all sufficiently large n*

$$x(n) \leq \frac{l^2}{4} \cdot g(n)$$

and let $y = y_{\mathcal{D}}$ be a positive integral function satisfying for all sufficiently large n

$$4l \cdot 2^l \cdot (3l-1) \cdot (h(n)-1) \le y(n) \le \frac{n}{(\log n)^2}.$$

Let $\mathcal{D} = \mathcal{D}(n,k,x,y)$ denote the set of all graphs G in $\mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{3l-1} \mathcal{Z}_i(n)$ which contain outside of $X_{\mathcal{C}}(G)$ at least $y(n)$ pairs of vertices s_i, t_i so that each pair is in one color–class and $y(n)$ pairwise disjoint k-cliques $S_i \subseteq (\Gamma(s_i) \cap \Gamma(t_i)) \setminus \bigcup_{j=1}^{y(n)} \{s_j, t_j\}$ such that for every $i = 1, \ldots, y(n)$:

$$|\{v \in V_n \setminus X_{\mathcal{C}}(G) \mid v \in \Gamma(s_i) \cap \Gamma(t_i) \text{ and } S_i \subseteq \Gamma(v)\}| \le 2^{6 \cdot l} \cdot h(n).$$

Then the following inequality is valid for all sufficiently large n:

$$\log \frac{|\mathcal{D}|}{|\mathcal{F}orb_{n-(k+2)y(n)}(K_{l+1})|} \le \left(\frac{l-1}{l} \cdot n - x(n) \right) \cdot y(n) \cdot (k+2)\,.$$

Proof. Construct all graphs $G \in \mathcal{D}$ as follows. First choose $y(n)$ pairs of vertices and $y(n)$ k-cliques (in at most $n^{(k+2)y(n)}$ ways) and edges between these vertices (at most $2^{y(n)^2 \cdot (k+2)^2}$ possibilities).
By Corollary 5.2.26 we know that for each graph $G \in \mathcal{D}$ there exists $X_{\mathcal{C}}(G) \subseteq V_n$ of size at most $(3l-1)\cdot(h(n)-1)$ such that $G[V_n \setminus X_{\mathcal{C}}(G)]$ is uniquely l-colorable and each color–class has size at most $\frac{n}{l} + \frac{l-1}{2} \cdot g(n)$. So choose such an $X_{\mathcal{C}}(G) \subseteq V_n$ (in at most $n^{(3l-1)(h(n)-1)}$ many ways). Then we may impose an l-colored graph on the remaining vertices with color–classes $C_1, \ldots, C_l$ each of size at most $\frac{n}{l} + \frac{l-1}{2} \cdot g(n)$. It is known (cf. e.g. Prömel, Steger [1992]) that the number of l-colored graphs is asymptotically the same as the number of l-colorable graphs. Hence, $2 \cdot |\mathcal{F}orb_{n-(k+2)y(n)}(K_{l+1})|$ is obviously an upper bound for the number of ways to choose an l-colored graph on the remaining vertices provided that n was chosen sufficiently large. To connect $X_{\mathcal{C}}(G)$ with this l-colored graph we have at most $2^{n \cdot (3l-1) \cdot (h(n)-1)}$ possibilities.
We still have to connect s_i, t_i and S_i for $i = 1, \ldots, y(n)$ with $\bigcup_{i=1}^{l} C_i \cup X_{\mathcal{C}}(G)$. To connect these vertices with $X_{\mathcal{C}}(G)$ we have at most $2^{(k+2) \cdot y(n) \cdot (3l-1) \cdot (h(n)-1)}$ choices. Next we extend the coloring to the vertices s_i, t_i and S_i. This can be done in at most $l^{\,y(n)} \cdot \left[(l-1) \cdot \ldots \cdot (l-k)\right]^{y(n)} \le l^{(l-1) \cdot y(n)}$ ways.
Fix i. There are at most $2^{6 \cdot l} \cdot h(n)$ vertices in $\Gamma(s_i) \cap \Gamma(t_i)$ which extend S_i to a $(k+1)$-clique. For this we have at most $2^{6 \cdot l} \cdot h(n) \cdot \binom{n}{2^{6 \cdot l} \cdot h(n)} < n^{2^{6 \cdot l} \cdot h(n)}$ choices. There are at most $2^{(k+1) \cdot \left(\frac{n}{l} + \frac{l-1}{2} \cdot g(n)\right)}$ possibilities to connect s_i, t_i and S_i to a color–class which contains some vertex of S_i. Moreover we have at most $2^{k \cdot \left(\frac{n}{l} + \frac{l-1}{2} \cdot g(n)\right)}$ possibilities to connect s_i, t_i and S_i to the color–class which contains s_i and t_i. Finally, there are at most $(2^{k+2}-1)^{\left(\frac{n}{l} + \frac{l-1}{2} \cdot g(n)\right)}$ possibilities to connect s_i, t_i and S_i to a color–class which contains neither s_i, t_i nor a vertex of S_i. Thus there are at most

$$\begin{aligned} & 2^{2^{6 \cdot l} \cdot h(n) \cdot \log n} \cdot \left[2^{k + (k+1) \cdot k} \cdot (2^{k+2}-1)^{(l-k-1)} \right]^{\frac{n}{l} + \frac{l-1}{2} \cdot g(n)} \\ \le\ & 2^{2^{6 \cdot l} \cdot h(n) \cdot \log n + \left[k + (k+1) \cdot k + (l-k-1) \cdot \left(k+2 - \frac{1}{2^{k+2}}\right)\right] \cdot \left(\frac{n}{l} + \frac{l-1}{2} \cdot g(n)\right)} \end{aligned}$$

choices to connect s_i, t_i and S_i to $\bigcup_{i=1}^{l} C_i$.
Putting things together and recalling that $g(n) \leq n/l^4 2^l$ and $h(n) \leq n/(\log n)^3$ we get for all n which are sufficiently large:

$$\begin{aligned}
&\log \frac{|\mathcal{D}|}{|\mathcal{F}orb_{n-(k+2)y(n)}(K_{l+1})|} \\
&\leq 1 + (k+2)\cdot y(n)\cdot \log n + y(n)^2 \cdot (k+2)^2 + (3l-1)\cdot (h(n)-1)\cdot \log n \\
&\quad + n\cdot (3l-1)\cdot (h(n)-1) + (k+2)\cdot y(n)\cdot (3l-1)\cdot (h(n)-1) \\
&\quad + (l-1)\cdot y(n)\cdot \log l + y(n)\cdot \Bigg(2^{6\cdot l}\cdot h(n)\cdot \log n \\
&\quad + \left[k + (k+1)k + (l-k-1)\cdot \left(k+2-\frac{1}{2^{k+2}}\right)\right]\left(\frac{n}{l} + \frac{l-1}{2}\cdot g(n)\right)\Bigg) \\
&\leq n(3l-1)(h(n)-1) + y(n)(k+2)\left(\frac{l-1}{l}n + \frac{l(l-1)g(n)}{2} + \frac{n}{2l^3 2^l}\right) - \frac{y(n)\cdot n}{l\cdot 2^l} \\
&\leq \left(\frac{l-1}{l}\cdot n - x(n)\right)\cdot y(n)\cdot (k+2).
\end{aligned}$$

□

Put $x_{\mathcal{D}}(n) = \left\lfloor \frac{l^2}{4}\cdot g(n)\right\rfloor$ and $y_{\mathcal{D}}(n) = (3l-1)\cdot 4l\cdot 2^l\cdot (h(n)-1)+1$. Then obviously, for $k = 0, \ldots, l-2$

$$\begin{aligned}
\mathcal{Z}_{3l+k}(n) &= \mathcal{D}(n, k, x_{\mathcal{D}}(n), y_{\mathcal{D}}(n)) \\
x_{3l+k}(n) &= x_{\mathcal{D}}(n) \\
y_{3l+k}(n) &= (k+2)\cdot y_{\mathcal{D}}(n)
\end{aligned}$$

satisfy that $x_{3l+k}(n) = \Theta(g(n))$, $y_{3l+k}(n) = \Theta(h(n))$ and fulfill condition (3) and condition (4) of Theorem 5.2.13.
Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{4l-2} \mathcal{Z}_i(n)$. Then by definition of $\mathcal{Z}_{3l+k}$, $k = 0, \ldots, l-2$, there are at most $y(n) - 1 = 4l\cdot 2^l\cdot (3l-1)\cdot (h(n)-1)$ many pairs of vertices and at most the same number of pairwise disjoint k-cliques so that the assertion of Lemma 5.2.28 is fulfilled. Denote these "bad" vertices by $D_k(G) \subseteq V_n \setminus X_{\mathcal{C}}(G)$ and put $D(G) = \bigcup_{k=0}^{l-2} D_k(G)$. Then a crude upper bound for $|D(G)|$ is $(2^{6\cdot l} - (3l-1))\cdot (h(n)-1)$. Now define $X(G)$ as $X(G) = X_{\mathcal{C}}(G) \cup D(G)$, then $|X(G)| \leq 2^{6\cdot l}\cdot (h(n)-1)$.
The remaining step is to show that the family $\mathcal{Z}_i(n)$, $i = 1, \ldots, 4l-2$ satisfies condition (2) of Theorem 5.2.13, i.e., the following corollary.

Corollary 5.2.29 *Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{4l-2} \mathcal{Z}_i(n)$ and let n be sufficiently large. Then there exists a set $X(G) \subseteq V_n$ of size at most $2^{6\cdot l}\cdot (h(n)-1)$ such that*
(1) $G[V_n \setminus X(G)]$ has the clique property,

(2) $G[V_n \setminus X(G)]$ *is linearly forcible* l*-colorable, and*

(3) $G[V_n \setminus X(G)]$ *has the* $g(n)$*-intersection property.*

Proof. Let $G \in \mathcal{F}orb_n(K_{l+1}) \setminus \bigcup_{i=1}^{4l-2} \mathcal{Z}_i(n)$ and let $X(G)$ be as defined above. Thus, $|X(G)| \leq 2^{6 \cdot l} \cdot (h(n) - 1)$.
By Corollary 5.2.23, choosing $Y = D(G)$, we get that $G[V_n \setminus X(G)]$ has the clique property. Corollary 5.2.26, putting $Y = D(G)$, assures that $G[V_n \setminus X(G)]$ is uniquely l-colorable. Since $G \notin \mathcal{Z}_{3l+k}(n)$ for $k = 0, \ldots, l-2$ we obtain from Lemma 5.2.28 that every k-clique $S_k \subseteq V_n \setminus X(G)$ which is in the common neighborhood of a pair x, y of identically colored vertices can be extended to a $(k+1)$-clique by more than $2^{6 \cdot l} \cdot h(n) > (2^{6 \cdot l} - (3l-1)) \cdot (h(n) - 1) + 1$ many vertices in $V_n \setminus X_{\mathcal{C}}$. Therefore, since $|D(G)| \leq (2^{6 \cdot l} - (3l-1)) \cdot (h(n) - 1)$ it follows that $G[V_n \setminus X(G)]$ is linearly forcible. Finally, the fact that $G[V_n \setminus X(G)]$ has the $g(n)$-intersection property is immediately implied by Corollary 5.2.27, choosing again $Y = D(G)$. □

This conludes the proof of 5.2.1. □

5.3 Random l-colorable graphs

Within the theory of random graphs the determination of the chromatic number of a random graph had been an open problem for over a decade until Bollobás [8] determined $\chi(G_{n,p})$ for constant p.

Theorem 5.3.1 [Bollobás, 1988] *Let* $0 < p < 1$ *be fixed and let* $q = 1 - p$ *and* $d = 1/q$*. Then almost surely*

$$\chi(G_{n,p}) = \frac{n}{2 \log_d n}(1 + o(1)).$$

For $p = 1/2$ every graph is equally likely, so Theorem 5.3.1 implies in particular:

Corollary 5.3.2 *The chromatic number of a random graph* $G_{n,\frac{1}{2}}$ *is almost surely* $\chi(G_n) = \frac{n}{2 \log n}(1 + o(1))$*.*

Luczak [61] improved Bollobás' result for edge probabilities $p = p(n)$ which are a function of n.

Theorem 5.3.3 [Luczak, 1991] *If* $d = d(n) = np \to \infty$ *and* $p = p(n) \to 0$ *then almost surely*

$$\chi(G_{n,p}) = \frac{d}{2 \ln d}(1 + o(1)).$$

Following Prömel, Steger (1993) we investigate in this section for which $l = l(n)$ almost all l-colorable graphs are uniquely l-colorable. Observe first that Theorem 5.3.1, in particular, implies that the theory of random graphs seems not to be an appropriate tool in order to study properties of l-colorable graphs. For example, if l is a constant almost all l-colorable graphs have $\Omega(n^2)$ many edges (we will make this precise later on), Bollobás' theorem on the other hand shows that almost all graphs with $\Omega(n^2)$ edges have a much higher chromatic number. Here we therefore study the class $\mathcal{C}ol_l(n)$ of all l-colorable graphs on n vertices directly. The main result of this section states that as long as $l = l(n)$ is asymptotically smaller as the chromatic number of a random graph $G_{n,\frac{1}{2}}$, almost all l-colorable graphs are uniquely l-colorable. Let G_n^l denote a uniformly chosen element of this class, i.e. a random l-colorable graph.

Theorem 5.3.4 *Let $\nu > 0$ be a constant. Then for all integer functions $l = l(n) \leq \frac{n}{(2+\nu)\log n}$ a random l-colorable graph G_n^l is almost surely uniquely l-colorable.*

Together with Theorem 5.3.1 we immediately obtain the following corollary.

Corollary 5.3.5 *The function $l = \frac{n}{(2+\nu)\log n}$ is the threshold for the property that a random l-colorable graph is uniquely l-colorable, i.e.*

if $\nu > 0$ then G_n^l is almost surely uniquely l-colorable, and

if $\nu < 0$ then G_n^l is almost surely not uniquely l-colorable.

□

Theorem 5.3.4 generalizes results of Bollobás [6], Turner [84] and Kučera [57] who showed that respectively for constant l, for $l \leq (1-\epsilon)\log n$, and for $l \leq \sqrt{n/(28\log n)}$ almost every l-colorable graph is uniquely l-colorable.

Within the theory of random graphs it suffices in most cases to study random graphs $G_{n,p}$ in order to learn something about random graphs $G_{n,m}$ on n vertices and exactly $m = p\binom{n}{2}$ edges. For l-colorable graphs, however, we again have to investigate the class $\mathcal{C}ol_l(n,m)$ of all l-colorable graphs on n vertices with $m = m(n)$ edges directly.

Theorem 5.3.6 *Let $0 < c < \frac{1}{2}$ be fixed and set $d = 2/\log\frac{1}{1-2c}$. Then for all integer functions $l = l(n) \leq \frac{n}{(d+\nu)\log n}$ a random l-colorable graph G_{n,cn^2}^l is almost surely uniquely l-colorable.*

For fixed p a random graph $G_{n,p}$ has with probability one $p\frac{n^2}{2} + o(n^2)$ edges. By the equivalence of the two models $G_{n,p}$ and $G_{n,m}$ (cf. Section 1.1) one observes that Theorem 5.3.1 again immediately implies the following corollary.

Corollary 5.3.7 *Let $0 < c < \frac{1}{2}$ be fixed and set $d = 2/\log\frac{1}{1-2c}$. Then the function $l = \frac{n}{(d+\nu)\log n}$ is the threshold for the property that a random l-colorable graph with $m = cn^2$ edges is uniquely l-colorable, i.e.*

if $\nu > 0$ then G_{n,cn^2}^l is almost surely uniquely l-colorable, and

if $\nu < 0$ then G_{n,cn^2}^l is almost surely not uniquely l-colorable.

□

For edge functions $m = m(n) = o(n^2)$ we prove the following weaker result.

Theorem 5.3.8 *If $l = l(n)$ and $m = m(n)$ are positive integer functions such that $m/l \geq n\ln n$ then a random l-colorable graph $G_{n,m}^l$ is almost surely uniquely l-colorable.*

If one compares this result with the corresponding result of Luczak [61] one observes that in this case the bound of Theorem 5.3.8 only corresponds up to a constant factor to the chromatic number of a random graph $G_{n,p}$ with $p = \frac{2m}{n^2}$. We believe that in this case the chromatic number of a random graph $G_{n,p}$ is indeed not a threshold for the property of being l-colorable. However, at present we are unable to prove this.

5.3.1 Proof of Theorem 5.3.4

An important notion in this section is the notion of an l-colored graph – in contrary to an l-colorable graph. The essential difference here is that every l-colorable graph which is not uniquely l-colorable counts several times as an l-colored graph. With $Col'_n(l)$ we denote the number of l-colored graphs on n vertices. (Note, that we count permutation of colorings only once!)

The proof of Theorem 5.3.4 is structured as follows. Firstly, we establish an explicit formula for the number of l-colored graphs. Secondly, we show that the number of l-colored graphs in which all color classes are "balanced", i.e. are of size about $\frac{n}{l}$, is asymptotically of the same order as the number of all l-colored graphs. Thirdly, we bound the probability that in a balancedly l-colored graph there exists a stable set of size roughly $\frac{n}{l}$ which "strongly" does not coincide with any of the color classes. Fourthly, we prove that for balancedly l-colored graphs the given l-coloring is with probability one the only proper l-coloring for this graph. Combining these facts, in particular the second and fourth one, we then easily complete the proof of Theorem 5.3.4.

For the remainder of this section we fix some notation. As usual n denotes the cardinality of the vertex set. The function $r = r(n,l)$ will denote the remainder of n and l, i.e. $n = l \cdot \lfloor \frac{n}{l} \rfloor + r$. Furthermore let $\mathcal{P}(k,l,n)$ denote the set of all ordered integer sequences $(p_1, \ldots, p_l)$ such that $\sum_{i=1}^{l} p_i = r$, $0 \le \lfloor \frac{n}{l} \rfloor + p_i \le n$ and $|p_i| \le k$ for all $i = 1, \ldots, l$, and $|p_j| = k$ for at least one $1 \le j \le l$.

With these definitions at hand an explicit formula for the number of l-colored graphs is easily obtained (cf. Read [73]).

Lemma 5.3.9

$$Col'_n(l) = \sum_{k \ge 0} \sum_{\mathcal{P}(k,l,n)} \frac{1}{l!} \cdot \binom{n}{\lfloor \frac{n}{l} \rfloor + p_1, \ldots, \lfloor \frac{n}{l} \rfloor + p_l} \cdot 2^{\binom{n}{2} - \sum_{i=1}^{l} \binom{\lfloor \frac{n}{l} \rfloor + p_i}{2}}. \qquad \square$$

Our first goal is to prove that in the above formula we may confine the first sum to values $k < \epsilon \log n$ without changing the order of magnitude of the right hand side.

Let $p = (p_1, \ldots, p_l)$. To shorten notation, we henceforth use the following abbreviations

$$f(n,p) = \binom{n}{\lfloor \frac{n}{l} \rfloor + p_1, \ldots, \lfloor \frac{n}{l} \rfloor + p_l},$$

and

$$e(n,p) = \binom{n}{2} - \sum_{i=1}^{l} \binom{\lfloor \frac{n}{l} \rfloor + p_i}{2} = \frac{l-1}{2l} n^2 + \frac{r^2}{2l} - \frac{1}{2} \sum_{i=1}^{l} p_i^2.$$

Lemma 5.3.10 *For all integer functions $l = l(n)$ such that $l \le \frac{n}{\log n}$ and all $\epsilon > 0$*

$$Col'_n(l) = (1 + o(1)) \cdot \sum_{0 \le k < \epsilon \log n} \sum_{\mathcal{P}(k,l,n)} \frac{1}{l!} \cdot f(n,p) \cdot 2^{e(n,p)}.$$

The proof hinges on the fact that the multinomial coefficient $f(n,p)$ as well as the exponential function $2^{e(n,p)}$ achieve their maximum when the p_i are as small as possible, i.e. $0 \le p_i \le 1$ for all $i = 1, \dots, l$. What remains to be shown is that both functions decrease fast enough.

We proceed as follows. First we establish a lower bound for $Col'_n(l)$ which allows us to show that one may confine the sum in Lemma 5.3.9 to those sequences p_i where the number of "big" p_i's is $o(l)$. By exhibiting an appropriate mapping between sequences $(p_1, \dots, p_l)$ we then show that also the sequences with at least one "big" entry are negligible.

For constant l the following proposition is an immediate corollary of the well-known Laplace approximation of the multinomial distribution. For $l = o(n)$ it can be proven by a straightforward application of Stirling's formula. We skip the calculations.

Proposition 5.3.11 *Let $l = l(n)$ be an integer function such that $l = o(n)$ and let $p_1, \dots, p_l$ be reals such that $\sum p_i = 0$ and such that $\frac{n}{l} + p_i$ are positive integers of size at least $\frac{n}{2l}$. Then for all sufficiently large n*

$$\binom{n}{\frac{n}{l} + p_1, \dots, \frac{n}{l} + p_l} \le (2\pi n)^{-\frac{l-1}{2}} \cdot l^{n+\frac{l}{2}} \cdot e^{\frac{l}{2n}\sum p_i^2 + \frac{l^2}{n^2}\sum p_i^3}$$

and

$$\binom{n}{\frac{n}{l} + p_1, \dots, \frac{n}{l} + p_l} \ge (2\pi n)^{-\frac{l-1}{2}} \cdot l^{n+\frac{l}{2}} \cdot e^{-\frac{l}{n}(\sum p_i^2 + \frac{l}{6})}. \qquad \square$$

Claim. *For all integer functions $l = l(n)$ such that $l = o(n)$ there exists a constant $C > 0$ such that*

$$Col'_n(l) \ge C \cdot \frac{1}{l!} \cdot n^{-l} \cdot l^n \cdot 2^{\frac{l-1}{2l}n^2 + \frac{r^2}{2l}}.$$

Proof. The idea of the proof is to just count the l-colored graphs where all parts are as equal as possible, which means that the lower bound will be roughly

$$\frac{1}{l!} \cdot \binom{n}{\frac{n}{l}; \dots; \frac{n}{l}} \cdot 2^{\frac{l-1}{2l}n^2}.$$

To be precise, we have to consider the case that n is not divisible by l. Let $q_i = 0$ for $i = 1, \dots, l - r$ and $q_i = 1$ for $i = l - r + 1, \dots, l$. Then certainly

$$Col'_n(l) \ge \frac{1}{l!} \cdot f(n,q) \cdot 2^{e(n,q)} = \frac{1}{l!} \cdot \binom{n}{\lfloor \frac{n}{l} \rfloor + q_1; \dots; \lfloor \frac{n}{l} \rfloor + q_l} \cdot 2^{\frac{l-1}{2l}n^2 + \frac{r^2}{2l} - \frac{r}{2}},$$

from which the desired inequality follows immediately by Proposition 5.3.11. □

Let $\mathcal{Q}(n,l)$ denote the set of all ordered integer sequences $p_1,\dots,p_l$ such that $\sum_{i=1}^l p_i = r$, $0 \le \lfloor \frac{n}{l} \rfloor + p_i \le n$ for all $i = 1,\dots,l$, and $\sum_{i=1}^l p_i^2 \ge 3l \log n$.

Claim. *For all integer functions $l = l(n)$ such that $l = o(n)$*

$$\sum_{\mathcal{Q}(n,l)} \frac{1}{l!} \cdot f(n,p) \cdot 2^{e(n,p)} = o(Col'_n(l)).$$

Proof. As the sum over $\mathcal{Q}(n,l)$ of the multinomial coefficients is less than l^n and $e(n,p) \le \frac{l-1}{2l} n^2 + \frac{r^2}{2l} - \frac{3}{2} l \log n$ for all $p \in \mathcal{Q}(n,l)$, the claim is an immediate consequence of the previous claim. □

Proof of Lemma 5.3.10. Let $\epsilon > 0$ be arbitrary, but fixed. By the previous claim, it is enough to show that

$$\sum \frac{1}{l!} \cdot f(n,p) \cdot 2^{e(n,p)} = o(Col'_n(l)), \tag{5.3.1}$$

where the sum is over all ordered integer sequences $(p_1,\dots,p_l)$ contained in the set $\bigcup_{k \ge \epsilon \log n} \mathcal{P}(k,l,n)$ satisfying $\sum_{i=1}^l p_i^2 < 3l \log n$.

Let for the moment $t \in \mathbb{N}$ and $1 \le i_1 < \dots < i_t \le l$ be fixed. Furthermore let $q_1,\dots,q_t$ be fixed integral values such that $0 \le \lfloor \frac{n}{l} \rfloor + q_j \le n$, $\epsilon \log n \le |q_j|$ for each $j = 1,\dots,t$, and $\sum_{j=1}^t q_j^2 < 3l \log n$. With respect to these values let $\mathcal{S}$ denote the set of all ordered integer sequences $p = (p_1,\dots,p_l)$ such that $p_{i_j} = q_j$ for all $j = 1,\dots,t$, $|p_i| < \epsilon \log n$ for all remaining i's, and $\sum_{i=1}^l p_i^2 < 3l \log n$.

The idea behind the following construction is to define a suitable mapping σ from the set $\mathcal{S}$ into the set of all sequences $(p_1,\dots,p_l) \in \bigcup_k \mathcal{P}(k,l,n)$ in such a way that for all $p \in \mathcal{S}$ the difference $e(n,\sigma(p)) - e(n,p)$ is "large enough", so that the ratios $2^{e(n,p)}/2^{e(n,\sigma(p))}$ are "small". By then summing over all t, possible choices of the i_j, possible choices of q_j, and $\mathcal{S}$ we will observe that the sum in (5.3.1) is indeed $o(Col'_n(l))$.

To simplify notation we assume in the sequel without loss of generality that $i_j = j$ for all $1 \le j \le t$ and we write p as $p = (q,r)$, where $q = (q_1,\dots,q_t)$ and $r = (r_1,\dots,r_{l-t})$. Let $Q = \sum_{j=1}^t q_j$. Observe that $\sum_{j=1}^t q_j^2 < 3l \log n$ and $|q_i| \ge \epsilon \log n$ implies that $t \le \frac{3}{\epsilon^2} \frac{l}{\log n}$ and that this together with $\sum_{j=1}^t q_j^2 \ge t \cdot \left(\frac{Q}{t}\right)^2$ implies that $|Q| \le \frac{3}{\epsilon} l$. Let $x = \frac{Q}{|Q|} \cdot \lceil \frac{9}{\epsilon} \rceil$, $y = Q - x \cdot \lfloor \frac{Q}{x} \rfloor$, and let $l' = \lfloor \frac{Q}{x} \rfloor + 1$. Observe that by definition $l' \le \frac{l}{3} + 1$. As the sum of the squares of the r_i is less than $3l \log n$, the number of r_i's with absolute value at least $\frac{\epsilon}{4} \log n$ is less than $\frac{48}{\epsilon^2} \frac{l}{\log n}$. By the pigeonhole principle we conclude that for sufficiently large n there exists an index $1 \le i_0 \le l$ so that starting at r_{i_0} from the next l' entries of r (here we view r as a cyclic vector, i.e. we assume that r_1 follows r_{l-t}) at most $\frac{1}{4} l'$ many entries have absolute value at least $\frac{\epsilon}{4} \log n$.

For $p \in \mathcal{S}$ and n sufficiently large define $\sigma(p)$ as follows. Replace all q_i by zero, and then starting at the smallest index i_0 having the above properties, replace the first

$l'-1$ of the r_i by r_i+x and $r_{i_0+l'-1}$ by $r_{i_0+l'-1}+y$. It is easy to see, that $\sigma(p)$ is indeed a sequence of $\bigcup_k \mathcal{P}(k,l,n)$.

We claim that for all sufficiently large n the following inequalities hold:

$$\frac{f(n,p)}{f(n,\sigma(p))} \leq 1, \tag{5.3.2}$$

$$e(n,\sigma(p)) - e(n,p) \geq t\frac{\epsilon^2}{40}\log^2 n, \tag{5.3.3}$$

and that the preimage of any sequence $p=(p_1,\ldots,p_l)$ is "small", namely

$$|\sigma^{-1}(p)| \leq l \qquad \text{for all } p=(p_1,\ldots,p_l) \in \bigcup_k \mathcal{P}(k,l,n). \tag{5.3.4}$$

Before we prove these claims, we show that they immediately conclude the proof of the lemma. For all sufficiently large n the left hand side of (5.3.1) is bounded from above by

$$\begin{aligned}
&\sum_{t\geq 1}\sum_{i_1<\ldots<i_t}\sum_{q_1,\ldots,q_t}\sum_{p\in\mathcal{S}}\frac{1}{l!}\cdot f(n,p)\cdot 2^{e(n,p)} \\
\leq\ &\sum_{t\geq 1}\sum_{i_1<\ldots<i_t}\sum_{q_1,\ldots,q_t}\sum_{p\in\mathcal{S}}\frac{1}{l!}\cdot f(n,\sigma(p))\cdot 2^{-t\frac{\epsilon^2}{40}\log^2 n}\cdot 2^{e(n,\sigma(p))} \\
\leq\ &\sum_{t\geq 1}\left[l^t\cdot n^t\cdot 2^{-t\frac{\epsilon^2}{40}\log^2 n}\cdot l\cdot Col'_n(l)\right] \\
\leq\ &Col'_n(l)\cdot\sum_{t\geq 1}2^{-t\frac{\epsilon^2}{50}\log^2 n} \\
=\ &o(Col'_n(l)).
\end{aligned}$$

We prove the above claims in reverse order. (5.3.4) follows immediately from the fact that there are at most $l-t$ choices for the index i_0. (5.3.3) follows by routine calculations from the definition of $\sigma(p)$:

$$\begin{aligned}
&e(n,\sigma(p))-e(n,p) \\
=\ &\frac{1}{2}\sum_{j=1}^{t}q_j^2+\frac{1}{2}\sum_{j=i_0}^{i_0+l'-2}(r_j^2-(r_j+x)^2)+\frac{1}{2}(r_{i_0+l'-1}^2-(r_{i_0+l'-1}+y)^2) \\
\geq\ &\frac{1}{2}\sum_{j=1}^{t}q_j^2-\frac{1}{2}\sum_{j=i_0}^{i_0+l'-1}(2|x||r_j|+x^2) \\
\geq\ &\frac{1}{2}\sum_{j=1}^{t}q_j^2-\frac{1}{2}\left[\frac{1}{4}l'\cdot(2|x|\epsilon\log n)+\frac{3}{4}l'(2|x|\frac{\epsilon}{4}\log n)+l'x^2)\right] \\
\geq\ &\frac{1}{32}\sum_{j=1}^{t}q_j^2+\frac{15}{32}\sum_{j=1}^{t}|q_j|\epsilon\log n-\frac{1}{2}l'(\frac{7}{8}|x|\epsilon\log n+x^2)
\end{aligned}$$

$$\geq \quad t\frac{\epsilon^2}{40}\log^2 n.$$

For the proof of (5.3.2) observe first that, if we abbreviate $\lfloor \frac{n}{l} \rfloor$ by a and assume without loss of generality that $i_0 = 1$, then (5.3.2) is equivalent to

$$\text{(5.3.5)} \qquad \frac{[a!]^t}{\prod_{i=1}^t (a+q_i)!} \cdot \frac{\prod_{i=1}^{l'-1}(a+r_i+x)!}{\prod_{i=1}^{l'-1}(a+r_i)!} \cdot \frac{(a+r_{l'}+y)!}{(a+r_{l'})!} \leq 1.$$

We start with some simple observations and reductions.
As $\frac{(a+r_i+x)!}{(a+r_i)!} \geq \frac{(a+r_i-1+x)!}{(a+r_i-1)!}$ if and only if $x \geq 0$, we may assume that all r_i, $1 \leq i \leq l'$, have the same sign as Q, that is as x and y.
Similarly, as $0 \leq x \leq q$ and $0 \geq x \geq q$ imply $\frac{a!}{(a+q)!} \leq \frac{a!a!}{(a+q-x)!(a+x)!}$, we may assume that $|q_i| \leq 2\epsilon \log n$ for each $1 \leq i \leq t$. Note, that this implies in particular, that $|q_i|a^{-1} \leq 2\epsilon$ for all $1 \leq i \leq t$.
We may assume that the q_i are ordered, i.e. that

$$q_1 \geq \ldots \geq q_m > 0 > q_{m+1} \geq \ldots \geq q_t.$$

We now distinguish two cases. Namely, that $Q \geq 0$ and that $Q < 0$. We only prove the claim for the first case and leave the similar proof of the second case to the reader. (5.3.5) is equivalent to

$$\prod_{i=m+1}^{t} \prod_{j=0}^{|q_i|-1} (a-j) \cdot \prod_{i=1}^{l'-1} \prod_{j=1}^{x} (a+r_i+j) \cdot \prod_{j=1}^{y} (a+r_{l'}+j) \leq \prod_{i=1}^{m} \prod_{j=1}^{q_i} (a+j).$$

If we divide every factor by a, take the logarithm of both sides, and recall that $\ln(1+x) \leq x$ for all $-1 < x < 1$ and $x - \frac{1}{2}x^2 \leq \ln(1+x)$ for all $0 \leq x < 1$, we observe that it suffices to show

$$-\sum_{i=m+1}^{t} \sum_{j=0}^{|q_i|-1} \frac{j}{a} + \sum_{i=1}^{l'-1} \sum_{j=1}^{x} \frac{r_i+j}{a} + \sum_{j=1}^{y} \frac{r_{l'}+j}{a} \leq \sum_{i=1}^{m} \sum_{j=1}^{q_i} \left(\frac{j}{a} - \frac{1}{2}\frac{j^2}{a^2} \right).$$

After some elementary calculations and using our various assumptions, we see that it suffices to show

$$-\frac{1}{2}\sum_{i=1}^{t} q_i - (\frac{1}{2} - \epsilon)\sum_{i=1}^{t} q_i^2 + x\sum_{i=1}^{l'} r_i + \frac{1}{2}l'(x^2+x) \leq 0,$$

which in turn can be reduced to

$$-(\frac{1}{2} - \epsilon) \cdot \epsilon \log n \cdot \max\{Q, \epsilon \log n\} + x\left[\frac{3}{16}l'\epsilon \log n + \frac{1}{4}l'\epsilon \log n\right] + \frac{1}{2}l'(x^2+x) \leq 0.$$

This last inequality, however, is easily seen to be true whenever ϵ is sufficiently small. □

Having shown that almost all l-colored graphs are "balanced", we now restrict our attention to these balanced graphs. We start with some notations and definitions.

For a fixed partition $\Pi = (A_1, \dots, A_l)$ of the vertex set V_n let $\mathcal{G}_\Pi$ denote the set of all graphs on V_n with the property that no two vertices which belong to the same A_i are connected by an edge. Furthermore G_Π denotes a random graph with respect to this partition, i.e.

$$\text{Prob}\,[G_\Pi = G] = |\mathcal{G}_\Pi|^{-1} \qquad \text{for each } G \in \mathcal{G}_\Pi.$$

Observe that G_Π is obtained by connecting each pair of vertices contained in different sets A_i, independently of all other pairs, by an edge with probability $\frac{1}{2}$.

A partition $\Pi = (A_1, \dots, A_l)$ of V_n is called *ϵ-balanced*, if and only if $(1-\epsilon)\frac{n}{l} \le |A_i| \le (1+\epsilon)\frac{n}{l}$ for all $i = 1, \dots, l$.

Lemma 5.3.12 *Let $\epsilon, \nu > 0$ be constants and let ω_n be a positive function tending to infinity arbitrarily slowly. Furthermore, let $l = l(n)$ be an integral function such that*

$$l(n) \le \frac{n}{(2+\nu)\log n}$$

and let $\Pi = (A_1, \dots, A_l)$ be an ϵ-balanced partition of the vertex set. Call a subset $S \subseteq V_n$ "bad", iff $(1-\epsilon)\frac{n}{l} \le |S| \le (1+\epsilon)\frac{n}{l}$ and S is stable in G_Π, but meets more than one class A_i in a substantial fraction, i.e. if $\max_{1\le i\le l} |S \cap A_i| \le |S| - \omega_n$. Then for all sufficiently small ϵ:

$$\text{Prob}\,[\textit{There exists a "bad" set } S \subseteq V_n \textit{ in } G_\Pi] = o(1).$$

Proof. Assume without loss of generality that $\omega_n = o(\log n)$ and that ϵ is sufficiently small, and let $\alpha, \delta > 0$ be appropriately defined constants (see below). For a set $S \subseteq V_n$ let

$$I_S := \{1 \le i \le l \mid |A_i \cap S| \ge \delta\frac{n}{l}\}$$

and

$$X_S = \bigcup_{i \notin I_S} (S \cap A_i).$$

We distinguish three types of sets S:

i) sets S such that $|X_S| \ge \alpha\frac{n}{l}$,

ii) sets S such that $\omega_n \le |X_S| \le \alpha\frac{n}{l}$,

iii) sets S such that $|X_S| < \omega_n$, and $\max_{1\le i\le l} |S \cap A_i| \le |S| - \omega_n$.

We show the statement of the lemma for every type separately. In all cases we can bound the number of sets $S \subseteq V_n$ of size $(1-\epsilon)\frac{n}{l} \le |S| \le (1+\epsilon)\frac{n}{l}$ as follows. There are at most $l^{|I_S|}$ possibilities to choose the classes A_i with which S has a large intersection and $2^{(1+\epsilon)\frac{n}{l}\cdot|I_S|}$ possibilities to choose the points of S contained in these classes. As $|I_S|$ is bounded by a constant, namely

$$|I_S| \le \frac{|S|}{\delta\frac{n}{l}} \le \frac{1+\epsilon}{\delta},$$

this shows that the number of sets S with a given cardinality $x = |X_S|$ is bounded from above by

$$2^{O(\frac{n}{l})} \cdot \binom{n}{x}.$$

The probability that a given set S is stable in G_Π is

$$\text{Prob}\,[S \ \mathit{stable}] = 2^{-\sum_{i<j}|A_i \cap S|\cdot|A_j \cap S|}. \tag{5.3.6}$$

That means, to get a good upper bound for this probability, we need a good lower bound for the number of pairs $\{u, v\}$ of vertices of S which belong to different classes A_i. Let for shortage of notation

$$Off(S) = \sum_{i<j} |A_i \cap S| \cdot |A_j \cap S|$$

denote this number. Observe that, if for the first type of sets S we can show that $Off(S) \geq (1+c)|X_S| \log n$ for some positive constant c, then

$$\begin{aligned} &\text{Prob}\,[\mathit{There\ exists\ a\ ''bad''\ set}\ S \subseteq V_n\ \mathit{of\ type\ i)\ in}\ G_\Pi] \\ &\leq \sum_{\alpha\frac{n}{l} \leq x} 2^{O(\frac{n}{l})} \cdot 2^{x \log n} \cdot 2^{-(1+c)x \log n} \ = \ o(1). \end{aligned}$$

As the minimal number of edges of a complete l-partite graph on n vertices with at most k vertices in each class is bounded from below by $\binom{\lfloor \frac{n}{k} \rfloor}{2} k^2$, we obtain

$$Off(S) \geq (|S| - |X_S|) \cdot |X_S| + \binom{\left\lfloor \frac{|X_S|}{\lfloor \delta \frac{n}{l} \rfloor} \right\rfloor}{2} \cdot \lfloor \delta \frac{n}{l} \rfloor^2 \geq \frac{1}{2}|S||X_S| - \frac{3}{2}\delta \frac{n}{l}|X_S|$$

and therefore

$$Off(S) \geq \frac{1-\epsilon-3\delta}{2}\frac{n}{l}|X_S| \geq \frac{(2+\nu)(1-\epsilon-3\delta)}{2}|X_S| \log n,$$

which suffices if ϵ and δ are chosen sufficiently small with respect to ν.

Next consider the type ii) sets. Observe that $|X_S| \leq \alpha \frac{n}{l}$ and $|S| \geq (1-\epsilon)\frac{n}{l}$ implies that $|X_S| \leq \frac{\alpha}{1-\epsilon}|S|$. Hence,

$$\begin{aligned} Off(S) &\geq (|S| - |X_S|) \cdot |X_S| \geq (1 - \frac{\alpha}{1-\epsilon})|S||X_S| \geq (1-\epsilon-\alpha)\frac{n}{l}|X_S| \\ &\geq \frac{1-\epsilon-\alpha}{2}\frac{n}{l}|X_S| + \frac{1-\epsilon-\alpha}{2}(2+\nu)|X_S| \log n. \end{aligned}$$

Together this gives, if ϵ and α are sufficiently small with respect to ν

$$\begin{aligned} &\text{Prob}\,[\mathit{There\ exists\ a\ ''bad''\ set}\ S \subseteq V_n\ \mathit{of\ type\ ii)\ in}\ G_\Pi] \\ &\leq \sum_{\omega_n \leq x \leq \alpha \frac{n}{l}} 2^{O(\frac{n}{l})} \cdot 2^{x \log n} \cdot 2^{-\frac{1-\epsilon-\alpha}{2}\frac{n}{l}x - \frac{1}{2}(1-\epsilon-\alpha)(2+\nu)x \log n} \end{aligned}$$

$$= \quad o(1).$$

Finally, observe that for all sets $S \subseteq V_n$ of type iii) the sets I_S have cardinality at least two. Therefore

$$Off(S) \geq \frac{1}{2}|S|\delta\frac{n}{l} \geq \frac{1}{4}\delta(1-\epsilon)\frac{n}{l}(2+\nu)\log n + \frac{1}{4}\delta(1-\epsilon)[(2+\nu)\log n]^2,$$

from which

$$\text{Prob}\,[There\ exists\ a\ "bad"\ set\ S \subseteq V_n\ of\ type\ iii)\ in\ G_\Pi] = o(1)$$

follows immediately. □

In the final step we show that for an ϵ-balanced partition Π almost every G_Π is uniquely l-colorable. To simplify notation we call a partition $\Pi' = (A'_1, \ldots, A'_l)$ a *valid partition* of the graph G_Π, if all classes A'_i induce in G_Π stable sets.

Proposition 5.3.13 *Let $\epsilon, \nu > 0$ be constants. Furthermore, let $l = l(n)$ be an integral function such that*

$$l(n) \leq \frac{n}{(2+\nu)\log n}$$

and let $\Pi = (A_1, \ldots, A_l)$ be an ϵ-balanced partition of the vertex set. Then for all sufficiently small ϵ with respect to ν every random graph G_Π satisfies

$$\text{Prob}\,[\exists i \neq j, S \subseteq A_i, v \in A_j\ s.t.\ |S| = (1-2\epsilon)\frac{n}{l}\ and\ \Gamma(v) \cap S = \emptyset] = o(1).$$

Proof. As there are l^2 choices for the indices i and j, at most $(1+\epsilon)\frac{n}{l}$ choices for the vertex v, and at most $\binom{(1+\epsilon)\frac{n}{l}}{(1-2\epsilon)\frac{n}{l}}$ choices for the set S, and as $\binom{(1+\epsilon)\frac{n}{l}}{(1-2\epsilon)\frac{n}{l}} \leq 2^{g(\epsilon)\cdot\frac{n}{l}}$ for some function $g(\epsilon)$ tending to zero if ϵ tends to zero,

$$\begin{aligned}
&\text{Prob}\,[\exists i \neq j, S \subseteq A_i, v \in A_j\ s.t.\ |S| = (1-2\epsilon)\frac{n}{l}\ and\ \Gamma(v) \cap S = \emptyset] \\
&\quad\leq\ l^2 \cdot (1+\epsilon)\frac{n}{l} \cdot 2^{g(\epsilon)\cdot\frac{n}{l}} \cdot 2^{-(1-2\epsilon)\frac{n}{l}} \\
&\quad\leq\ 2^{2\log n - (1-2\epsilon-g(\epsilon))(2+\nu)\log n} \\
&\quad\leq\ 2^{-\frac{\nu}{2}\log n},
\end{aligned}$$

if ϵ is sufficiently small compared to ν. □

Lemma 5.3.14 *Let $\epsilon, \nu > 0$ be constants. Furthermore, let $l = l(n)$ be an integral function such that*

$$l(n) \leq \frac{n}{(2+\nu)\log n}$$

and let $\Pi = (A_1, \ldots, A_l)$ be an ϵ-balanced partition of the vertex set. Then for all sufficiently small ϵ with respect to ν:

$$\text{Prob}\,[There\ exists\ a\ valid\ partition\ \Pi' \neq \Pi\ of\ G_\Pi] = o(1).$$

Proof. Let ω_n be a function tending to infinity such that $\omega_n = o(\frac{n}{l})$. Choose ϵ sufficiently small so that Lemma 5.3.12 and Proposition 5.3.13 hold. Assume a partition $\Pi' \neq \Pi$ is a valid partition. If G_Π contains no "bad" set with respect to ω_n and ϵ, then every class B_i of size $|B_i| \geq (1-\epsilon)\frac{n}{l}$ satisfies $|B_i \cap A_{\sigma(i)}| \geq |B_i| - \omega_n$, where $\sigma(i)$ is defined by $|B_i \cap A_{\sigma(i)}| = \max_{1\leq j\leq l} |B_i \cap A_j|$. As Π is an ϵ-balanced partition, this implies that there has to exist a stable set S of size $|S| \geq (1-\epsilon)\frac{n}{l}$ such that there exists a partition $S = S_1 \cup S_2$ and an index i with $S_1 \subseteq A_i$, $S_2 \cap A_i = \emptyset$ and $1 \leq |S_2| \leq \omega_n$. Therefore

$$\begin{aligned}
&\text{Prob}\left[\textit{There exists a valid partition } \Pi' \neq \Pi \textit{ of } G_\Pi\right] \\
\leq\ & \text{Prob}\left[\textit{There exists a "bad" set } S \subseteq V_n \textit{ in } G_\Pi\right] + \\
& \text{Prob}\left[\exists i \neq j, S \subseteq A_i, v \in A_j \textit{ s.t. } |S| = (1-2\epsilon)\frac{n}{l} \textit{ and } \Gamma(v) \cap S = \emptyset\right] \\
=\ & o(1) + o(1).
\end{aligned}$$

□

Proof of Theorem 5.3.4. Choose $\epsilon > 0$ sufficiently small so that Lemma 5.3.12 and Lemma 5.3.14 hold simultaneously. Following an idea of Turner [84] let $\mathcal{C}$ denote the set of all l-colorable graphs on V_n and let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the set of all uniquely l-colorable graphs. We want to show $|\mathcal{C}_0|/|\mathcal{C}| \to 1$.

Denote with $\mathcal{P}$ the set of all partitions of V_n into l classes and with $\mathcal{P}_0 \subseteq \mathcal{P}$ the set of all ϵ-balanced partitions. We define a bipartite graph on $\mathcal{C} \cup \mathcal{P}$ as follows: Connect a partition $\Pi \in \mathcal{P}$ with a graph $G \in \mathcal{C}$ by an edge if and only if Π is a valid partition of G. For subsets $A \subseteq \mathcal{C}$ and $B \subseteq \mathcal{P}$ let $e(A, B)$ denote the number of edges between A and B.

In this notation the results of Lemma 5.3.10 and Lemma 5.3.14 state

$$\frac{e(\mathcal{C}, \mathcal{P}_0)}{e(\mathcal{C}, \mathcal{P})} \to 1 \qquad \text{and} \qquad \frac{e(\mathcal{C} \setminus \mathcal{C}_0, \mathcal{P}_0)}{e(\mathcal{C}, \mathcal{P}_0)} \to 0.$$

Both results together, however immediately imply the theorem:

$$\frac{|\mathcal{C}_0|}{|\mathcal{C}|} \geq \frac{e(\mathcal{C}_0, \mathcal{P}_0)}{e(\mathcal{C}, \mathcal{P})} = \frac{e(\mathcal{C}, \mathcal{P}_0)}{e(\mathcal{C}, \mathcal{P})} - \frac{e(\mathcal{C} \setminus \mathcal{C}_0, \mathcal{P}_0)}{e(\mathcal{C}, \mathcal{P})} \to 1.$$

□

5.3.2 Proof of Theorems 5.3.6 and 5.3.8

Let $Col'_{n,m}(l)$ denote the number of l-colored graphs on n vertices with m edges. Similarly as for $Col'_n(l)$ an explicit formula for $Col'_{n,m}(l)$ is easily obtained (here $e(n,p)$ denotes the function of the preceding section):

Lemma 5.3.9'

$$Col'_{n,m}(l) = \sum_{k\geq 0} \sum_{\mathcal{P}(k,l,n)} \frac{1}{l!} \cdot \binom{n}{\lfloor\frac{n}{l}\rfloor + p_1, \ldots, \lfloor\frac{n}{l}\rfloor + p_l} \cdot \binom{e(n,p)}{m}.$$

□

Comparing Lemma 5.3.9' with Lemma 5.3.9 indicates where we have to modify the proof of Theorem 5.3.4: instead of the exponential function $2^{e(n,p)}$ we now have to deal with the binomial coefficient $\binom{e(n,p)}{m}$. It is well known that especially for $m = e(n,p)/2$ both function behave very similarly. It is therefore not surprising that the proof of Theorem 5.3.4 carries over to Theorem 5.3.6 and 5.3.8. In coping with the arising asymptotic estimations the following inequalities turn out to be useful.

Proposition 5.3.15 *For all integer functions $m = m(n)$ and $x = x(n)$ such that $m < n$*

$$\frac{\binom{n-x}{m}}{\binom{n}{m}} \leq e^{-\frac{xm}{n}},$$

if in addition $x \leq \frac{1}{2}(n-m)$ one also has

$$\frac{\binom{n-x}{m}}{\binom{n}{m}} \geq e^{-\frac{xm}{n-m}-\frac{mx^2}{(n-m)^2}}.$$

Moreover for $m = cn$ and $x \leq \frac{1}{2}(n-m)$

$$\frac{\binom{n-x}{cn}}{\binom{n}{cn}} = e^{-x(\ln\frac{1}{1-c}+O(\frac{x}{n}))}.$$

□

Lemma 5.3.10' *For all integer functions $l = l(n)$ and $m = m(n)$ such that $m \leq \frac{l-1}{2l}n^2 - \frac{n^2}{\ln n}$ and $\frac{m}{l} \geq \frac{1}{2}n\ln n$ and all $\epsilon > 0$*

$$Col'_{n,m}(l) = (1+o(1)) \cdot \sum_{0\leq k<\frac{\epsilon n\ln n}{\sqrt{m}}} \sum_{\mathcal{P}(k,l,n)} \frac{1}{l!} f(n,p)\binom{e(n,p)}{m}$$

Sketch. As before one uses Stirling's approximation and Proposition 5.3.15 to first show that

$$Col'_{n,m}(l) \geq \frac{1}{l!} \cdot n^{-l} \cdot l^n \cdot \binom{\frac{l-1}{2l}n^2 + \frac{r^2}{2l}}{m}$$

and to deduce

$$\sum_{\mathcal{Q}(n,m,l)} \frac{1}{l!} \cdot f(n,p) \cdot \binom{e(n,p)}{m} = o(Col'_{n,m}(l)),$$

where $\mathcal{Q}(n,m,l)$ denotes the set of all ordered integer sequences $p_1, \ldots, p_l$ such that $\sum_{i=1}^{l} p_i = r$, $0 \leq \lfloor\frac{n}{l}\rfloor + p_i \leq n$ for all $i = 1, \ldots, l$, and $\sum_{i=1}^{l} p_i^2 \geq 3l\frac{n^2}{m}\ln n$.

In the last part of the proof we substitute $\epsilon \log n$ by $\frac{\epsilon n \ln n}{\sqrt{m}}$ and define the mapping σ similarly as in the proof of Lemma 5.3.10: we let $x = \frac{Q}{|Q|}\lceil\frac{9}{\epsilon}\frac{n}{\sqrt{m}}\rceil$ and define y and l' as before. Then (5.3.2) and (5.3.4) continue to hold, while (5.3.3) is replaced by oldtheequation=5.3.6

$$e(n,\sigma(p)) - e(n,p) \geq t\frac{\epsilon^2 n^2 \ln n}{40m}. \tag{5.3.3'}$$

To complete the proof it suffices to observe that (5.3.3') and Proposition 5.3.15 imply

$$\frac{\binom{e(n,p)}{m}}{\binom{e(n,\sigma(p))}{m}} \le e^{-(e(n,\sigma(p))-e(n,p))\frac{2m}{n^2}} \le e^{-t\frac{\epsilon^2}{40}\ln^2 n}.$$

□

For a fixed partition $\Pi = (A_1, \ldots, A_l)$ of the vertex set V_n let $\mathcal{G}_\Pi^m$ denote the set of all graphs on V_n with m edges and the property that no two vertices which belong to the same A_i are connected by an edge. As before let G_Π^m denote a random graph with respect to this partition, i.e.

$$\text{Prob}\,[G_\Pi^m = G] = |\mathcal{G}_\Pi^m|^{-1} \qquad \text{for each } G \in \mathcal{G}_\Pi^m.$$

Observe that if $e(\Pi) = \binom{n}{2} - \sum_{i=1}^{l}\binom{|A_i|}{m}$ denotes the number of edges of the complete l-partite graph with classes A_i, then

$$\text{Prob}\,[G_\Pi^m = G] = \frac{1}{\binom{e(\Pi)}{m}} \qquad \text{for each } G \in \mathcal{G}_\Pi^m.$$

The couterpart of Lemma 5.3.12 then states (observe that the definition of a "bad" set is changed in the case of $m = o(n^2)$):

Lemma 5.3.12' *Let $\epsilon, \nu > 0$ be constants and let ω_n be a positive function tending to infinity arbitrarily slowly. Furthermore, let $l = l(n)$ and $m = m(n)$ be an integral functions, so that $\frac{m}{l} \ge (1+\nu)n\ln n$ or $m = cn^2$ and $l \le \frac{n}{(2/\log\frac{1}{1-2c}+\nu)\log n}$ for some constant $c > 0$, and let $\Pi = (A_1, \ldots, A_l)$ be an ϵ-balanced partition of the vertex set. Call a subset $S \subseteq V_n$ "bad", iff $(1-\epsilon)\frac{n}{l} \le |S| \le (1+\epsilon)\frac{n}{l}$ and S is stable in G_Π, but meets more than one class A_i in a substantial fraction, i.e. if $\max_{1\le i\le l}|S\cap A_i| \le |S| - \omega_n \cdot \frac{n^2}{m}$. Then for all sufficiently small ϵ:*

$$\text{Prob}\,[\textit{There exists a bad set } S \subseteq V_n \textit{ in } G_\Pi] = o(l^{-1}).$$

Sketch. Following the proof of Lemma 5.3.12, we only have to change (5.3.6) to

$$\text{Prob}\,[S\ \textit{stable}] = \frac{\binom{e(\Pi)-Off(S)}{m}}{\binom{e(\Pi)}{m}} \le e^{-Off(S)\frac{2m}{n^2}}, \tag{5.3.6'}$$

if $\frac{m}{l} \ge (1+\nu)n\ln\frac{m}{n}$ and to

$$\text{Prob}\,[S\ \textit{stable}] = \frac{\binom{e(\Pi)-Off(S)}{m}}{\binom{e(\Pi)}{m}} \le e^{-Off(S)\cdot(\ln\frac{1}{1-2c}+o(1))}, \tag{5.3.6''}$$

if $m = cn^2$ and $l \le \frac{n}{(2/\log\frac{1}{1-2c}+\nu)\log n}$ (cf. Proposition 5.3.15).

With these inequalities the proof can be completed as before. □

If $m = cn^2$ Proposition 5.3.13 carries over in a straightforward way, for smaller m, however, we have to change the statement of the proposition slightly. Both cases can be proven along the lines of Proposition 5.3.13. We omit the details.

Proposition 5.3.13' *Let $\epsilon, \nu > 0$ be constants. Furthermore, let $l = l(n)$ and $m = m(n)$ be integral functions such that $m = cn^2$ and $l \le \frac{n}{(2/\log\frac{1}{1-2c}+\nu)\log n}$ for some constant $c > 0$ and let $\Pi = (A_1, \ldots, A_l)$ be an ϵ-balanced partition of the vertex set. Then for all sufficiently small ϵ with respect to ν every random graph G_Π^m satisfies*

$$\text{Prob}\left[\exists i \ne j, S \subseteq A_i\, v \in A_j\, s.t.\ |S| = (1-2\epsilon)\frac{n}{l} \text{ and } \Gamma(v) \cap S = \emptyset\right] = o(1). \qquad \square$$

Proposition 5.3.13" *Let $\epsilon, \nu > 0$ be constants. Furthermore, let $l = l(n)$ and $m = m(n)$ be integral functions such that $\frac{m}{l} \ge (1+\nu)n \ln n$ and let $\Pi = (A_1, \ldots, A_l)$ be an ϵ-balanced partition of the vertex set. Then for all sufficiently small ϵ with respect to ν every random graph G_Π^m satisfies*

$$\begin{array}{lll}\text{Prob}\,[& \exists i \ne j, S \subseteq A_i, S' \subseteq A_j \text{ such that} & \\ & S| = (1-2\epsilon)\frac{n}{l},\ |S'| = \frac{n^2}{m} \text{ and } \Gamma(v) \cap S = \emptyset\ ] & = \quad o(1). \end{array} \qquad \square$$

As in the previous section Lemma 5.3.12' and Proposition 5.3.13', respectively Proposition 5.3.13", now imply that for every m and l and every ϵ-balanced partition Π each random graph G_Π^m is with probability one uniquely l-colorable. We leave the analogous proof to the reader.

Lemma 5.3.14' *Let $\epsilon, \nu > 0$ be constants and let ω_n be a positive function tending to infinity arbitrarily slowly. Furthermore, let $l = l(n)$ and $m = m(n)$ be an integral functions, so that $\frac{m}{l} \ge n \ln n$ or $m = cn^2$ and $l \le \frac{n}{(2/\log\frac{1}{1-2c}+\nu)\log n}$, for some constant $c > 0$ and let $\Pi = (A_1, \ldots, A_l)$ be an ϵ-balanced partition of the vertex set. Then for all sufficiently small ϵ:*

$$\text{Prob}\,[\textit{There exists a valid partition } \Pi' \ne \Pi \textit{ of } G_\Pi^m] = o(1). \qquad \square$$

Proof of Theorems 5.3.6 and 5.3.8. The proof of Theorem 5.3.4 carries over word by word, if we let $\mathcal{C}$ denote the set of all l-colorable graphs on n vertices with m edges and let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the set of all such graphs which are uniquely l-colorable. (Recall that $\frac{n}{l} + \frac{\epsilon n \ln n}{\sqrt{m}} \le (1+\epsilon)\frac{n}{l}$.) $\square$

5.3.3 Counting l-colorable graphs

The following two corollaries on the number of l-colorable graphs are immediate consequences of the main theorems of this section.

Corollary 5.3.16 *For all integer functions $l = l(n)$ such that $l \le \frac{n}{(2+\nu)\log n}$ for some constant $\nu > 0$ and all $\epsilon > 0$:*

$$|\mathcal{C}ol_n(l)| = (1+o(1)) \cdot \sum_{0 \le k < \epsilon \log n} \sum_{\mathcal{P}(k,l,n)} \frac{\binom{n}{\lfloor\frac{n}{l}\rfloor + p_1, \ldots, \lfloor\frac{n}{l}\rfloor + p_l}}{l!} \cdot 2^{\frac{l-1}{2l}n^2 + \frac{r^2}{2l} - \frac{1}{2}\sum_{i=1}^l p_i^2},$$

where $\mathcal{P}(k,l,n)$ denotes the set of all ordered integer sequences $(p_1,\dots,p_l)$ such that $\sum_{i=1}^l p_i = n - l\lfloor\frac{n}{l}\rfloor$, $0 \le \lfloor\frac{n}{l}\rfloor + p_i \le n$ and $|p_i| \le k$ for all $i = 1,\dots,l$, and $|p_j| = k$ for at least one $1 \le j \le l$.

If in addition $m = m(n)$ is an integer functions so that $m/l \ge n\ln n$, then

$$|\mathcal{C}ol_{n,m}(l)| = (1+o(1)) \cdot \sum_{0\le k<\frac{\epsilon n\ln n}{\sqrt{m}}} \sum_{\mathcal{P}(k,l,n)} \frac{\binom{n}{\lfloor\frac{n}{l}\rfloor+p_1,\dots,\lfloor\frac{n}{l}\rfloor+p_l}}{l!} \cdot \binom{\frac{l-1}{2l}n^2+\frac{r^2}{2l}-\frac{1}{2}\sum_{i=1}^l p_i^2}{m},$$

Proof. Follows immediately from Theorems 5.3.4, 5.3.6 and 5.3.8 and Lemmas 5.3.10 and 5.3.10'. □

Corollary 5.3.17 *Let $l \ge 2$ be fixed. Then there exist constants $c_0,\dots,c_{l-1}$ such that*

$$|\mathcal{C}ol_n(l)| = (c_r + o(1)) \cdot 2^{\frac{l-1}{2l}n^2+n\log l-\frac{l-1}{2}\log n}$$

for $n = r \bmod l$. In addition, for every $0 < c < \frac{1}{2}$ there exist constants $d_0,\dots,d_{l-1}$ such that with $h(x) = -x\log x - (1-x)\log(1-x)$

$$|\mathcal{C}ol_{n,cn^2}(l)| = (d_r + o(1)) \cdot 2^{\frac{l-1}{2l}\cdot h(\frac{2lc}{l-1})n^2+n\log l-\frac{l}{2}\log n}$$

for $n = r \bmod l$.

Proof. Let k be an integer of size at most $\epsilon\log n$, and let $(p_1,\dots,p_l)$ be a sequence contained in $\mathcal{P}(k,l,n)$, which is as defined in Corollary 5.3.16. It follows from Proposition 5.3.11 that then

$$\binom{n}{\lfloor\frac{n}{l}\rfloor+p_1,\dots,\lfloor\frac{n}{l}\rfloor+p_l} = (1+o(1)) \cdot (2\pi n)^{-\frac{l-1}{2}} \cdot l^{n+\frac{l}{2}}.$$

Therefore, by Corollary 5.3.16, letting $c'_r = l^{\frac{l}{2}} \cdot 2^{\frac{r^2}{2l}} \cdot (2\pi)^{-\frac{l-1}{2}} \cdot (l!)^{-1}$

$$|\mathcal{C}ol_n(l)| = (c'_r + o(1)) \cdot 2^{\frac{l-1}{2l}n^2+n\log l-\frac{l-1}{2}\log n} \cdot \sum_{0\le k<\epsilon\log n} \sum_{\mathcal{P}(k,l,n)} 2^{-\frac{1}{2}\sum_{i=1}^l p_i^2}.$$

As it is easily seen that the sum converges, the first part of the corollary follows.

For the second part we observe that by Proposition 5.3.15

$$\binom{\frac{l-1}{2l}n^2+\frac{r^2}{2l}-\frac{1}{2}\sum_{i=1}^l p_i^2}{cn^2} = (1+o(1)) \cdot \binom{\frac{l-1}{2l}n^2}{cn^2} \cdot e^{(\frac{r^2}{2l}-\frac{1}{2}\sum_{i=1}^l p_i^2)\cdot\ln(1-\frac{2lc}{l-1})^{-1}}.$$

Using that

$$\binom{n}{cn} = (1+o(1)) \cdot 2^{h(c)x} \cdot (2\pi c(1-c)n)^{-\frac{1}{2}}$$

(which can be easily obtained from Stirling's formula) we observe that by Corollary 5.3.16, letting $d'_r = \frac{1}{2}\ln(1-\frac{2lc}{l-1})^{-1}$ and $d''_r = l^{\frac{l}{2}} \cdot e^{\frac{d''_r\cdot r^2}{l}} \cdot (2\pi)^{-\frac{l-1}{2}} \cdot (l!)^{-1}$

$$|\mathcal{C}ol_{n,cn^2}(l)| = (d''_r + o(1)) \cdot 2^{\frac{l-1}{2l}\cdot h(\frac{2lc}{l-1})\cdot n^2+n\log n-\frac{l-2}{2}\log n} \cdot \sum_{0\le k<\epsilon\ln n} \sum_{\mathcal{P}(k,l,n)} e^{-d''_r\sum_{i=1}^l p_i^2},$$

which concludes the proof of Corollary 5.3.17. □

5.4 Kučera's algorithm

For small enough l, the results of the previous section can be summarized in the statement: "almost all l-colorable graphs are uniquely l-colorable". In fact, however, we have shown considerably more. Namely, we exhibited the structure of almost all l-colorable graphs. Rougly speaking this may be phrased as follows: "almost all l-colorable graphs are uniquely l-colorable, and this coloring is ϵ-balanced". From an algorithmic point of view, this additional clause is of great importance: It allows us, to decide whether two vertices should be colored with the same color by just considering the local structure of the graph. More precisely, consider two vertices in the same color class of an ϵ-balanced coloring. The expected number of vertices connected to both of them is at least $\frac{1}{2}(l-1)(1-\epsilon)n/l$. On the other hand, for two vertices in different color classes this number is at most $\frac{1}{2}(l-2)(1+\epsilon)n/l$. Clearly, for small enough l (and ϵ) the difference in these two expectations is large compared to their variance. The following algorithm of Kučera (1989) is based on these facts.

Algorithm COLOR
Input: A graph $G = (V_n, E)$;
Output: A coloring of G with l colors or failure;
begin
  for each pair $e = \{u, v\}$ of vertices of V_n **do** $d(e) := |\Gamma(u) \cap \Gamma(v)|$;
  Order all pairs of vertices so that $d(e_1) \geq d(e_2) \geq \ldots$;
  $F := \emptyset$; $i := 1$;
  repeat
    $F := F \cup \{e_i\}$; $i := i + 1$;
  until $F \cup e_i$ has only $l - 1$ components;
  Color each component of (V_n, F) with one color;
  if coloring is proper **then** output coloring
  else stop with failure;
end.

Theorem 5.4.1 *There exists a constant $c > 0$ such that for all integral functions $l \leq \sqrt{cn/\ln n}$ the algorithm* COLOR *finds a proper l-coloring for almost all l-colorable graphs.*

Proof. With foresight let $c = \frac{1}{163}$ and $\epsilon < \frac{1}{27}$. By Theorem 5.3.4 we know that for $l \leq \sqrt{cn/\ln n}$ almost all l-colorable graphs are uniquely l-colorable and that this unique coloring is ϵ-balanced. It thus suffices to consider an arbitrary but fixed ϵ-balanced partition $\Pi = (A_1, \ldots, A_l)$ of V_n and to show that for a random graph G_Π the algorithm COLOR finds with probability one an optimal coloring. (See Section 5.3, page 100, for the definition of G_Π.)

As outlined above the algorithm is based on the fact that with high probability vertices in the same color class have significantly more joint neighbors than vertices in different color classes. We make this precise by showing that

$$\text{(5.4.7)} \qquad \text{Prob}\,[\min_i\{d(u,v) \mid u, v \in A_i\} > \max_{i \neq j}\{d(u,v) \mid u \in A_i, v \in A_j\}] = 1 - o(1),$$

where $d(u,v)$ denotes the number of common neighbors of the vertices u and v, i.e. $d(u,v) = |\Gamma(u) \cap \Gamma(v)|$. Obviously (5.4.7) will conclude the proof of Theorem 5.4.1.

Let $a_{ij} := n - |A_i \cup A_j|$. For $u \in A_i$ and $v \in A_j$ the number $d(u,v)$ is binomial distributed with parameters $n = a_{ij}$ and $p = \frac{1}{4}$. Therefore for all sufficiently large n (cf. Theorem B.1)

$$\mathrm{Prob}\left[|d(u,v) - \tfrac{1}{4}a_{ij}| > \tfrac{1}{9}\tfrac{n}{l}\right] \leq e^{-\frac{1}{81}n^2/(l^2 a_{ij})}.$$

As $a_{ij} \leq n$ and $l \leq \sqrt{cn/\ln n}$ this implies

$$\mathrm{Prob}\left[|d(u,v) - \tfrac{1}{4}a_{ij}| > \tfrac{1}{9}\tfrac{n}{l}\right] \leq e^{-\frac{1}{81c}\ln n},$$

and therefore

$$\mathrm{Prob}\left[\exists u, v \in V_n \text{ s.t. } |d(u,v) - \tfrac{1}{4}a_{ij}| > \tfrac{1}{9}\tfrac{n}{l}\right] \leq \binom{n}{2} \cdot e^{-\frac{1}{81c}\ln n} = o(1).$$

Consider now vertices $u, v \in A_i$, $x \in A_j$ and $y \in A_k$, for $j \neq k$. Then almost surely

$$\begin{aligned} d(u,v) - d(x,y) &\geq \left(\mathbb{E}[d(u,v)] - \tfrac{1}{9}\tfrac{n}{l}\right) - \left(\mathbb{E}[d(x,y)] + \tfrac{1}{9}\tfrac{n}{l}\right) \\ &= \tfrac{1}{4}(a_{ii} - a_{jk}) - \tfrac{2}{9}\tfrac{n}{l} \\ &\geq \tfrac{1}{4}\left(2(1-\epsilon)\tfrac{n}{l} - (1+\epsilon)\tfrac{n}{l}\right) - \tfrac{2}{9}\tfrac{n}{l} > 0, \end{aligned}$$

which concludes the proof of (5.4.7). □

Let us close this section with the remark that it is not too difficult to modify algorithm Color in such a way that Color always ends with a proper coloring (which may use more than l colors). Namely, one successively considers the pairs e_i and adds e_i to F if and only if no component of $F \cup \{e_i\}$ contains two vertices that are adjacent in G. We omit the details.

Chapter 6

Sparse Graphs

A well-studied problem in graph theory is the question to what extent the local sparseness of a graph, indicated for example by the fact that a graph is H-free, implies a simple global structure, indicated, for example, by a small chromatic number. A first (negative) step in this direction was undertaken by W. Tutte who published, under the pseudonym Blanche Descartes, a construction of a triangle-free graph, whose chromatic number is four. This result was further generalized by Zykov (1949), who showed that in fact for every $k \in \mathbb{N}$ there exists a graph without K_3-subgraph whose chromatic number is at least k. The sizes of the graphs G from Zykov's contructions grow exponentially in k. Using probabilistic arguments Erdős (1961) was able to prove the existence of k-chromatic triangle-free graphs on $ck^2(\log k)^2$ vertices. The main advantage of Erdős' probabilistic construction is, however, not only the reduction in the size of the graphs, but the fact that his approach provides, for each k, exponentially many k-chromatic triangle-free graphs on $ck^2(\log k)^2$ many vertices.

In view of the Erdős, Kleitman, Rothschild result that almost all triangle-free graphs are bipartite (cf. Theorem 3.1.1), this seems to be very surprising. Note, however, that the triangle-free k-chromatic graphs from Erdős' construction are very sparse. The extremal triangle-free graph on the other hand is very dense. This indicates that the validity of Theorem 3.1.1 crucially depends on the fact that a triangle-free graph chosen randomly from $\mathcal{F}orb_n(K_3)$ is very dense.

The aim of this section is to study these phenomena more closely. In particular, in Section 6.1 we will partially answer the question, when a graph randomly drawn from $\mathcal{F}orb_{n,m}(K_3)$, the class of all triangle-free graphs on n vertices with exactly m edges, is almost surely bipartite.

6.1 Triangle free-graphs

It is well-known (and not difficult to show) that for $p \ll 1/n$ a random graph $G_{n,p}$ is with high probability a tree. Observe, that this implies in particular, that for all $m \ll n$ a triangle-free graph on n vertices with m edges is almost surely bipartite. For much higher edge densities a similar phenomenon follows from Theorem 3.1.1 of Erdős, Kleitman and Rothschild. Using the structure of a maximum bipartite graph, one easily deduces that their result implies that for $m \geq n^2/8$ almost every triangle-free graph on n vertices with m edges is bipartite.

While these observations make it tempting to speculate that the same phenomena holds in fact for all $1 \leq m \leq \binom{n}{2}$, it is not too difficult to show that in the range $n \ll m \ll n^{3/2}$ in fact almost *no* triangle-free graph on n vertices with m edges is bipartite. The aim of this section is to prove the following result of Prömel and Steger [72], which exhibits the structure of a random triangle-free graph on n vertices with m edges for a wide range of functions $m = m(n)$.

Theorem 6.1.1 *Let $\mathcal{F}orb_{n,m}(K_3)$ denote the class of all triangle-free graphs on n vertices with $m = m(n)$ edges. Then there exists constants $c_1, c_2, c_3 > 0$ such that*

$$\begin{aligned} &\text{Prob}\left[G \in \mathcal{C}ol_n(2) \mid G \in \mathcal{F}orb_{n,m}(K_3)\right] = 1 - o(1), && \textit{for all} \ \ m \ \geq \ c_1 n^{7/4}\log n, \\ &\text{Prob}\left[G \in \mathcal{C}ol_n(2) \mid G \in \mathcal{F}orb_{n,m}(K_3)\right] = o(1), && \textit{for all} \ \ c_2 n \ \leq \ m \ \leq \ c_3 n^{3/2}. \end{aligned}$$

It as an interesting open problem to determine, whether there exists a threshold function $t(n)$ such that whenever $m \gg t(n)$ then almost every triangle-free graph is bipartite, while for $c_2 n \leq m \ll t(n)$ almost every triangle-free graph is not bipartite. Our general approach to proving Theorem 6.1.1 is to use a variation of the Kleitman-Rothschild method. Recall, that in order to apply this method one needs good bounds on the size of the negligible sets in terms of the size of the whole class on some smaller vertex set. As the density of a subgraph induced by a subset of the vertex set usually differs from the density of the whole graph, this step, however, does not work if instead of classes $\mathcal{F}orb_n(H)$ one considers classes $\mathcal{F}orb_{n,m}(H)$. In this section we will develop a method for resolving these difficulties.

6.1.1 Proof of the second part of Theorem 6.1.1

All we need for the proof of the second part of Theorem 6.1.1 are good bounds for the cardinality of $\mathcal{F}orb_{n,m}(K_3)$ and for the number of bipartite graphs on n vertices with m edges. These, however, were already obtained in Theorem 1.4.2 together with (1.4.8), and in Corollary 5.3.16, respectively. Instead of the later result one can also use the bounds of the following lemma, which is easy to prove, but suffices for the purpose of proving Theorem 6.1.1.

Lemma 6.1.2 *For all m with $m >> n\log n$ and n sufficiently large*

$$2^{n-2\log n} \cdot \binom{\frac{1}{4}n^2}{m} \leq |\mathcal{C}ol_{n,m}(2)| \leq 2^n \cdot \binom{\frac{1}{4}n^2}{m}.$$

Proof. The upper bound is obvious, as there are exactly 2^n ways to choose a two-coloring and at most $\binom{\frac{1}{4}n^2}{m}$ ways two choose m edges respecting a given two-coloring. Observe that this upper bound also implies that for all sufficiently large n

$$\sum_{1 \leq p \leq \frac{n}{2}} \binom{n}{p} \cdot 2^p \cdot 2^{n-p} \cdot \binom{\frac{1}{4}p^2 + \frac{1}{4}(n-p)^2}{m} \leq 2^n \cdot \binom{\frac{1}{4}n^2}{m} e^{-\frac{m}{n}}$$

is an upper bound for the number of bipartite graphs which are *not* connected. Therefore

$$|\mathcal{C}ol_{n,m}(2)| \geq \binom{n}{\frac{n}{2}}\binom{\frac{1}{4}n^2}{m} - 2^n \cdot \binom{\frac{1}{4}n^2}{m} e^{-\frac{m}{n}} \geq 2^{n-\log n}\binom{\frac{1}{4}n^2}{m} - 2^n \cdot \binom{\frac{1}{4}n^2}{m} e^{-\frac{m}{n}}.$$

□

Proof of the second part of Theorem 6.1.1. Observe that by Theorem 1.4.2 there exists a constant $c > 0$ such that

$$|\mathcal{F}orb_{n,m}(K_3)| \geq e^{-c\,m^3 n^{-3}} \cdot \binom{\binom{n}{2}}{m} \qquad \text{for all } m \leq n^{3/2}.$$

On the other hand, by 6.1.2

$$|\mathcal{C}ol_{n,m}(2)| \leq 2^n \cdot \binom{\frac{1}{4}n^2}{m}.$$

As for $m \leq n^{3/2}$ we have

$$\binom{\binom{n}{2}}{m} \geq e^{\frac{m}{3}}\binom{\frac{1}{4}n^2}{m},$$

this easily concludes the proof for an appropriate choice of c_2 and c_3. □

6.1.2 Proof of the first part of Theorem 6.1.1

We proceed in three steps. First we define sets $\mathcal{A}(n,m)$, $\mathcal{B}(n,m)$, $\mathcal{C}(n,m)$ and and $\mathcal{D}(n,m)$ such that

$$\mathcal{F}orb_{n,m}(K_3) \subseteq \mathcal{C}ol_{n,m}(2) \cup \mathcal{A}(n,m) \cup \mathcal{B}(n,m) \cup \mathcal{C}(n,m) \cup \mathcal{D}(n,m),$$

and establish upper bounds for the cardinalities of these sets. In a second part we show inductively that there exists a constant $c > 0$ such that

$$|\mathcal{F}orb_{n,m}(K_3)| \leq 2^{c n^{3/2}\log n} \cdot |\mathcal{C}ol_{n,m}(2)| \qquad \text{for all } n, m \in \mathbb{N}.$$

In the final step we subsequently use this inequality to show that $|\mathcal{F}orb_{n,m}(K_3)| = (1+o(1)) \cdot |\mathcal{C}ol_{n,m}(2)|$ whenever m is sufficiently large. Let

$\mathcal{A}(n,m) =$ set of all graphs $G = (V_n, E)$ in $\mathcal{F}orb_{n,m}(K_3)$ which contain a vertex v of degree less than $\sqrt{n}$.

$\mathcal{B}(n,m) =$ set of all graphs $G = (V_n, E)$ in $\mathcal{F}orb_{n,m}(K_3)$ which contain a set Q of size $\sqrt{n}$ such that $|\Gamma(Q)| \leq \frac{1}{2}n - n^{3/4}$.

$\mathcal{C}(n,m) =$ set of all graphs $G = (V_n, E)$ in $\mathcal{F}orb_{n,m}(K_3)$ which contain a vertex v and a set Q_v of size $\sqrt{n}$ contained in the neighborhood of v such that $|\Gamma(Q_v)| \geq \frac{1}{2}n + n^{3/4}$.

$\mathcal{D}(n,m) =$ set of all graphs $G = (V_n, E)$ in $\mathcal{F}orb_{n,m}(K_3) \setminus (\mathcal{B}(n,m) \cup \mathcal{C}(n,m))$ which contain two vertices v_1 and v_2 connected by an edge and two sets Q_1 and Q_2 of size $\sqrt{n}$ contained in the neighborhood of v_1 and v_2, respectively, such that $|\Gamma(Q_1) \cap \Gamma(Q_2)| \geq \frac{1}{30}n$.

Lemma 6.1.3 *For all sufficiently large n*

$$\mathcal{F}orb_{n,m}(K_3) \setminus (\mathcal{A}(n,m) \cup \mathcal{B}(n,m) \cup \mathcal{C}(n,m) \cup \mathcal{D}(n,m)) \ \subseteq \ \mathcal{C}ol_{n,m}(2).$$

Proof. Let $G = (V_n, E)$ be an arbitrary but fixed graph in $\mathcal{F}orb_{n,m}(K_3) \setminus (\mathcal{A}(n,m) \cup \mathcal{B}(n,m) \cup \mathcal{C}(n,m) \cup \mathcal{D}(n,m))$. For every vertex v fix a set $Q_v \subseteq \Gamma(v)$ of size $\sqrt{n}$ and let $R_v = \Gamma(Q_v)$. (By definition of $\mathcal{A}(n,m)$ every vertex has degree at least $\sqrt{n}$.) Then $|R_v| \geq \frac{1}{2}n - n^{\frac{3}{4}}$ for all $v \in V_n$ and $|R_x \cap R_y| \leq \frac{1}{30}n$ for all vertices x, y connected by an edge (cf. the definition of $\mathcal{B}(n,m)$ and $\mathcal{D}(n,m)$). One easily checks that this implies

$$|R_x \cap R_v| \geq \frac{13}{30}n - 3n^{\frac{3}{4}} \qquad \text{for all } v \in V_n \text{ and } x \in R_v. \tag{6.1.1}$$

Fix an edge $\{v_1, v_2\}$ and let $T_i = \{v \in V_n \mid |R_v \cap R_{v_i}| \geq \frac{7}{24}n\}$. Routine calculations show that for all sufficiently large n (6.1.1) implies that $V_n = T_1 \cup T_2$ and that T_1 and T_2 are stable. $\square$

Lemma 6.1.4

$$|\mathcal{A}(n,m)| \leq 2^{2\sqrt{n}\log n} \cdot \sum_{e=0}^{\sqrt{n}-1} |\mathcal{F}orb_{n-1,m-e}(K_3)|,$$

for n sufficiently large.

Proof. Construct all graphs in $\mathcal{A}(n,m)$ as follows. First choose the vertex v (at most n choices). For all $0 \leq e < \sqrt{n}$ choose then a triangle-free graph with $m - e$ edges on the remaining vertices ($|\mathcal{F}orb_{n-1,m-e}(K_3)|$ ways). Finally, choose the e neighbors of v (in $\binom{n-1}{e} \leq 2^{\sqrt{n}\log n}$ ways).
Together, this gives

$$|\mathcal{A}(n,m)| \leq 2^{\log n + \sqrt{n}\log n} \cdot \sum_{e=0}^{\sqrt{n}-1} |\mathcal{F}orb_{n-1,m-e}(K_3)|.$$

$\square$

Lemma 6.1.5

$$|\mathcal{B}(n,m)| \leq 2^{2n} \cdot \sum_{e_q=0}^{\binom{\sqrt{n}}{2}} \sum_{e_{qr}=0}^{\sqrt{n}(\frac{1}{2}n - n^{3/4})} \binom{\sqrt{n}(\frac{1}{2}n - n^{3/4})}{e_{qr}} \cdot |\mathcal{F}orb_{n-\sqrt{n},m-e_q-e_{qr}}(K_3)|,$$

for n sufficiently large.

Proof. Construct all graphs in $\mathcal{B}(n,m)$ as follows. First choose the set Q (at most $\binom{n}{\sqrt{n}} \leq 2^{\sqrt{n}\log n}$ ways). For all $0 \leq e_q \leq \binom{\sqrt{n}}{2}$ and $0 \leq e_{qr} \leq \sqrt{n}(\frac{1}{2}n - n^{3/4})$ choose then a triangle-free graph with $m - e_q - e_{qr}$ edges on the remaining vertices ($|\mathcal{F}orb_{n-\sqrt{n},m-e_q-e_{qr}}(K_3)|$ ways), choose e_q edges in Q (in $\binom{\binom{\sqrt{n}}{2}}{e_q} \leq 2^{\frac{1}{2}n}$ ways), choose the set $\Gamma(Q)$ (less than 2^n ways), and finally choose the e_{qr} edges connecting Q and $\Gamma(Q)$ (at most $\binom{\sqrt{n}(\frac{1}{2}n-n^{3/4})}{e_{qr}}$ ways).
Together with $2^{\sqrt{n}\log n+\frac{1}{2}n+n} \leq 2^{2n}$ this completes the proof of the lemma. □

Lemma 6.1.6

$$|\mathcal{C}(n,m)| \leq 2^{2\sqrt{n}\log n} \cdot \sum_{e_v=0}^{n} \binom{\frac{1}{2}n - n^{3/4}}{e_v} \cdot |\mathcal{F}orb_{n-1,m-e_v}(K_3)|,$$

for n sufficiently large.

Proof. Construct all graphs in $\mathcal{C}(n,m)$ as follows. First choose a vertex v (n ways). For all $0 \leq e_v \leq n$ choose then a triangle-free graph with $m-e_v$ edges on the remaining vertices ($|\mathcal{F}orb_{n-1,m-e_v}(K_3)|$ ways) and choose an appropriate set Q_v (less than $\binom{n}{\sqrt{n}} \leq 2^{\sqrt{n}\log n}$ ways). Finally, connect v to the remaining vertices. To do this observe, that v may not be connected to any vertex in $\Gamma(Q_v)$. Hence, the number of ways to connect v is bounded by $\binom{\frac{1}{2}n-n^{3/4}}{e_v}$.
Together, this gives

$$|\mathcal{C}(n,m)| \leq 2^{\log n+\sqrt{n}\log n} \cdot \sum_{e_v=0}^{n} \binom{\frac{1}{2}n - n^{3/4}}{e_v} \cdot |\mathcal{F}orb_{n-1,m-e_v}(K_3)|.$$

□

Lemma 6.1.7

$$\begin{aligned} |\mathcal{D}(n,m)| \quad &\leq \quad 2^{3\sqrt{n}\log n} \cdot \max_{\frac{1}{30}n \leq I \leq n} \left[\sum_{0\leq e_1,e_2<n} \sum_{\substack{0\leq i_1\leq e_1\\ 0\leq i_2\leq e_2}} \binom{I+2n^{3/4}}{i_1\,;\,i_2} \right. \\ &\qquad \left. \binom{\frac{1}{2}n+n^{3/4}-I}{e_1-i_1}\binom{\frac{1}{2}n+n^{3/4}-I}{e_2-i_2} \cdot |\mathcal{F}orb_{n-2,m-e_1-e_2-1}(K_3)| \right], \end{aligned}$$

for n sufficiently large.

Proof. Construct all graphs in $\mathcal{D}(n,m)$ as follows. First choose two vertices v_1 and v_2 (less than n^2 ways). For all $0 \leq e_1, e_2 < n$ choose then an appropriate triangle-free graph with $m - e_1 - e_2 - 1$ edges on the remaining vertices ($|\mathcal{F}orb_{n-2,m-e_1-e_2-1}(K_3)|$ ways). Then choose the sets Q_1 and Q_2 (at most $\binom{n}{\sqrt{n}}^2 \leq 2^{2\sqrt{n}\log n}$ ways). Finally, connect v_1 and v_2 to the remaining vertices. To do this observe, that v_i cannot be connected to any vertex in $\Gamma(Q_i)$ and that no vertex in $V_n \setminus (\Gamma(Q_1) \cup \Gamma(Q_2))$ can be connected to both v_1 and v_2. Recall also that by definition of $\mathcal{B}(n,m)$ and $\mathcal{C}(n,m)$ we have $\frac{1}{2}n - n^{3/4} \leq |\Gamma(Q_i)| \leq \frac{1}{2}n + n^{3/4}$. Hence, there are at most

$$\begin{aligned}
&\sum_{\substack{0\leq i_1\leq e_1\\0\leq i_2\leq e_2}} \binom{n-|\Gamma(Q_1)\cup\Gamma(Q_2)|}{i_1\,;\,i_2}\binom{|\Gamma(Q_2)\setminus\Gamma(Q_1)|}{e_1-i_1}\binom{|\Gamma(Q_1)\setminus\Gamma(Q_2)|}{e_2-i_2}\\
&\leq \sum_{\substack{0\leq i_1\leq e_1\\0\leq i_2\leq e_2}} \binom{|\Gamma(Q_1)\cap\Gamma(Q_2)|+2n^{3/4}}{i_1\,;\,i_2}\binom{\frac{1}{2}n+n^{3/4}-|\Gamma(Q_1)\cap\Gamma(Q_2)|}{e_1-i_1}\cdot\\
&\qquad\binom{\frac{1}{2}n+n^{3/4}-|\Gamma(Q_1)\cap\Gamma(Q_2)|}{e_2-i_2}
\end{aligned}$$

ways to connect v_1 and v_2 to the remaining vertices.
Together, this gives

$$\begin{aligned}
|\mathcal{D}(n,m)| \leq\ & 2^{2\log n+2\sqrt{n}\log n}\cdot \max_{\frac{1}{30}n\leq I\leq n}\Bigg[\sum_{0\leq e_1,e_2<n}\ \sum_{\substack{0\leq i_1\leq e_1\\0\leq i_2\leq e_2}}\binom{I+2n^{3/4}}{i_1\,;\,i_2}\\
&\binom{\frac{1}{2}n+n^{3/4}-I}{e_1-i_1}\binom{\frac{1}{2}n+n^{3/4}-I}{e_2-i_2}\cdot|\mathcal{F}orb_{n-2,m-e_1-e_2-1}(K_3)|\Bigg],
\end{aligned}$$

for all sufficiently large n. $\square$

Some useful inequalities. At this point we like to remind the reader of the inequalities in Appendix A which we use in the sequel, partly without further reference, and the following two inequalities: $\binom{x}{\gamma-1;\delta}\binom{a-x}{c-\gamma+1} \leq \binom{x}{\gamma;\delta}\binom{a-x}{c-\gamma}$ for all $a,c,x,\gamma,\delta\in\mathbb{N}$ such that $\gamma \leq c\cdot\frac{x}{a}$, and $\binom{x}{a;b} < e^{-\frac{ab}{x}}\cdot\binom{x}{a}\binom{x}{b}$. for all $a,b,x\in\mathbb{N}$.

Lemma 6.1.8 *For all $0\leq\delta\leq 1$ and $a,b,c\in\mathbb{N}$ with $c\leq\frac{2}{3}a$:*

$$\sum_{x=0}^{b} e^{\delta x}\binom{b}{x}\binom{a-b}{c-x} \leq e^{6\delta\frac{bc}{a}}\binom{a}{c}.$$

Proof. One easily checks that $e^{\delta x}\cdot\binom{b}{x} \leq \binom{e^{\delta}\cdot b}{x}$ for all $\delta>0$ and $b,x\in\mathbb{N}$. Hence,

$$\begin{aligned}
\sum_{x=0}^{b} e^{\delta x}\binom{b}{x}\binom{a-b}{c-x} &\leq \sum_{x=0}^{b}\binom{e^{\delta}\cdot b}{x}\binom{a-b}{c-x}\\
&= \binom{a+(e^{\delta}-1)b}{c}\\
&\leq \binom{a}{c}e^{(e^{\delta}-1)\frac{bc}{a-c}}.
\end{aligned}$$

The observations that for $0\leq\delta\leq 1$ one has $e^{\delta}-1\leq 2\delta$ and that $\frac{1}{a-c}\leq\frac{3}{a}$ by assumption on a and c conclude the proof. $\square$

We continue by applying this lemma to the sums occuring in Lemma 6.1.5 - 6.1.7.

Lemma 6.1.9 *For n sufficiently large and $m \geq \frac{1}{2}n^{3/2}$:*

$$\sum_{e_q=0}^{\binom{\sqrt{n}}{2}} \sum_{e_{qr}=0}^{\sqrt{n}(\frac{1}{2}n-n^{3/4})} \binom{\sqrt{n}(\frac{1}{2}n - n^{3/4})}{e_{qr}} \cdot \binom{\frac{1}{4}(n-\sqrt{n})^2}{m - e_q - e_{qr}} \cdot e^{\delta(e_q+e_{qr})}$$
$$\leq \binom{\frac{1}{4}n^2}{m} \cdot e^{-n^{1/4}\frac{m}{n}},$$

for all $0 \leq \delta \leq \frac{1}{14}n^{-1/4}$.

Proof. We first consider the case $m \geq \frac{1}{7}n^2$. Then for n sufficiently large

$$\begin{aligned}
&\sum_{e_q=0}^{\binom{\sqrt{n}}{2}} \sum_{e_{qr}=0}^{\sqrt{n}(\frac{1}{2}n-n^{3/4})} \binom{\sqrt{n}(\frac{1}{2}n - n^{3/4})}{e_{qr}} \cdot \binom{\frac{1}{4}(n-\sqrt{n})^2}{m - e_q - e_{qr}} \cdot e^{\delta(e_q+e_{qr})} \\
\leq\ & e^{\frac{1}{2}\delta n^{3/2}} \cdot \sum_{e_q=0}^{\binom{\sqrt{n}}{2}} \binom{\frac{1}{4}n^2 + \frac{1}{4}n - n^{5/4}}{m - e_q} \\
\leq\ & e^{\frac{1}{2}\delta n^{3/2}} \cdot \binom{\frac{1}{4}n^2 + \frac{3}{4}n - \frac{\sqrt{n}}{2} - n^{5/4}}{m} \\
\leq\ & e^{\frac{1}{2}\delta n^{3/2}} \cdot \binom{\frac{1}{4}n^2}{m} \cdot e^{-2n^{1/4}\frac{m}{n}} \\
\leq\ & \binom{\frac{1}{4}n^2}{m} \cdot e^{-n^{1/4}\frac{m}{n}}.
\end{aligned}$$

Now assume that $m \leq \frac{1}{7}n^2$. Then according to Lemma 6.1.8

$$\begin{aligned}
&\sum_{e_q=0}^{\binom{\sqrt{n}}{2}} \sum_{e_{qr}=0}^{\sqrt{n}(\frac{1}{2}n-n^{3/4})} \binom{\sqrt{n}(\frac{1}{2}n - n^{3/4})}{e_{qr}} \cdot \binom{\frac{1}{4}(n-\sqrt{n})^2}{m - e_q - e_{qr}} \cdot e^{\delta(e_q+e_{qr})} \\
\leq\ & \sum_{e_q=0}^{\binom{\sqrt{n}}{2}} e^{\delta e_q} \cdot \binom{\frac{1}{4}n^2 - n^{5/4} + \frac{1}{4}n}{m - e_q} \cdot e^{6\delta(\frac{1}{2}n^{3/2} - n^{5/4})\frac{m-e_q}{\frac{1}{4}n^2 - n^{5/4} + \frac{1}{4}n}} \\
\leq\ & \binom{\sqrt{n}}{2} \cdot e^{\delta\binom{\sqrt{n}}{2}} \cdot \binom{\frac{1}{4}n^2}{m} \cdot e^{-2n^{1/4}\frac{m}{n}} \cdot e^{\frac{13}{14}n^{1/4}\frac{m}{n}} \\
\leq\ & \binom{\frac{1}{4}n^2}{m} \cdot e^{-n^{1/4}\frac{m}{n}}.
\end{aligned}$$

□

Similarly we obtain

Lemma 6.1.10 *For n sufficiently large*

$$\sum_{e_v=0}^{n} \binom{\frac{1}{2}n - n^{3/4}}{e_v} \cdot \binom{\frac{1}{4}(n-1)^2}{m-e_v} \cdot e^{\delta e_v} \le \binom{\frac{1}{4}n^2}{m} \cdot e^{-n^{3/4}\frac{m}{n^2}},$$

for all $0 \le \delta \le \frac{1}{5}n^{-1/4}$.

$\square$

For the $\mathcal{D}$-sets the proof is a bit more envolved.

Lemma 6.1.11 *For n sufficiently large*

$$\begin{aligned}
\max_{\frac{1}{30}n \le I \le n} \Big[\sum_{0 \le e_1, e_2 < n} \sum_{\substack{0 \le i_1 \le e_1 \\ 0 \le i_2 \le e_2}} & \binom{I + 2n^{3/4}}{i_1\,;\,i_2} \binom{\frac{1}{2}n + n^{3/4} - I}{e_1 - i_1} \binom{\frac{1}{2}n + n^{3/4} - I}{e_2 - i_2} \cdot \\
& \binom{\frac{1}{4}(n-2)^2}{m - e_1 - e_2 - 1} e^{\delta(e_1+e_2+1)} \Big] \\
\le \quad & \binom{\frac{1}{4}n^2}{m} \cdot e^{60n^{-1/4}\frac{m}{n}} \cdot e^{-\frac{1}{5}\frac{m^2}{n^3}},
\end{aligned}$$

for all $0 \le \delta \le \frac{1}{12}n^{-1/4}$ *and* $m \ge 5\delta n^2$

Proof. Let D_I denote the expression within the brackets of the statement of the lemma. We first consider the case $m \le \frac{1}{7}n^2$. Changing the order of summation and applying Lemma 6.1.8 twice we obtain for all n sufficiently large

$$D_I \quad \le \quad \sum_{0 \le i_1, i_2 < n} \binom{I + 2n^{3/4}}{i_1\,;\,i_2} \binom{\frac{1}{4}n^2 + 2n^{3/4} - 2I + 1}{m - i_1 - i_2 - 1} \cdot e^{\delta(i_1+i_2) + 25\delta\frac{m}{n}}$$

Using now that $\binom{x}{\gamma-1;\delta}\binom{a-x}{c-\gamma+1} \le \binom{x}{\gamma;\delta}\binom{a-x}{c-\gamma}$ for all a, c, x, γ, δ with $\gamma \le c \cdot \frac{x}{a}$, we observe that for every fixed i_2 the maximum summand occurs for $i_1 \ge (m - i_2 - 1)\frac{I+2n^{3/4}}{\frac{1}{4}n^2+4n^{3/4}-I+1} \ge \frac{3Im}{n^2}$. Arguing analogously for i_2 we obtain

$$\begin{aligned}
D_I \quad & \le \quad 2n \cdot \max_{\frac{3Im}{n^2} \le i_1, i_2 < n} \underbrace{\binom{I + 2n^{3/4}}{i_1\,;\,i_2}}_{\le e^{-\frac{i_1 i_2}{I+2n^{3/4}}}\binom{I+2n^{3/4}}{i_1}\binom{I+2n^{3/4}}{i_2}} \binom{\frac{1}{4}n^2 + 2n^{3/4} - 2I + 1}{m - i_1 - i_2 - 1} \cdot e^{\delta(i_1+i_2)+25\delta\frac{m}{n}} \\
& \le \quad 2n \cdot \max_{\frac{3Im}{n^2} \le i_1, i_2 < n} e^{-\frac{i_1 i_2}{I+2n^{3/4}} + \delta(i_1+i_2)} \cdot \binom{\frac{1}{4}n^2 + 6n^{3/4} + 1}{m - 1} \cdot e^{25\delta\frac{m}{n}} \\
& \le \quad 2n \cdot e^{-\frac{\left(\frac{3Im}{n^2}\right)^2}{I+2n^{3/4}} + 2\delta\frac{3Im}{n^2}} \cdot \binom{\frac{1}{4}n^2}{m} e^{57n^{3/4}\frac{m}{n^2}} \cdot e^{25\delta\frac{m}{n}} \\
& \le \quad \binom{\frac{1}{4}n^2}{m} \cdot e^{-\frac{1}{5}\frac{m^2}{n^3} + 60n^{-1/4}\frac{m}{n}}.
\end{aligned}$$

The case $m \geq \frac{1}{7}n^2$ follows similarly. The main differences are that in the beginning we bound $e^{\delta(e_1+e_2+1)}$ by $e^{2\delta n}$ and use the ordinary Vandermonde identity instead of Lemma 6.1.8, and that later on the inequality $\binom{\frac{1}{4}n^2+6n^{3/4}+1}{m-1} \leq \binom{\frac{1}{4}n^2}{m} \cdot e^{(6n^{3/4}+2)\cdot\ln\frac{1}{4}n^2}$ is used. $\square$

A crude upper bound. We are now in a position to establish an upper bound for the size of $\mathcal{F}orb_{n,m}(K_3)$, which is valid for all $m \in \mathbb{N}$.

Lemma 6.1.12 *There exists a constant $c \geq 47$ such that*

$$|\mathcal{F}orb_{n,m}(K_3)| \leq 2^{cn^{3/2}\log n} \cdot |\mathcal{C}ol_{n,m}(2)| \qquad \textit{for all } 0 \leq m \leq \tbinom{n}{2} \textit{ and } n \in \mathbb{N}.$$

Proof. Observe that by Theorem 6.1.2 it suffices to show

$$|\mathcal{F}orb_{n,m}(K_3)| \leq \binom{\frac{1}{4}n^2}{m} \cdot 2^{cn^{3/2}\log n} \qquad \text{for all } 0 \leq m \leq \tbinom{n}{2} \text{ and } n \in \mathbb{N}.$$

Choose $n_0 \geq 2^{300}$ in such a way that Lemma 6.1.4 - 6.1.7, Theorem 6.1.2 and Lemma 6.1.9 - 6.1.11 hold for all $n \geq n_0$ and such that for all $n \geq n_0$:

$$n^{3/2}\log n \leq \frac{1}{16}n^2$$

and

$$2^{(n-x)^{3/2}\log(n-x)} \leq 2^{n^{3/2}\log n - x\sqrt{n}\log n} \qquad \text{for all } 0 \leq x \leq \sqrt{n}.$$

Subsequently choose $c \geq 47$ in such a way that the statement of the theorem is true for all $n \leq n_0$. We proceed by induction on n. So assume the statement of the theorem holds for all $n' < n$ for some $n \in \mathbb{N}$. We will show that it then also holds for n. Let m be an arbitrary integer between 0 and $\binom{n}{2}$. If $m \leq 11 \cdot n^{3/2}\log n$ then trivially

$$|\mathcal{F}orb_{n,m}(K_3)| \leq \binom{\binom{n}{2}}{m} \leq \binom{\frac{n^2}{4}}{m} \cdot e^{\frac{n^2}{4}\frac{m}{\frac{1}{4}n^2-m}} \leq \binom{\frac{n^2}{4}}{m} \cdot e^{\frac{4}{3}m} \leq \binom{\frac{n^2}{4}}{m} \cdot 2^{47n^{3/2}\log n},$$

so there is nothing to show. So we may assume that $m \geq 11 \cdot n^{3/2}\log n$.
As Lemma 6.1.3 implies that

$$|\mathcal{F}orb_{n,m}(K_3)| \leq |\mathcal{A}(n,m)| + |\mathcal{B}(n,m)| + |\mathcal{C}(n,m)| + |\mathcal{D}(n,m)| + |\mathcal{C}ol_{n,m}(2)|$$

it suffices to show that the five summands on the right hand sight are all smaller or equal than

$$\frac{1}{5} \cdot \binom{\frac{1}{4}n^2}{m} \cdot 2^{cn^{3/2}\log n}.$$

For the last term this is an immediate consequence of Theorem 6.1.2. For the first term Lemma 6.1.4, induction and the choice of n_0 implies

$$|\mathcal{A}(n,m)| \quad \leq \quad 2^{2\sqrt{n}\log n} \cdot \sum_{e=0}^{\sqrt{n}-1} |\mathcal{F}orb_{n-1,m-e}(K_3)|$$

$$
\begin{aligned}
&\leq 2^{2\sqrt{n}\log n} \cdot \sum_{e=0}^{\sqrt{n}-1} \binom{\frac{1}{4}(n-1)^2}{m-e} \cdot 2^{c\cdot(n-1)^{3/2}\cdot\log(n-1)} \\
&\leq 2^{2\sqrt{n}\log n} \cdot \binom{\frac{1}{4}n^2}{m} \cdot 2^{c\cdot n^{3/2}\log n - c\cdot\sqrt{n}\log n} \\
&\leq \frac{1}{5} \cdot \binom{\frac{1}{4}n^2}{m} \cdot 2^{c\cdot n^{3/2}\cdot\log n}.
\end{aligned}
$$

Combining Lemma 6.1.5 and Lemma 6.1.9 (for $\delta = 0$) and the assumption on n_0 one easily gets

$$
|\mathcal{B}(n,m)| \quad \leq \quad 2^{2n} \cdot \binom{\frac{1}{4}n^2}{m} \cdot e^{-n^{1/4}\frac{m}{n}} \cdot 2^{c\cdot n^{3/2}\cdot\log n} \cdot 2^{-cn\log n}
$$

which immediately implies

$$
|\mathcal{B}(n,m)| \quad \leq \quad \frac{1}{5} \cdot \binom{\frac{1}{4}n^2}{m} \cdot 2^{c\cdot n^{3/2}\cdot\log n}.
$$

Similarly we obtain

$$
|\mathcal{C}(n,m)| \quad \leq \quad 2^{2\sqrt{n}\log n} \cdot e^{-n^{3/4}\frac{m}{n^2}} \cdot \binom{\frac{1}{4}n^2}{m} \cdot 2^{c\cdot n^{3/2}\cdot\log n} \cdot 2^{-c\cdot\sqrt{n}\log n}
$$

and

$$
|\mathcal{D}(n,m)| \quad \leq \quad 2^{3\sqrt{n}\log n} \cdot e^{60n^{-1/4}\frac{m}{n}} \cdot e^{-\frac{1}{5}\frac{m^2}{n^3}} \cdot \binom{\frac{1}{4}n^2}{m} \cdot 2^{c\cdot n^{3/2}\cdot\log n} \cdot 2^{-2c\cdot\sqrt{n}\log n}
$$

from Lemma 6.1.6 together with Lemma 6.1.10 (for $\delta = 0$) and from Lemma 6.1.7 together with Lemma 6.1.11 (for $\delta = 0$), respectively. We now again easily deduce that

$$
|\mathcal{C}(n,m)|, \quad |\mathcal{D}(n,m)| \quad \leq \quad \frac{1}{5} \cdot \binom{\frac{1}{4}n^2}{m} \cdot 2^{c\cdot n^{3/2}\cdot\log n}
$$

(To see the latter inequality use that for $m \leq n^{7/4}\log n$ one has $60n^{-1/4}\frac{m}{n} \leq 60\sqrt{n}\log n$ and that for $m \geq n^{7/4}\log n$ and $n \geq n_0 \geq 2^{300}$ one has $60n^{-1/4}\frac{m}{n} \leq \frac{1}{5}\frac{m^2}{n^3}$). $\Box$

The final step. To prove our main theorem we have to refine the error function of Lemma 6.1.12. The main idea is to substitute the function $n^{3/2}\log n$ by a function which depends also on m. This function should tend to zero if $m = m(n)$ is "large". For "small" m however the new function should dominate our old function, so that in this case we already know that the statement is true.

Theorem 6.1.13 *There exist a constant $a > 0$ such that for all $n \in \mathbb{N}$ and all $0 \leq m \leq \binom{n}{2}$:*

$$|\mathcal{F}orb_{n,m}(K_3)| \leq (1 + a2^{-\sqrt{n}} + 2^{(2-\frac{m}{14cn^{7/4}\log n})\cdot cn^{3/2}\log n}) \cdot \mathcal{C}ol_{n,m}(2),$$

where c is the constant from Lemma 6.1.12.

Proof. Choose $n_0 \geq 2^{400}$ in such a way that Theorem 6.1.2, Lemma 6.1.4 - 6.1.7 and Lemma 6.1.9 - 6.1.11 hold for all $n \geq n_0$ and such that for all $n \geq n_0$:

$$(6.1.2)\quad (n-x)^{3/2}\log(n-x) \leq n^{3/2}\log n - x\sqrt{n}\log n \qquad \text{for all } 1 \leq x \leq \sqrt{n}.$$

Subsequently choose $a \geq 2$ in such a way that the statement of the theorem is true for all $n \leq n_0$. We proceed by induction on n. So assume the theorem holds for all $n' < n$ for some $n \in \mathbb{N}$. We will show that it then also holds for n. Let m be an arbitrary integer between 0 and $\binom{n}{2}$. Observe that if $2 - \frac{m}{14cn^{7/4}\log n} \geq 1$ then by Lemma 6.1.12 there is nothing to show. So we may assume without loss of generality that $m \geq 14cn^{7/4}\log n$.
Our main goal is to proceed as in the proof of Lemma 6.1.12. To be able to do this, we have to convert the bounds on the size of $\mathcal{F}orb_{n-x,m-y}(K_3)$ given by the induction hypothesis into a somewhat nicer form. First observe that for all $1 \leq x \leq \sqrt{n}$ and $0 \leq y \leq xn$

$$\mathcal{C}ol_{n-x,m-y}(2) \leq 2^{n-x} \cdot \binom{\frac{1}{4}(n-x)^2}{m-y}$$

(cf. Theorem 6.1.2) and that

$$2^{(2-\frac{m-y}{14c(n-x)^{7/4}\log(n-x)})\cdot c(n-x)^{3/2}\cdot\log(n-x)} \leq 2^{(2-\frac{m}{14cn^{7/4}\log n})\cdot cn^{3/2}\log n + \frac{yn}{14\,n^{5/4}}}.$$

To see the last inequality observe that by assumption (6.1.2)

$$\begin{aligned}
&(2 - \frac{m-y}{14c(n-x)^{7/4}\log(n-x)}) \cdot (n-x)^{3/2}\log(n-x) \\
\leq\ & 2n^{3/2}\log n - 2x\sqrt{n}\log n - \frac{m-y}{14c(n-x)^{1/4}} \\
\leq\ & n^{3/2}\log n \cdot (2 - \frac{m}{14cn^{7/4}\log n}) + \frac{y}{14c\,n^{1/4}}.
\end{aligned}$$

We therefore conclude that for all $1 \leq x \leq \sqrt{n}$ and $0 \leq y \leq xn \leq n^{3/2}$:

$$\begin{aligned}
|\mathcal{F}orb_{n-x,m-y}(K_3)| &\leq (1 + a + 2^{(2-\frac{m}{14cn^{7/4}\log n})\cdot cn^{3/2}\log n + \frac{y}{14\,n^{1/4}}}) \cdot 2^{n-x} \cdot \binom{\frac{1}{4}(n-x)^2}{m-y} \\
&\leq (2a + 2^{(2-\frac{m}{14cn^{7/4}\log n})\cdot cn^{3/2}\log n + \frac{y}{14\,n^{1/4}}}) \cdot 2^{n-x} \cdot \binom{\frac{1}{4}(n-x)^2}{m-y}.
\end{aligned}$$

Recall that by Lemma 6.1.3

$$|\mathcal{F}orb_{n,m}(K_3)| \leq |\mathcal{A}(n,m)| + |\mathcal{B}(n,m)| + |\mathcal{C}(n,m)| + |\mathcal{D}(n,m)| + \mathcal{C}ol_{n,m}(2).$$

Hence, it suffices to show that each of the first four summands on the right hand sight is smaller or equal than

$$\frac{1}{4}\left(a2^{-\sqrt{n}} + 2^{(2-\frac{m}{14cn^{7/4}\log n})\cdot c\cdot n^{3/2}\log n}\right)\cdot \mathcal{C}ol_{n,m}(2).$$

Moreover, by Theorem 6.1.2 it even suffices to show that each of the first four summands on the right hand sight is smaller or equal than

$$\frac{1}{4}\left(a2^{-\sqrt{n}} + 2^{(2-\frac{m}{14cn^{7/4}\log n})\cdot c\cdot n^{3/2}\log n}\right)\cdot 2^{n-\log^2 n}\cdot\binom{\frac{1}{4}n^2}{m}.$$

For the first term Lemma 6.1.4, induction and the choice of n_0 implies

$$\begin{aligned}
|\mathcal{A}(n,m)| &\leq 2^{2\sqrt{n}\log n}\cdot\sum_{e=0}^{\sqrt{n}-1}|\mathcal{F}orb_{n-1,m-e}(K_3)| \\
&\leq 2^{2\sqrt{n}\log n}\cdot\sum_{e=0}^{\sqrt{n}-1}(2a+2^{(2-\frac{m}{14cn^{7/4}\log^n})\cdot cn^{3/2}\log n+\frac{e}{14\,n^{1/4}}})\cdot 2^{n-1}\cdot\binom{\frac{1}{4}(n-1)^2}{m-e} \\
&\leq 2^{2\sqrt{n}\log n}\cdot\binom{\frac{1}{4}(n-1)^2}{m}\cdot(2a+2^{(2-\frac{m}{14cn^{7/4}\log n})\cdot cn^{3/2}\log n+\frac{n^{1/2}}{14\,n^{1/4}}}) \\
&\leq 2^{2\sqrt{n}\log n}\cdot\binom{\frac{1}{4}n^2}{m}\cdot e^{-\frac{m}{n}}\cdot(2a+2^{(2-\frac{m}{14cn^{7/4}\log n})\cdot cn^{3/2}\log n+\frac{1}{14}n^{1/4}}),
\end{aligned}$$

from which the desired bound easily follows (recall that $m \geq 14cn^{7/4}\log n$). Combining Lemma 6.1.5 and Lemma 6.1.9 and the assumption on n_0 one easily gets

$$|\mathcal{B}(n,m)| \leq 2^{2n}\cdot\binom{\frac{1}{4}n^2}{m}\cdot e^{-n^{1/4}\frac{m}{n}}\cdot(2a+2^{(2-\frac{m}{14cn^{7/4}\log^n})\cdot cn^{3/2}\log n})\cdot 2^{n-\sqrt{n}}$$

from which the desired bound is immediately deduced.
Similarly we obtain

$$|\mathcal{C}(n,m)| \leq 2^{2\sqrt{n}\log n}\cdot\binom{\frac{1}{4}n^2}{m}\cdot e^{-n^{3/4}\frac{m}{n^2}}\cdot\left(2a+2^{(2-\frac{m}{14cn^{7/4}\log^n})\cdot cn^{3/2}\log n}\right)\cdot 2^{n-1}$$

and

$$|\mathcal{D}(n,m)| \leq 2^{3\sqrt{n}\log n}\cdot\binom{\frac{1}{4}n^2}{m}\cdot e^{60n^{-1/4}\frac{m}{n}}\cdot e^{-\frac{1}{5}\frac{m^2}{n^3}}\cdot\left(2a+2^{(2-\frac{m}{14cn^{7/4}\log^n})\cdot cn^{3/2}\log n}\right)\cdot 2^{n-2}$$

from Lemma 6.1.6 together with Lemma 6.1.10 and from Lemma 6.1.7 together with Lemma 6.1.11, respectively. Using $m \geq 14cn^{7/4}\log n$ and $c \geq 47$, both inequalities again easily imply the desired bound. □

As the first part of Theorem 6.1.1 is an immediate consequence of Theorem 6.1.13, this concludes the proof of Theorem 6.1.1.

6.2 Cliques and other graphs

A basic problem in random graph theory is the study of the evolution process of random graphs. Namely, starting with edge-probability $p = 0$ one slowly increases p and asks for every p what a typical graph with edge-probability p looks like. It is well-known that for most natural graph problems at a particular edge-probabilty p_0 the structure of the typical graph changes dramatically, that is a phase transitions occurs. It would be extremly interesting to determine whether a similar phase transitions phenomena holds for H-free graphs as well. The results of the previous section indicate that this might indeed be the case. However, even if so, Theorem 6.1.1 also implies, that then at least two pahse transitions should occur during the evolution from $m = 0$ to $\binom{n}{2}$. Unfortunately, the methods used in the proof of Theorem 6.1.1 seem far too weak to establish any further results in this direction.

For graphs H different from a triangle, the picture looks even worse. While for graphs containing a color-critical vertex, so in particular for cliques, it seems very plausible that the technique of the proof of Theorem 4.2.2 can be modified in a similar way than that of the proof of Theorem 3.1.1 to yield a result of the same flavour as the one for triangles, one clearly needs new ideas to attack the problem in general.

Chapter 7

Induced Subgraphs

The classical result of Turán states that the (unique) extremal K_{l+1}−free graph is the complete l-partite graph where all parts are chosen as equal as possible. As we have seen in Chapter 2, Erdős, Stone and Simonovits extended Turán's result to arbitrary graphs H with chromatic number $\chi(H) \geq 3$ by showing that

$$ex_n(H) = ex_n(K_{\chi(H)})(1 + o(1)).$$

The aim of this chapter is to develop a similar notion of "extremal graph" for the class of graphs not containing a graph H as an *induced* subgraph. Obviously, asking just for a graph with maximal number of edges does not lead to a meaningful concept: in this case the complete graph would be the (unique) extremal graph for all graphs H different from a clique. That is, we have to generalize some other notion inherent to extremal graphs.

Note that in the case of excluding weak subgraphs every subgraph of an H-free graph is also H-free, i.e. the property of being H-free is *hereditary.* As a consequence extremal graphs in the class $\mathcal{F}orb_n(H)$ of H-free graphs on n vertices are exactly those graphs in $\mathcal{F}orb_n(H)$ which give rise to a maximal number of H-free graphs by taking subgraphs. In particular, the cardinality of $\mathcal{F}orb_n(H)$ therefore depends mainly on the chromatic number of H (cf. Theorem 3.4.1).

As it turns out similar results are true for induced subgraphs. The role of the chromatic number therein is taken over by a different parameter, which is a common generalization of the chromatic number $\chi(H)$ and the clique covering number $\sigma(H)$.

After defining this parameter, denoted $\tau(H)$, we show that fundamental results of extremal graph theory (for weak subgraphs), as for example the Erdős, Stone, Simonovits result, carry over to induced subgraphs, if one replaces $\chi(H)$ by $\tau(H)$ in the statement of these theorems. Later, in Section 7.2, we will also show that the definitions and results of this section can be used to generalize Theorem 3.4.1 of Erdős, Frankl and Rödl to induced subgraphs.

7.1 Extremal graphs

The following definition of extremal graphs for induced subgraphs captures the hereditary property indicated above.

Definition 7.1.1 *Let $ex_n^*(H)$ denote the maximal number of edges a graph $G = (V_n, E)$ may have such that there exists a graph $G_0 = (V_n, E_0)$ with $E \cap E_0 = \emptyset$ such that $(V_n, E_0 \cup X)$ does not contain an induced $H-$subgraph for all $X \subseteq E$. We call such a graph G an extremal graph for the class $\mathcal{F}orb_n^*(H)$ of graphs on n vertices not containing an induced H-subgraph and G_0 a supplemental graph with respect to the extremal graph G.*

As an example consider the case $H = C_4$. With $K^*(\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ we denote the complete bipartite graph $K(\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ on n vertices in which the first class is replaced by a clique. Observe that as well $K^*(\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ as all subgraphs arising from $K^*(\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ by deleting edges from the original $K(\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ do not contain an induced C_4. In fact, $K(\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ is the unique extremal graph:

Lemma 7.1.2

$$ex_n^*(C_4) = \lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil,$$

and for $n \geq 6$ the graph $K(\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ is the only extremal graph for $\mathcal{F}orb_n^*(C_4)$.

Proof of Lemma 7.1.2. Consider first the case $n = 4$. Since $ex(4, C_4) = 4$, one easily checks that there are exactly two extremal graphs for $\mathcal{F}orb_4^*(C_4)$. Similarly, for the case $n = 5$ and there are exactly five extremal graphs.

Assume now $n \geq 6$ and complete the proof by induction on n. Let G_{ex} be an extremal graph for $\mathcal{F}orb_n^*(C_4)$ and let G_0 be a supplemental graph for G_{ex}. By the above remark on $K^*(\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ we know that $|E(G_{ex})| \geq \lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil$. Since $ex_n(C_4) \leq \frac{1}{4}(n + n\sqrt{4n-3})$ (Reiman [1958], cf. also Lemma 2.2.3) this implies (for $n \geq 6$) that G_{ex} contains a C_4 as a (weak) subgraph. One now easily checks that for the edges in the complement of the C_4 there are no choices: one has to be in $E(G_0)$, the other may be neither in $E(G_{ex})$ nor in $E(G_0)$. This is the case since in all other cases an appropriate subgraph of G_{ex} together with G_0 would always contain an induced C_4.

Let G'_{ex} be the graph G_{ex} without the C_4. By a similar argument as above one derives that vertices in G'_{ex} may only be connected in G_{ex} to at most two vertices of the C_4 and if a vertex is connected to two vertices these have to be either u and v or x and y, assuming that the C_4 consists of the edges $\{u, x\}$, $\{x, v\}$, $\{v, y\}$, and $\{y, x\}$. On the other hand using the fact that

$$|E(G_{ex})| \geq \lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil \text{ and } |E(G'_{ex})| \leq \lfloor\frac{n-4}{2}\rfloor\lceil\frac{n-4}{2}\rceil$$

where the latter follows from the induction hypothesis, implies that every vertex in G'_{ex} has to be connected to exactly two vertices of the C_4. Hence, $V(G'_{ex}) = V_1 \cup V_2$ where V_1 consists of the vertices connected to u and v, while V_2 consists of the vertices connected to x and y. Furthermore this easily implies that V_1 together with x and y forms a clique in G_0, while V_2 together with u and v forms a stable set in $G_0 \cup G_{ex}$. The induction hypothesis on G'_{ex} and the above remarks on edges of G_0 then easily imply that $G_{ex} = K(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ and that furthermore

$$G_{ex} \cup G_0 = K^*(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil) \text{ or } G_{ex} \cup G_0 = K^*(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor).$$

□

In Prömel, Steger [64] it is shown that in fact almost all graphs without induced C_4 can be constructed by replacing one color class of a bipartite graph by a clique. This implies that there exist constants c_0 and c_1 such that

$$|\mathcal{F}orb_n^*(C_4)| = (c_r + o(1))2^{\frac{n^2}{4}+n-\frac{1}{2}\log n},$$

where $r = n \bmod 2$.

As a motivation for the definition of $\tau(H)$, which replaces the role of the chromatic number in the theory of induced subgraphs, consider the following definitions of $\chi(H)$ and $\sigma(H)$:

$\chi(H)$ is the largest integer k such that no $(k-1)$-partite graph contains H as an induced subgraph,

$\sigma(H)$ is the largest integer k such that no $(k-1)$-partite graph, in which *all* classes are replaced by a clique, contains H as an induced subgraph.

As a generalization of both we define:

$\tau(H)$ as the largest integer k for which there exists an integer $\tilde{k}$ between 0 and $k-1$ such that no $(k-1)$-partite graph, in which $\tilde{k}$ classes are replaced by a clique, contains H as an induced subgraph.

The following statements are easily seen to be true for every graph H:

$$\tau(H) \geq \chi(H), \qquad \tau(H) \geq \sigma(H), \quad \text{and} \quad \tau(H) < \chi(H) + \sigma(H).$$

The counterpart of the Erdős, Stone, Simonovits result for induced subgraphs states as follows:

Theorem 7.1.3 *Let H be a fixed (nonempty) subgraph. Then*

$$ex_n^*(H) = \left(1 - \frac{1}{\tau(H)-1}\right)\frac{n^2}{2} + o(n^2).$$

The proof of Theorem 7.1.3 follows, similarly as the corresponding result for weak subgraphs, essentially from the Erdős and Stone Theorem 2.2.4. However, here we also need the following bound on the Ramsey-numbers:

Theorem 7.1.4 [Erdős, Szekeres, 1935] *Let $R(n,n)$ denote the smallest integer such that every graph on $R(n,n)$ vertices contains either a clique or a stable set on n vertices. Then*

$$R(n,n) \leq \binom{2n-2}{n-1} \leq 4^n \text{ for all } n \geq 2.$$

□

Proof of Theorem 7.1.3. Observe that the definition of $\tau(H)$ implies

$$ex_n^*(H) \geq \left(1 - \frac{1}{\tau(H)-1}\right)\frac{n^2}{2} + o(n^2)$$

(consider for example the Turán graph $T_n(\tau(H)-1)$ with an appropriate number of classes replaced by cliques). To show the upper bound assume there exists some $\epsilon > 0$ such that

$$ex_n^*(H) \geq \left(1 - \frac{1}{\tau(H)-1}\right)\frac{n^2}{2} + \epsilon n^2$$

for all n greater than some n_0. Consider any extremal graph $G_{ex} = (V_n, E)$ with supplemental graph $G_0 = (V_n, E_0)$ for n sufficiently large. Applying Theorem 2.2.4 to $G_{ex} = (V_n, E)$ shows that G_{ex} contains a complete $\tau(H)-$partite graph with parts of size $4^{|V(H)|}$. Consider now the subgraphs induced by these parts in $G_0 = (V_n, E_0)$. By Theorem 7.1.4 we can find in each part either a clique or a stable set of size $|V(H)|$. Together this shows that G_{ex} contains a (sufficiently large) complete $\tau(H)-$partite graph such that all parts either form a stable set or a clique in G_0. The definition of $\tau(H)$ together with the definition of G_{ex} now gives the intended contradiction. □

The structural theorems of Extremal Graph Theory (like the Asymptotic Structure Theorem or the First Stability Theorem) also remain true for induced subgraphs after straightforward modifications along the lines indicated above. For details we refer the interested reader to Prömel, Steger [71]

7.2 Asymptotic Structure

Let $\mathcal{F}orb_n^*(H)$ denote the class of all (labelled) graphs on n vertices which do not contain H as an *induced* subgraph. In Theorem 3.4.1 we have seen that the cardinality of the class $\mathcal{F}orb_n(H)$ (defined by excluding *weak* subgraphs) mainly depends only on the chromatic number. In this section we show a result of Prömel, Steger [65] stating that a similar result holds for the class $\mathcal{F}orb_n^*(H)$ – if one replaces the chromatic number $\chi(H)$ by the parameter $\tau(H)$ introduced in the previous section.

Theorem 7.2.1 *Let H be a fixed (nonempty) subgraph with $\tau(H) \geq 3$. Then*

$$|\mathcal{F}orb_n^*(H)| = 2^{(1-\frac{1}{\tau(H)-1})\frac{n^2}{2}+o(n^2)} = 2^{ex_n^*(H)(1+o(1))}.$$

Note that excluding a graph H with $\tau(H) = 2$ as an induced subgraph corresponds in a sense to excluding a graph H with $\chi(H) = 2$ as a weak subgraph. However, since all graphs H with $\tau(H) = 2$ must satisfy $\chi(H) \leq 2$ and $\tau(H) \leq 2$ there do exist only five such graphs. They are shown in Figure 7.1. For H_0 and H_1 there trivially

H_0 H_1 H_2 H_3 H_4

Figure 7.1: All graphs H with $\tau(H) = 2$

exists only one graph on n vertices not containing H_0 respectively H_1 as an induced subgraph. Since H_2 and H_3 are complementary it suffices to consider H_2. Here one easily observes that a graph $G = (V, E)$ does not contain H_2 as an induced subgraph if and only if E can be partitioned into pairwise disjoint cliques. That is,

$$|\mathcal{F}orb_n^*(H_2)| = \sum_{k=1}^{n} S_{n,k},$$

where $S_{n,k}$ denote the Stirling numbers of the second kind. Hence, the only nontrivial "bipartite" case is the graph H_4. Here $\mathcal{F}orb_n^*(H_4)$ is the well studied class of cographs.

The main tools for the proof of Theorem 7.2.1 are the two regularity theorems 3.3.1 and 3.3.2 and some additional tools based on the Erdős, Szekeres bound on the Ramsey number $R(n, n)$ (cf. Theorem 7.1.4). Let $G = (V, E)$ be a graph. A subset $A \subseteq V$ is called *homogeneous*, if A induces in G either a clique or a stable set. An r-*homogeneous* set is a homogeneous set of size r.

Lemma 7.2.2 *For a graph G let $k_r(G)$ denote the number of r-homogeneous sets in G. Then there exist for every $r \in \mathbb{N}$ an $n_0 = n_0(r)$ and a $c_r > 0$ such that all graphs $G = (V, E)$ on $n \geq n_0$ vertices satisfy $k_r(G) \geq c_r n^r$.*

Proof. Let $R = R(r, r)$ denote the diagonal Ramsey-numbers (cf. Theorem 7.1.4). By the definition of R we know that every subset W of V of size R contains at least one r-homogeneous set. On the other hand each r-homogeneous set is contained in at most $\binom{n-r}{R-r}$ different such subsets W. Hence, one has

$$k_r(G) \geq \frac{\binom{n}{R}}{\binom{n-r}{R-r}},$$

implying the statement of the lemma e.g. for $n_0 = R$ and $c_r = \frac{1}{R^r}$. $\square$

Lemma 7.2.3 *Let $\mathcal{H} = (V, F)$ be an r-uniform hypergraph on n vertices and $|F| \geq cn^r$ edges. Furthermore let $c > \epsilon > 0$ and let $W \subseteq V$ be a vertex set with $|W| \leq \epsilon|V|$. Then there exists a vertex $v_1 \in V \setminus W$ which is contained in at least $\frac{r(c-\epsilon)}{1-\epsilon} n^{r-1}$ many edges of $F \cap \binom{V \setminus W}{r}$.*

Proof. Each vertex in W is contained in at most $\binom{n-1}{r-1} < n^{r-1}$ many edges of F. Hence, at least $(c - \epsilon)n^r$ many edges of F contain only vertices from $V \setminus W$. For a vertex $v \in V \setminus W$ let $f(v)$ denote the number of edges of F which contain v and $r - 1$ additional vertices from $V \setminus W$. By the above remark we have

$$\sum_{v \in V \setminus W} f(v) \geq r(c - \epsilon)n^r.$$

This immediately implies the existence of v_1. □

An analogous argument shows:

Corollary 7.2.4 *Let $\mathcal{H} = (V, F)$ be an r-uniform hypergraph on n vertices and let $v_1, ..., v_k$, $1 \leq k < r$ be pairwise different vertices from V such that $|\{f \in F \mid \{v_1, ..., v_k\} \subseteq f\}| \geq cn^{r-k}$. Furthermore let ϵ be a constant such that $c > \epsilon > 0$ and let $W \subseteq V \setminus \{v_1, ..., v_k\}$ be a set of vertices such that $|W| \leq \epsilon|V|$. Then there exists a vertex $v_{k+1} \in V \setminus W$ different from $v_1, \ldots, v_k$ such that*

$$|\{f \in F| \ \{v_1, ..., v_{k+1}\} \subseteq f\}| \geq \frac{(r-k)(c-\epsilon)}{1-\epsilon} n^{r-k-1}.$$ □

Lemma 7.2.5 *For every $\epsilon, c > 0$ and $r \in \mathbb{N}, r \geq 2$, there exist positive integers $n_0 = n_0(\epsilon, c, r)$ and $m = m(\epsilon, c, r)$ such that every r-uniform hypergraph $\mathcal{H} = (V, F)$ on $n \geq n_0$ vertices with $|F| \geq cn^r$ many edges contains pairwise disjoint sets $A_1, ..., A_r \subseteq V$ which satisfy:*

(i) $\frac{n}{m} \leq |A_1| = ... = |A_r| \leq \epsilon \frac{n}{r^2}$,

(ii) $(A_1, ..., A_r)$ *is ϵ-regular and*

(iii) $d(A_1, ..., A_r) \geq \frac{c-3\epsilon}{1-\epsilon}$.

Proof. Let n_0 and m' be the constants of Theorem 3.3.2 for ϵ, r and $l = \lceil \frac{r^2}{\epsilon} \rceil$. Let $m = \frac{m'}{1-\epsilon}$. Then Theorem Theorem 3.3.2 guarantees that every $r-$uniform hypergraph $H = (V, F)$ on $n \geq n_0$ vertices has a ϵ-uniform partition $V = A_0 \cup A_1 \cup ... \cup A_k$, such that all $A_1, \ldots, A_k$ satisfy (i). Without loss of generality let $(A_1, ..., A_r)$ be a ϵ-regular tuple which has highest density among all such ϵ-regular tuples. Set $d = d(A_1, ..., A_r)$. This implies that

$$\begin{aligned} cn^r &\leq d \cdot |A_1|^r \cdot (\tbinom{k}{r} - \epsilon k^r) + |A_1|^r \epsilon k^r \\ &\quad + \sum_{i=1}^{r-1} |A_1|^r (r-i)^i \tbinom{k}{r-i} + |A_0| \cdot n^{r-1}, \end{aligned}$$

since the terms on the right hand side form an upper bound for the number of edges in ϵ-regular tuples, in ϵ-irregular tuples, for the number of edges containing at least two vertices of a set A_i, $i \geq 1$, and for the number of edges containing at least one vertex of A_0, respectively. Using $|A_0| \leq \epsilon n$ and $k|A_1| \leq n$ this implies

$$\begin{aligned} cn^r &\leq dn^r(1-\epsilon) + \epsilon n^r + \sum_{i=1}^{r-1}(\frac{r}{k})^i \cdot n^r + \epsilon n^r \\ &\leq d(1-\epsilon)n^r + (2\epsilon + \frac{r^2}{k})n^r \end{aligned}$$

and hence because of $k \geq l \geq \frac{r^2}{\epsilon}$ also $d \geq \frac{c-3\epsilon}{1-\epsilon}$. □

The following lemma forms the core of the proof of Theorem 7.2.1. It shows in particular that *every* sufficiently regular and sufficiently large k-partite graph contains *every* k-partite graph of a fixed size as an induced subgraph.

Lemma 7.2.6 *For all $k, t \in \mathbb{N}$ and $0 < \delta < \frac{1}{2}$ there exists $0 < \epsilon_0 < \frac{1}{2}$ such that for all $\epsilon \leq \epsilon_0$ there exists $n_0 = n_0(\epsilon)$ so that for all $n \geq n_0$ the following is true. If $G = (V, E)$ is graph, and $V = \bigcup_{i=1}^k A_i$ is a partition of the vertex set such that $|A_1| = \cdots = |A_k| = n$ and for all $1 \leq i < j \leq k$ the pair (A_i, A_j) is ϵ-regular and has density between δ and $1-\delta$, then G contains a universal k-partite graph, i.e., a k-partite graph that contains every k-partite graph on at most t vertices as a subgraph, whose parts are all homogeneous in G.*

Proof. Let us first outline the main idea of the proof. We will construct the desired universal graph iteratively. To do this we construct in a first step t vertices $v_1^1, ..., v_t^1$ in A_1 that form a homogeneous set in G. In a second step we construct for each of the 2^t possible adjacency structures to the vertices $v_1^1, ..., v_t^1$ t vertices in A_2 that also form a homogeneous set in G. In a third step we construct for each of the $2^t \cdot 2^{t \cdot 2^t}$ adjacency structures to the already constructed $t + t2^t$ many vertices t vertices in A_3 that form in G a homogeneous set, and so on. It is easy to see that the so constructed k-partite graph contains every k-partite graph with at most t vertices in each class as a subgraph.

Let $\alpha(i)$ denote the number of vertices which have to be constructed in $A_i, 1 \leq i \leq k$, and let $\beta(i)$ denote the number of different adjacency structures to the vertices constructed in $A_1, ..., A_i$. Then the $\alpha(i)$ and $\beta(i)$ satisfy the following equations:

$$\begin{array}{ll} \alpha(1) = t, & \beta(1) = 2^{\alpha(1)}, \\ \alpha(2) = t \cdot \beta(1), & \beta(2) = \beta(1) \cdot 2^{\alpha(2)}, \\ \ldots & \ldots \\ \ldots & \ldots \\ \alpha(k-1) = t \cdot \beta(k-2), & \beta(k-1) = \beta(k-2) \cdot 2^{\alpha(k-1)}, \\ \alpha(k) = t \cdot \beta(k-1). & \end{array}$$

Let $r = \alpha(k)$ and denote with n_r and c_r the corresponding constants from Lemma 7.2.2. Choose $\delta_0 > 0$ in such a way that $\frac{c_r - 3\delta_0}{1-\delta_0} > \frac{c_r}{2}$ and let $\epsilon_s := \min\{(\frac{\delta}{2})^r, \delta_0, \frac{c_r}{4}\}$.

Let $n_s = n_s(\epsilon_s, c_r, r)$ and $m_s = m_s(\epsilon_s, c_r, r)$ denote the constants from Lemma 7.2.5. Choose $t_0 \in \mathbb{N}$ in such a way that $(\frac{\delta}{2})^{t_0} \leq \frac{1}{m_s}$ and choose $t_1 \in \mathbb{N}$ in such a way that $\frac{x-2kr(\frac{\delta}{2})^{t_1}}{1-2kr(\frac{\delta}{2})^{t_1}} \geq \frac{x}{2}$ for all $x \geq \frac{c_r}{2^{r+2}}$. Let $\epsilon_0 := (\frac{\delta}{2})^{t_0+t_1+r}$.
Choose an arbitrary $\epsilon < \epsilon_0$ and fix it for the rest of the proof. Choose $n_0 = n_0(H, \delta, \epsilon, l)$ as follows: Let $n'_s = n_s(\epsilon, l)$ and $m'_s = m_s(\epsilon, l)$ the constants of Theorem 3.3.1 and choose n_1 in such a way that $\frac{c_r}{4}(\frac{\delta}{2})^{r(t_0+r)} n_1 \geq 1$. Let

$$n_0 = \max\{n'_s, \frac{m'_s n_r}{1-\epsilon}, \frac{m'_s n_s}{1-\epsilon}, \frac{m_G n_1}{1-\epsilon}\}.$$

For the rest of the proof assume that G is a graph satisfying the assumptions of the lemma with respect to some $\epsilon \leq \epsilon_0$ and $n \geq n_0(\epsilon)$. From Lemma 7.2.2 we deduce that each class A_i contains at least $c_r|A_i|^r$ many r-homogeneous sets. Consider now the hypergraphs $H(A_i)$, that contain A_i as vertex set and all r-homogeneous sets of A_i as edges. Lemma 7.2.5 implies that in particular for $i = 2, ..., k$ every class A_i contains an ϵ_s-regular tupel $(X_i^1(0), ..., X_i^r(0))$ in such a way that

(1) $|X_i^j(0)| \geq \frac{|A_i|}{m_s} \geq (\frac{\delta}{2})^{t_0}|A_i|$ for all $2 \leq i \leq k, 1 \leq j \leq r$ and

(2) $d_{H(A_i)}(X_i^1(0), ..., X_i^r(0)) \geq \frac{c_r - 3\epsilon_s}{1-\epsilon_s} \geq \frac{c_r}{2}$ for all $2 \leq i \leq k$.

The idea of construction of the vertices $v_1^1, ..., v_{\alpha(1)}^1, v_1^2, ..., v_{\alpha(2)}^2, ..., v_1^k, ..., v_{\alpha(k)}^k$ is to keep at the same time a sequence of sets $X_i^j(k), 2 \leq i \leq k, 1 \leq j \leq r$ and $1 \leq k \leq \sum_{s=1}^{i-1} \alpha(s)$ in such a way that the following four conditions are satisfied:

(3) $X_i^j(k) \subseteq X_i^j(k-1)$,

(4) $|X_i^j(k)| \geq \frac{\delta}{2}|X_i^j(k-1)|$,

(5) if for an $i' < i$ and for $1 \leq s \leq \alpha(i')$

$$\Gamma(v_s^{i'}) \cap X_i^j(k) \neq \emptyset \text{ for } k = \sum_{t=1}^{i'-1} \alpha(t) + s,$$

then also $X_i^j(k) \subseteq \Gamma(v_s^{i'})$,

and finally

(6) if one denotes for $i' < i$ and $1 \leq s \leq \alpha(i')$ and $1 \leq j \leq r$:

$$a_{i',s}(i,j) = \begin{cases} 0, & \text{if } X_i^j(\sum_{t=1}^{i'} \alpha(t)) \cap \Gamma(v_s^{i'}) = \emptyset \\ 1, & \text{otherwise} \end{cases}$$

and

$$b_{i'}(i,j) = \sum_{t=1}^{i'} \sum_{s=1}^{\alpha(t)} a_{t,s}(i,j) 2^{\sum_{k=1}^{t-1} \alpha(k)+s-1},$$

then one has $b_{i'}(i,j_1)=b_{i'}(i,j_2)$ if and only if

$$j_1 \equiv j_2 \pmod{\beta(i')}.$$

Note that (4) implies that

(7) $|X_i^j(k)| \geq (\frac{\delta}{2})^k |X_i^j(0)|$

and hence together with (1) also

(8) $|X_i^j(k)| \geq (\frac{\delta}{2})^{t_0+k} |A_i|$.

All pairs $(A_{i_1}, A_{i_2}), 1 \leq i_1 < i_2 \leq k$ are by assumption ϵ-regular. Therefore $\epsilon < \epsilon_0 = (\frac{\delta}{2})^{t_0+t_1+r}$ implies together with (8) for all $1 \leq i' < i \leq k$ and $A \subseteq A_{i'}$ with $|A| \geq \epsilon |A_{i'}|$:

$$\delta - \epsilon \leq d(A, X_i^j(k)) \leq 1 - \delta + \epsilon.$$

If in addition $|A| \geq (\frac{\delta}{2})^{t_0+k}|A_{i'}|$ then the pair $(A, X_i^j(k))$ is even $(\frac{\delta}{2})^{t_1+r-k}$–regular. This observation can be used to show that for a set $A \subseteq A_{i'}$ with $|A| \geq \epsilon |A_{i'}|$ "most" of its vertices have "many" but not "too many" neighbors in $X_i^j(k)$. Namely, let

$$A' = \{a \in A \mid (\delta - \epsilon - (\frac{\delta}{2})^{t_1+r-k})|X_i^j(k)| \geq |\Gamma(a) \cap X_i^j(k)|\}$$

and

$$A'' = \{a \in A \mid |\Gamma(a) \cap X_i^j(k)| \geq (1 - \delta + \epsilon + (\frac{\delta}{2})^{t_1+r-k})|X_i^j(k)|\}$$

one therefore has

$$|A'| \leq (\frac{\delta}{2})^{t_1+r-k}|A| \leq (\frac{\delta}{2})^{t_1}|A|$$

and

$$|A''| \leq (\frac{\delta}{2})^{t_1+r-k}|A| \leq (\frac{\delta}{2})^{t_1}|A|.$$

Note that for $\delta - \epsilon - (\frac{\delta}{2})^{t_1+r-k}$ one has

(9) $\delta - \epsilon - (\frac{\delta}{2})^{t_1+r-k} \geq \delta - (\frac{\delta}{2})^{t_0+t_1+r} - (\frac{\delta}{2})^{t_1+r-k} \geq \frac{\delta}{2}$.

We now describe the construction of the $\alpha(1)$ vertices in A_1. By the above remarks the set

$$\begin{aligned} X_1(1) \quad = \quad & \{x \in A_1 \mid (\delta - \epsilon - (\frac{\delta}{2})^{t_1+r})|X_i^j(0)| \leq |\Gamma(x) \cap X_j^i(0)| \\ & \leq (1 - \delta + \epsilon + (\frac{\delta}{2})^{t_1+r})|X_j^i(0)| \text{ for all } 2 \leq i \leq k, 1 \leq j \leq r\} \end{aligned}$$

satisfies the inequality

$$|X_1(1)| \geq (1 - 2kr(\frac{\delta}{2})^{t_1})|A_1|.$$

Since A_1 contains at least $c_r|A_1|^r$ many r-homogeneous sets, Lemma 7.2.3 implies that the set $X_1(1)$ contains a vertex v_1 contained in at least $\frac{c_r - 2kr(\frac{\delta}{2})^{t_1}}{1-2kr(\frac{\delta}{2})^{t_1}} n^{r-1} \geq \frac{c_r}{2} n^{r-1}$ (cf. choice of t_1) many r-homogeneous sets of $X_1(1)$.
Set now for all $2 \leq i \leq k$ and $1 \leq j \leq r$:

$$X_i^j(1) := \begin{cases} X_i^j(0) \setminus \Gamma(v_1), & \text{if } j-1 \equiv 0 \pmod 2 \\ X_i^j(0) \cap \Gamma(v_1), & \text{otherwise.} \end{cases}$$

For the $X_i^j(1)$ (3) and (4) is valid by the definition of $X_1(1)$ and by (9).
Assume now that we have already constructed $s < \alpha(1)$ vertices $v_1^1, ..., v_s^1$ in A_1 and sets $X_1(s) \subseteq A_1$ and $X_i^j(s)$ such that besides (3) und (4) one also has:

$v_1^1, ..., v_s^1$ are contained in at least $\frac{c_r}{2^s} n^{r-s}$ many r-homogeneous sets in $X_1(s)$.

Now we construct v_{s+1}^1. To do this put

$$\begin{aligned} X_1(s+1) &= \{x \in X_1(s) \mid (\delta - \epsilon - (\frac{\delta}{2})^{t_1+r-s})|X_i^j(s)| \leq |\Gamma(x) \cap X_j^i(s)| \\ &\leq (1 - \delta + \epsilon + (\frac{\delta}{2})^{t_1+r-s})|X_j^i(s)| \text{ for all } 2 \leq i \leq k, 1 \leq j \leq r\}. \end{aligned}$$

Again

$$|X_1(s+1)| \geq (1 - 2kr(\frac{\delta}{2})^{t_1})|X_1(s)|$$

and 7.2.4 imply the existence of a vertex v_{s+1}^1 in $X_1(s+1)$ such that $v_1^1, ..., v_{s+1}^1$ are contained in at least $\frac{\frac{c_r}{2^s} - 2kr(\frac{\delta}{2})^{t_1}}{1-2kr(\frac{\delta}{2})^{t_1}} n^{r-s-1} \geq \frac{c_r}{2^{s+1}} n^{r-s-1}$ many r-homogeneous sets in $X_1(s+1)$.
For all $2 \leq i \leq k$ and $1 \leq j \leq r$ set

$$X_i^j(s+1) := \begin{cases} X_i^j(s) \setminus \Gamma(v_{s+1}), & \text{if } \lfloor \frac{j-1}{2^s} \rfloor \equiv 0 \pmod 2 \\ X_i^j(s) \cap \Gamma(v_{s+1}), & \text{otherwise.} \end{cases}$$

Repeating this construction finally yields $\alpha(1)$ vertices $v_1^1, ..., v_{\alpha(1)}^1 \in A_1$ which are contained in $\frac{c_r}{2^{\alpha(1)}} n^{r-\alpha(1)} \geq 1$ many r-homogeneous sets. Hence $v_1^1, ..., v_{\alpha(1)}^1$ form an homogeneous set. Furthermore the construction implies that the sets $X_i^j(\alpha(1))$ for $2 \leq i \leq k$ and $1 \leq j \leq r$ satisfy besides (3) and (4) also (5) and (6) for $i' = 1$.
Hence it suffices to construct $\alpha(2)$ vertices $v_1^2, ..., v_{\alpha(2)}^2$ in A_2 in such a way that $v_j^2 \in X_2^j(\alpha(1))$. The only remaining difficulty is to make sure that the vertices v_j^2 induce in G a homogeneous set. To guarantee this we use the fact that $(X_2^1(0), ..., X_2^j(0))$ are ϵ_s-regular with respect to the r-homogeneous sets in G and has density at least $\frac{c_r}{2}$ (cf. (2)). Therefore by (7) $(X_2^1(\alpha(1)), ..., X_2^r(\alpha(1)))$ contains at least

$$\left(\frac{c_r}{2} - \epsilon_s\right) \prod_{i=1}^{r} |X_2^i(\alpha(1))| \geq \frac{c_r}{4} \cdot \prod_{i=1}^{r} |X_2^i(\alpha(1))|$$

many r-homogeneous sets with exactly one vertex from every set $X_2^j(\alpha(1))$. Now define

$$Y_2^1(\alpha(1)) = \{x \in X_2^1(\alpha(1)) \mid (\delta - \epsilon - (\frac{\delta}{2})^{t_1+r-\alpha(1)})|X_i^j(\alpha(1))|$$

$$\leq |\Gamma(x) \cap X_i^j(\alpha(1))| \leq (1-\delta+\epsilon+(\frac{\delta}{2})^{t_1+r-\alpha(1)}|X_i^j(\alpha(1))|$$
$$\text{for all } 3 \leq i \leq k, 1 \leq j \leq r\}.$$

Again we obtain

$$|Y_2^1(\alpha(1))| \geq (1-2kr(\frac{\delta}{2})^{t_1})|X_2^1(\alpha(1))|.$$

A similar argument as in the proof of Lemma 7.2.3 shows that there exists at least one vertex $v_1^2 \in Y_2^1(\alpha(1))$ which is contained in at least $\frac{\frac{c_r}{4}-2kr(\frac{\delta}{2})^{t_1}}{1-2kr(\frac{\delta}{2})^{t_1}} \cdot \prod_{j=2}^{r} |X_2^j(\alpha(1))|$ $\geq \frac{c_r}{8} \prod_{j=2}^{r} |X_2^j(\alpha(1))|$ many r-homogeneous sets which contain besides v_1^2 exactly one vertex from every set $X_2^j(\alpha(1))$, $j \geq 2$. For $3 \leq i \leq k$ and $1 \leq j \leq r$ let

$$X_i^j(\alpha(1)+1) := \begin{cases} X_i^j(\alpha(1)) \setminus \Gamma(v_1^2), & \text{if } \lfloor \frac{j-1}{2^{\alpha(1)}} \rfloor \equiv 0 \pmod 2 \\ X_i^j(\alpha(1)) \cap \Gamma(v_1^2), & \text{otherwise.} \end{cases}$$

Obviously, the $X_i^j(\alpha(1)+1)$ satisfy (3) and (4) and (5) for $i'=2$ and $s=1$.
We now construct then vertices $v_2^2, ..., v_{\alpha(2)}^2$ similarly as the vertices $v_2^1, ..., v_{\alpha(1)}^1$. For all $2 \leq s \leq \alpha(2)$ let

$$X_i^j(\alpha(1)+s) := \begin{cases} X_i^j(\alpha(1)+s-1) \setminus \Gamma(v_s^2), & \text{if } \lfloor \frac{j-1}{2^{\alpha(1)+s-1}} \rfloor \equiv 0 \pmod 2 \\ X_i^j(\alpha(1)+s-1) \cap \Gamma(v_s^2), & \text{otherwise.} \end{cases}$$

It is easy to see that the so constructed sets $X_i^j(\alpha(1)+\alpha(2))$ satisfy conditions (3), (4) and (5), (6) for $i'=2$.

Continuing in this way for $3 \leq i \leq k-1$ we obtain vertices $v_j^i \in X_i^j\left(\sum_{t=1}^{i-1}\alpha(t)\right)$, $1 \leq j \leq \alpha(i)$.
To obtain the new vertices $v_1^k, \ldots, v_r^k$ in A_k we exploit our knowledge on the sets $X_k^1(\sum_{t=1}^{k-1}\alpha(t)), \ldots, X_k^r(\sum_{t=1}^{k-1}\alpha(t))$. Because of (7) and (8), by the ϵ_s-regularity of $(X_k^1(0), ..., X_k^r(0))$ and by the definition of ϵ_s and n_0 these sets contain at least

$$\frac{c_r}{4}\prod_{j=1}^{r}|X_i^j(\textstyle\sum_{t=1}^{k-1}\alpha(t))| \geq \frac{c_r}{4}\Big((\frac{\delta}{2})^{t_0+\sum_{t=1}^{k-1}\alpha(t)} \cdot n_1\Big)^r \geq 1$$

many r-homogeneous sets with exactly one vertex in each of the sets $X_k^j(\sum_{t=1}^{k-1}\alpha(t))$. The vertices of such an r-homogeneous set are the vertices $v_1^k, ..., v_{\alpha(k)}^k$. The construction, in particular (6), implies immediately that the vertices v_i^j contain every k-partite graph on at most t many vertices in each class as an induced subgraph. □

Proof of Theorem 7.2.1. Let $\frac{1}{2} > \gamma > 0$ be a constant. Choose $\delta \leq \gamma$ in such a way that all sufficiently large n satisfy $n\binom{n}{\delta n} < 2^{\gamma n}$. Let $l = \lceil \gamma^{-1} \rceil$ and $\epsilon = \min\{\delta, \epsilon_0\}$, where $\epsilon_0 = \epsilon_0(H, \delta, l)$ is the constant from Lemma 7.2.6. Let $m = m(\epsilon, l)$ denote the constant from Theorem 3.3.1 and let $n_0 = n_0(H, \delta, \epsilon, l)$ be the maximum of the corresponding constants from Theorem 3.3.1 and Lemma 7.2.6. By Theorem 3.3.1 we can construct all graphs $G = (V, E)$ in $\mathcal{F}orb_n^*(H)$ for $n \geq n_0$ as follows: First choose a partition $V = A_0 \cup A_1 \cup \ldots \cup A_k$ of V with $|A_0| < \epsilon n, |A_1| = \ldots = |A_k|$ and $l \leq k \leq m$. This can be done in at most n^n different ways. Then choose the edges which are completely contained in some class $A_i, 1 \leq i \leq k$ and connect the vertices in A_0 to all other vertices. This can be done in at most $2^{k \cdot \binom{|A_1|}{2}} \cdot 2^{|A_0| \cdot n} \leq 2^{\frac{n^2}{2k} + \epsilon n^2}$ ways. Next decide for every pair $(A_i, A_j), 1 \leq i < j \leq k$ whether it is

(I) ϵ-irregular,

(II) ϵ-regular with density less that δ,

(III) ϵ-regular with density larger that $1 - \delta$, or

(IV) ϵ-regular with density between δ and $1 - \delta$.

There are $4^{\binom{k}{2}} < 2^{k^2}$ ways to do this. Finally, insert those edges in G which are contained in some pair $(A_i, A_j), 1 \leq i < j \leq k$: By Theorem 3.3.1 we know that there are at most ϵk^2 many ϵ-irregular pairs. Therefore all edges which are contained in type-I pairs can be inserted in at most $2^{|A_1|^2 \epsilon k^2} \leq 2^{\epsilon n^2}$ different ways. For a type-II or type-III pairs there are by choice of δ for sufficiently large n at most $\sum_{i=1}^{\delta |A_1|^2} \binom{|A_1|^2}{i} \leq \delta |A_1|^2 \binom{|A_1|^2}{\delta |A_1|^2} < 2^{\gamma \frac{n^2}{k^2}}$ ways to choose the edges contained in this pair. Therefore there are at most $2^{\gamma n^2}$ ways to insert all edges contained in some type-II or type-III pair. Finally, we conclude from Lemma 7.2.6 that the type-IV cannot contain all $\binom{\tau(H)}{2}$ pairs between any $\tau(H)$ sets A_i. In particular, this shows that the edges of G contained in these pairs pairs may not contain a $K_{\tau(H)}$. That is, there are at most $|\mathcal{F}orb_n(K_{\tau(H)})|$ ways to insert these edges.

Together with Theorem 3.4.1 this implies that for all sufficiently large n

$$\begin{aligned} |\mathcal{F}orb_n^*(H)| &\leq n^n 2^{\frac{n^2}{2k} + \epsilon n^2} 2^{k^2} 2^{\epsilon n^2} 2^{\gamma n^2} |\mathcal{F}orb_n(K_{\tau(H)})| \\ &\leq 2^{(1 - \frac{1}{\tau(H) - 1}) \frac{n^2}{2} + 4\gamma n^2}. \end{aligned}$$

Since γ was chosen arbitrarily small this implies that

$$|\mathcal{F}orb_n^*(H)| \leq 2^{(1 - \frac{1}{\tau(H) - 1}) \frac{n^2}{2} + o(n^2)}.$$

On the other hand the definition of an extremal graph implies immediately that

$$|\mathcal{F}orb_n^*(H)| \geq 2^{ex_n^*(H)}$$

and therefore together with Theorem 7.1.3 also

$$|\mathcal{F}orb_n^*(H)| = 2^{(1 - \frac{1}{\tau(H) - 1}) \frac{n^2}{2} + o(n^2)}.$$

$\square$

7.3 Hereditary graph properties

Let $\mathcal{P}$ be a graph property. The property is called *hereditary*, if every induced subgraph of a graph having property $\mathcal{P}$ also has property $\mathcal{P}$. We denote by $\mathcal{P}_n$ the set of all graphs on n vertices having property $\mathcal{P}$. A natural question to ask is that to determining the growth rate of $|\mathcal{P}_n|$ for arbitrary hereditary properties $\mathcal{P}$. The solution is, in fact, an immediate consequence of the results in the previous sections.

Theorem 7.3.1 *Let $\mathcal{P}$ be a hereditary property of graphs. Then $\lim_{n\to\infty} \log|\mathcal{P}_n|/\binom{n}{2}$ exists and equals $1-\frac{1}{r}$ for some integer $r \in \mathbb{N}$.*

Theorem 7.3.1 answers a question of Scheinermann and Zito [74], and was independently proven by Alekseev [1] and Bollobás and Thomason [14].

In order to prove Theorem 7.3.1 we generalize the parameter $\tau(H)$ introduced in Section 7.1 to (not necessarily finite) families $\mathcal{H}$ of (finite) graphs.

Definition 7.3.2 *Let $\mathcal{H}$ be a family of finite graphs. Then $\tau(\mathcal{H})$ is the largest integer k for which there exists an integer $\tilde{k}$ between 0 and $k-1$ such that no $(k-1)$-partite graph, in which $\tilde{k}$ classes are replaced by a clique, contains any graph from $\mathcal{H}$ as an induced subgraph.*

A trivial observation is that $\tau(\mathcal{H})$ is bounded from above by the values of $\tau(H)$ for all $H \in \mathcal{H}$:

$$\tau(\mathcal{H}) \;\leq\; \min\{\tau(H) \mid H \in \mathcal{H}\}. \tag{7.3.1}$$

Remark. Observe that the inequality in (7.3.1) may be strict. An example for this is $\mathcal{H} = \{C_4, K_4 \setminus e\}$. In this respect the behaviour of properties hereditary with respect to the induced subgraph relation differs considerably from those which are hereditary with respect to the weak subgraph relation: For the latter the corresponding parameter would be $\chi(\mathcal{H})$, which *is* in fact given by $\chi(\mathcal{H}) = \min\{\chi(H) \mid H \in \mathcal{H}\}$.

With these definitions at hand we can now state and prove a slightly stronger version of Theorem 7.3.1.

Theorem 7.3.3 *Let $\mathcal{P}$ be a hereditary property of graphs and let $\mathcal{H}$ denote the set of all those graphs which do* not *have property $\mathcal{P}$. Then*

$$|\mathcal{P}_n| \;=\; 2^{(1-\frac{1}{\tau(\mathcal{H})-1})\frac{n^2}{2}+o(n^2)}.$$

Proof. We proceed in exactly the same way as in the proof of Theorem 7.2.1. The only difference is the observation that Lemma 7.2.6 together with the definition of $\tau(\mathcal{H})$ in fact guarantees that for any $\tau(\mathcal{H})$ sets A_i not all $\binom{\tau(\mathcal{H})}{2}$ pairs can be of type IV. As before, this implies in particular that the edges of G contained in type-IV pairs cannot contain a $K_{\tau(\mathcal{H})}$. The construction of the proof of Theorem 7.2.1 gives therefore

$$|\mathcal{P}_n| \;\leq\; 2^{(1-\frac{1}{\tau(\mathcal{H})-1})\frac{n^2}{2}+o(n^2)}.$$

As the definition of $\tau(\mathcal{H})$ implies that, trivially, also

$$|\mathcal{P}_n| \;\geq\; 2^{(1-\frac{1}{\tau(\mathcal{H})-1})\frac{n^2}{2}-\mathcal{O}(n)},$$

this concludes the proof of Theorem 7.3.3. □

We close this section by generalizing the algorithm of Section 5.1 to design fast expected time algorithms for the membership problem given by hereditary graph properties, i.e., for the problem *"Does a graph G has property $\mathcal{P}$?"*.

In the rest of this section $\mathcal{P}$ denotes an arbitrary but fixed hereditary property and H denotes a minimal obstruction for $\mathcal{P}$, i.e. a graph with minimal number of vertices which does not have the property $\mathcal{P}$. Furthermore we assume that Exact-Test(G) is an algorithm which tests whether a given graph G has property $\mathcal{P}$ and that $\text{run}_{\mathcal{P}}(n)$ denotes its worst-case running time on graphs of with n vertices.

```
Algorithm TEST-PROPERTY(G)
Input:   Graph G with n vertices;
Output: TRUE if G has property P, FALSE otherwise.
begin
    k := 1;
    while k ≤ n do begin
        if G[1, k] contains an induced H-subgraph then  return FALSE
                                                  else   k := k + 1;
    end while
    return EXACT-TEST(G)
end.
```

Similarli, as in Section 5.1 we immediately observe that Test-Property(G) is correct and that the average running time $\text{ave}_T(n)$ on graphs with n vertices is bounded by

$$\text{ave}_T(n) \;\leq\; 2^{-\binom{n}{2}} \cdot \left(\sum_{t=1}^{n} \mathcal{O}(t^{|V(H)|}) \cdot |\mathcal{F}orb_t^*(H)| \cdot 2^{\binom{n}{2}-\binom{t}{2}} + \text{run}_{\mathcal{P}}(n) \cdot |\mathcal{F}orb_n^*(H)|\right).$$

Together with Theorem 7.2.1 this implies the following theorem.

Theorem 7.3.4 *Every hereditary property of graphs that has a membership test whose worst case running time is bounded by $2^{o(n^2)}$ can be tested in $\mathcal{O}(1)$ expected time.* □

Corollary 7.3.5 *The following hereditary properties of graphs can be tested in constant expected time: k-colorability, perfectness, monochromatic triangle, partition into cliques, partial k-tree.* □

(Compare Garey, Johnson [46] for a precise definition of these problems.)

Chapter 8

Applications in Graph Theory

The meeting on graph theory in Halle an der Saale in 1960, organized by Herbert Grötsch, was very special in several respects. Not only was it the first meeting on graph theory in Germany and one of the very first worldwide, it gave birth to two conjectures which heavily influenced the development of graph theory during the years to come.

The conjecture of Hajós stated that the so-called topological clique number is an upper bound for the chromatic number. This conjecture was open for almost 20 years, and was generally believed to be true. In 1979, however, Catlin [17] exhibited a counterexample, and shortly after that Erdős and Fajtlowicz [30] proved that Hajós' conjecture is in fact almost always false!

At the same meeting, Claude Berge introduced the notion of a "perfect" graph: a graph $G = (V, E)$ is *perfect* if G and each of its induced subgraphs have the property that the chromatic number equals the size of a maximum clique. He also suggested a a possible characterization of perfect graphs in terms of forbidden subgraphs, which became known as the "*Strong perfect graph conjecture*". Until today, this conjecture has resisted all attacks, and is considered one of the outstanding open problems in graph theory. Last year Prömel and Steger [66] showed that it holds at least almost always.

8.1 The Hajós conjecture

A graph G is a *subdivision* of a graph H, iff G can be constructed from H by inserting vertices on the edges of H. The *topological clique number* $\omega_t(G)$ is the maximal l for which G contains a subdivision of K_l. Obviously,

$$\omega(G) \le \omega_t(G)$$

for all graphs. As every 3-chromatic graph contains an odd cycle, and therefore in particular a subdivision of K_3, all graphs with chromatic number $\chi(G) = 3$ satisfy $\chi(G) \le \omega_t(G)$. For 4-chromatic graphs Dirac (1952) was able to prove a similar result.

Theorem 8.1.1 *Let G be a graph with chromatic number $\chi(G) \le 4$. Then*

$$\chi(G) \le \omega_t(G). \qquad \square$$

Eight years later, the Hungarian mathematician Hajós [49] conjectured that this inequality should be valid for all graphs.

Conjecture 8.1.2 (Hajós) *All graphs G satisfy:*

$$\chi(G) \le \omega_t(G).$$

In the following years this conjecture became well-known. One reason for this was that it contained the 4-color conjecture as a special case: as a planar graph contains no subdivision of K_5 (and $K_{3,3}$), the truth of Hajós conjecture for $\chi(G) = 5$ would imply that every planar graph has chromatic number at most four. Like the 4-color conjecture, Conjecture 8.1.2 was generally believed to be true. In 1979, however, Catlin disproved it by exhibiting a counterexample. Shortly thereafter Erdős and Fajtlowicz (1981) showed that Hajós' conjecture is in fact almost never true.

Theorem 8.1.3 *Almost all graphs satisfy*

$$\chi(G) > \frac{1}{6}\frac{\sqrt{n}}{\log n} \cdot \omega_t(G).$$

The proof is based on the following proposition, which bounds the size of $\omega_t(G)$ in terms of the density of G.

Proposition 8.1.4 *If $G = (V_n, E)$ is a graph such that all subsets $S \subseteq V_n$ of size $|S| = \omega_t(G)$ satisfy $|E(G[S])| \le (1-\gamma)\frac{|S|^2}{2}$, then $\omega_t(G) \le \sqrt{2n/\gamma}$.*

Proof. By the definition of $\omega_t(G)$, G contains a set $S_0 \subseteq V_n$ of size $|S_0| = \omega_t(G)$ such that every two vertices of S_0 are joined by pairwise internally disjoint paths. In particular, this implies that every missing edge in $E(G[S_0])$ gives rise to at least one extra vertex outside of S_0. Therefore

$$n \;=\; |V(G)| \;\ge\; |S_0| + \binom{|S_0|}{2} - |E(G[S_0])| \;\ge\; \frac{1}{2}\gamma \cdot \omega_t(G)^2.$$

$\square$

Proof of Theorem 8.1.3. Consider a random graph G_t on t vertices with edge probability $\frac{1}{2}$. By Bernstein's inequality (cf. Corollary B.2) we now that

$$\text{Prob}\left[|E(G_t)| \ge \tfrac{2}{3}\tbinom{t}{2}\right] \le e^{-\frac{1}{54}\binom{t}{2}}.$$

That is, a random graph G_n contains almost surely no subgraph on $t \ge 110 \log n$ vertices with more that $\frac{2}{3}\binom{t}{2}$ edges:

$$\text{Prob}\left[\exists S \subseteq V_n \text{ s.t. } |S| \ge 110\log n,\ |E(G_n[S])| \ge \tfrac{2}{3}\tbinom{|S|}{2}\right] \;\le\; \sum_{t \ge 110 \log n} \tbinom{n}{t} \cdot e^{-\frac{1}{54}\binom{t}{2}} \;=\; o(1).$$

By Proposition 8.1.4 this implies that

$$\text{Prob}\,[\omega_t(G_n) \le \sqrt{6n}] = 1 - o(1).$$

As on the other hand also

$$\text{Prob}\,[\chi(G_n) \ge \frac{n}{2.1\ \log n}] = 1 - o(1)$$

(cf. Theorem 5.3.1), this concludes the proof of Theorem 8.1.3. □

8.2 Almost all Berge graphs are perfect

A graph $G = (V, E)$ is *perfect* if and only if $\omega(H) = \chi(H)$ for all induced subgraphs H of G. Since their introduction by Berge in the beginning of the 60s, the class of perfect graphs became more and more important and well-understood. Various striking results on perfect graphs were proven, among them the surprising fact that perfect graphs can be colored in polynomial time, cf. Grötschel, Lovász and Schrijver (1981). A possible characterization of perfect graphs in terms of forbidden subgraphs, also posed by Berge (1961), has remained open and become one of the outstanding open problems in graph theory.

Conjecture 8.2.1 (Strong perfect graph conjecture) *A graph $G = (V, E)$ is perfect if and only if neither G nor its complement $\overline{G}$ contains an odd cycle of length at least five as an induced subgraph.*

The aim of this section is to show that the strong perfect graph conjecture is at least asymptotically true. More precisely, let $Perf(n)$ denote the set of all (labelled) perfect graphs on n vertices and let $Berge(n)$ denote the set of all (labelled) graphs on n vertices which contain neither an odd cycle of length at least five nor the complement of such a cycle as an induced subgraph (these graphs are known as *Berge* graphs). Trivially, all perfect graphs are Berge graphs.

Theorem 8.2.2 [Prömel, Steger, 1992] *Almost all Berge graphs are perfect, i.e.*

$$\lim_{n\to\infty} \frac{|Perf(n)|}{|Berge(n)|} = 1.$$

In fact, we will not prove Theorem 8.2.2 directly: instead of dealing with Berge graphs, we consider the larger class of all graphs which do not contain an induced C_5, a cycle of length five.

The main idea is to find a structural characterization of almost all C_5-free graphs. More precisely, we shall identify a subset $\mathcal{S}(n) \subseteq \mathcal{F}orb_n^*(C_5)$ such that on the one hand

$$\mathcal{S}(n) \subseteq Perf(n) \subseteq Berge(n) \subseteq \mathcal{F}orb_n^*(C_5)$$

and on the other hand

$$\frac{|\mathcal{S}(n)|}{|\mathcal{F}orb_n^*(C_5)|} \longrightarrow 1 \quad \text{for } n \to \infty.$$

Clearly, this suffices to prove Theorem 8.2.2. As we will see the class $\mathcal{S}(n)$ consists of all generalized split graphs.

8.2.1 Generalized split graphs

A graph $G = (V, E)$ is a *generalized split graph*, if there exists a partition $V = V_1 \cup V_2$ of the vertex set such that either $G[V_1]$ is a union of pairwise disjoint cliques and V_2 induces a clique in G or $\overline{G}[V_1]$ is a union of pairwise disjoint cliques and V_2 is a stable set in G. Observe that if V_1 induces a clique in G and V_2 is a stable set then G is just a split graph.

Hayward [50] proved that weakly triangulated graphs, i.e. graphs which contain neither a cycle of length at least 5 nor the complement of such a cycle as an induced subgraph, are perfect. As generalized split graphs are weakly triangulated the result of Hayward implies that they are perfect.

Lemma 8.2.3 *Generalized split graphs are perfect.* $\square$

By $\mathcal{S}(n)$ we denote the class of generalized split graphs on n vertices.

Lemma 8.2.4

$$|\mathcal{S}(n)| = 2^{\frac{n^2}{4}+\frac{n}{2}\log n-\frac{n}{2}\log\ln n-\frac{n}{2}\log\frac{e}{2}+O(\frac{n}{\log\log n})}.$$

Proof. We first show the upper bound. Obviously

$$|\mathcal{S}(n)| \leq 2 \cdot \sum_{0\leq i\leq n} \binom{n}{i} 2^{i(n-i)} B_i,$$

where B_n denotes the nth Bell number, i.e. the number of partitions of an n-element set. It is well known that (cf. e.g. [22])

$$B_n = 2^{n\log n-n\log\ln n-n\log e+O(n\frac{\log\log n}{\log n})}.$$

It is also not difficult to see that this implies that e.g. for $\delta \leq n^{\frac{2}{3}}$ one has

$$B_{\frac{n}{2}\pm\delta} = 2^{\frac{n}{2}\log n-\frac{n}{2}\log\ln n-(1+\log e)\frac{n}{2}+O(n\frac{\log\log n}{\log n})}.$$

Hence,

$$|\mathcal{S}(n)| \quad \leq \quad 2\binom{n}{\frac{n}{2}} \left[2 \sum_{i=0}^{\frac{n}{2}-n^{\frac{2}{3}}-1} 2^{i(n-i)} B_n + \sum_{i=\frac{n}{2}-n^{\frac{2}{3}}}^{\frac{n}{2}+n^{\frac{2}{3}}} 2^{i(n-i)} B_i \right]$$

$$\begin{aligned}
&\leq\quad 2\binom{n}{\frac{n}{2}}\left[2(\frac{n}{2}-n^{\frac{2}{3}})2^{(\frac{n}{2}-n^{\frac{2}{3}})(\frac{n}{2}+n^{\frac{2}{3}})}B_n+2n^{\frac{2}{3}}2^{\frac{n^2}{4}}\max_{|\delta|\leq n^{\frac{2}{3}}}B_{\frac{n}{2}+\delta}\right]\\
&\leq\quad 2^n\left[2^{\frac{n^2}{4}-\frac{1}{2}n^{\frac{4}{3}}}+2^{\frac{n^2}{4}+\frac{n}{2}\log n-\frac{n}{2}\log\ln n-(1+\log e)\frac{n}{2}+o(\frac{n}{\log\log n})}\right]\\
&\leq\quad 2^{\frac{n^2}{4}+\frac{n}{2}\log n-\frac{n}{2}\log\ln n-\frac{n}{2}\log\frac{e}{2}+o(\frac{n}{\log\log n})}.
\end{aligned}$$

To see the lower bound recall that almost all 2-colorable graphs are in fact uniquely 2-colorable and have the property that the degree of every vertex exceeds $\frac{1}{100}n$. (This follows, for example, from the proof of the Erdős, Kleitman, Rothschild Theorem 3.1.1.) Hence, we can bound the number of generalized split graphs from below by the number of such graphs times the number of ways to partition the larger color class into cliques of size $\ln\frac{n}{2}$:

$$|\mathcal{S}(n)|\geq(1+o(1))\cdot P_2(n)\cdot\frac{1}{\left(\frac{\frac{n}{2}}{\ln\frac{n}{2}}\right)!}\cdot\binom{\frac{n}{2}}{\ln\frac{n}{2};\ldots;\ln\frac{n}{2}},$$

where $\mathcal{C}ol_n(2)$ denotes the number of bipartite graphs. Using the asymptotic formula

$$\mathcal{C}ol_n(2)=\Theta(2^{\frac{n^2}{4}+n-\frac{1}{2}\log n})$$

(cf. Corollary 5.3.17) we obtain that

$$|\mathcal{S}(n)|\quad\geq\quad 2^{\frac{n^2}{4}+n+o(\frac{n}{\log\log n})}\cdot 2^{\frac{n}{2}\log n-\frac{n}{2}\log\ln n-(1+\log e)\frac{n}{2}+O(\frac{n}{\log\log n})}.$$

□

An immediate consequence of Lemma 8.2.4 is a bound on the growth rate of the class $\mathcal{S}(n)$.

Corollary 8.2.5 *There exists a constant $D\in\mathbb{R}$ such that*

$$\log\frac{|\mathcal{S}(n)|}{|\mathcal{S}(n+1)|}\leq-\frac{n}{2}+D\frac{n}{\log\log n}\quad\textit{for all } n\in\mathbb{N}.$$ □

Note that the asymptotic formula of Lemma 8.2.4 can easily be sharpened, however for our purposes this version suffices.

8.2.2 Outline of the proof

We shall prove Theorem 8.2.2 by suitably generalizing the Kleitman-Rothschild method and combining it with Szemerédi's uniformity lemma.

Theorem 8.2.6 (The Kleitman-Rothschild Method – Version III) *Let $0 < \epsilon < 1$ and $k, n_0 \in \mathbb{N}$ be constants. If there exist k families $(\mathcal{Y}_i(n,q))_{q \in I_i(n)}$, $i = 1, \dots, k$ and associated functions $y_i(n,q)$ such that for all $n \geq n_0$ the following three conditions are satisfied:*

(i) $\mathcal{F}orb^*_n(C_5) \subseteq \mathcal{S}(n) \cup \bigcup_{i=1}^{k} \bigcup_{q \in I_i(n)} \mathcal{Y}_i(n,q)$,

(ii) $1 \leq y_i(n,q) \leq \epsilon n$ *and* $|I_i(n)| \leq n$ *for all* $i = 1, \dots, k$ *and*

(iii) $\log \frac{|\mathcal{Y}_i(n,q)|}{|\mathcal{F}orb^*_{n-y_i(n,q)}(C_5)|} \leq (\frac{1}{2} - \epsilon) n y_i(n,q)$ *for all* $q \in I_i(n)$ *and* $i = 1, \dots, k$.

Then there exist $c \geq 1$ and $\gamma > 1$, such that for all $n \in \mathbb{N}$:

$$|\mathcal{F}orb^*_n(C_5)| \leq (1 + c \cdot \gamma^{-n})|\mathcal{S}(n)|.$$

Proof. Let $\gamma = 2^{\frac{\epsilon}{4}}$ and choose $n_1 \geq n_0$ large enough such that

$$2^{-\frac{\epsilon}{4}n + \frac{1}{4} + D\frac{n}{\log\log n}} \leq 1 \qquad \text{for all } n \geq n_1,$$

where D is the constant from Corollary 8.2.5, and such that

$$2^{-\frac{\epsilon}{2}n} \leq \frac{1}{2kn}\gamma^{-n} \qquad \text{for all } n \geq n_1.$$

Subsequently choose $c \geq 1$ large enough such that

$$|\mathcal{F}orb^*_n(C_5)| \leq (1 + c \cdot \gamma^{-n})|\mathcal{S}(n)| \qquad \text{for all } n \leq n_1.$$

We prove the theorem by induction on n. So assume that the theorem holds for some $n - 1 \geq n_1$. By assumption (i) we have that

$$|\mathcal{F}orb^*_n(C_5)| \leq \sum_{i=1}^{k} \sum_{q \in I_i(n)} |\mathcal{Y}_i(n,q)| + |\mathcal{S}(n)|.$$

Therefore, it suffices to show that

$$\frac{|\mathcal{Y}_i(n,q)|}{|\mathcal{S}(n)|} \leq \frac{c}{k|I_i(n)|} \cdot \gamma^{-n}$$

holds for all $i = 1, ..., k$ and $q \in I_i(n)$.

Using assumption (iii), Corollary 8.2.5 and the induction hypothesis we conclude that

$$\begin{aligned} \frac{|\mathcal{Y}_i(n,q)|}{|\mathcal{S}(n)|} &= \frac{|\mathcal{Y}_i(n,q)|}{|\mathcal{F}orb^*_{n-y_i(n,q)}(C_5)|} \cdot \frac{|\mathcal{F}orb^*_{n-y_i(n,q)}(C_5)|}{|\mathcal{S}(n - y_i(n,q))|} \cdot \prod_{j=1}^{y_i(n,q)} \frac{|\mathcal{S}(n-j)|}{|\mathcal{S}(n+1-j)|} \\ &\leq 2^{(\frac{1}{2}-\epsilon)ny_i(n,q)} \cdot (1 + c\gamma^{-(n-y_i(n,q))}) \cdot 2^{\sum_{j=1}^{y_i(n,q)}(-\frac{1}{2}(n-j) + D\frac{n-j}{\log\log(n-j)})}. \end{aligned}$$

Since $\gamma^{-(n-y_i(n,q))} \leq 1$ and $c \geq 1$ we have that

$$1 + c\gamma^{-(n-y_i(n,q))} \leq 2c.$$

For the third factor $y_i(n,q) \leq \epsilon n$ implies that

$$\begin{aligned} 2^{\sum_{j=1}^{y_i(n,q)}(-\frac{1}{2}(n-j)+D\frac{n-j}{\log\log(n-j)})} &\leq 2^{-\frac{1}{2}\cdot y_i(n,q)\cdot n+\frac{1}{4}y_i(n,q)^2+\frac{1}{4}y_i(n,q)+D\frac{n}{\log\log n}y_i(n,q)} \\ &\leq 2^{-\frac{1}{2}\cdot y_i(n,q)\cdot n+y_i(n,q)(\frac{1}{4}\epsilon n+\frac{1}{4}+D\frac{n}{\log\log n})}. \end{aligned}$$

By the assumptions on n_1 and (ii) therefore

$$\begin{aligned} \frac{|\mathcal{Y}_i(n,q)|}{|\mathcal{S}(n)|} &\leq 2^{-\frac{\epsilon}{2}y_i(n,q)n}\cdot(2c)\cdot 2^{(-\frac{\epsilon}{4}n+\frac{1}{4}+D\frac{n}{\log\log n})y_i(n,q)} \\ &\leq 2c\cdot\frac{1}{2kn}\gamma^{-n} \\ &\leq \frac{c}{k|I_i(n)|}\gamma^{-n}. \end{aligned}$$

□

In order to apply Theorem 8.2.6 for proving Theorem 8.2.2 the main work is to define suitable families $\mathcal{Y}_i(n,q)$. To do this we heavily rely on Szemerédi's uniformity lemma. In fact, on a slight generalization which we are going to proof now.

Let $G = (V,E)$ be a graph and P be any partition of the vertex set, say $V = V_1 \cup \ldots \cup V_t$. A partition $V = A_0 \cup A_1 \cup \ldots \cup A_{kt}$ is called an *ϵ–uniform refinement of P*, if $|A_0| < \epsilon|V|$, $|A_{(i-1)k+1}| = \ldots = |A_{ik}|$ for all $i = 1,\ldots,t$, $A_{(i-1)k+j} \subseteq V_i$ for all $i = 1,\ldots,t$ and $j = 1,\ldots,k$, and at most $\epsilon(kt)^2$ pairs (A_i,A_j), $1 \leq i < j \leq kt$, are ϵ–irregular.

Theorem 8.2.7 *For every $\epsilon > 0$ there exist positive integers $n_0 = n_0(\epsilon)$ and $m = m(\epsilon)$ such that the following is true. If $G = (V,E)$ is a graph and P a partition $V = V_1 \cup \ldots \cup V_t$ of the vertex set such that $|V_i| \geq n_0$ for all $i = 1,\ldots,t$ then there exists an ϵ-uniform refinement $V = A_0 \cup \ldots \cup A_{kt}$ of P with $k \leq m$.*

Proof. Recall that in Section 3.3 Szemerédi's uniformity lemma is proven as follows. To every partition P an index $\text{ind}(P)$ is assigned. In Lemma 3.3.3 it is then shown that if P is any partition with more than ϵk^2 ϵ-irregular pairs, then there exists an ϵ-uniform refinement Q of P such that $\text{ind}\,Q \geq \text{ind}\,P + \frac{\epsilon^2}{20}$. As the definition of the index directly implies that the index of every partition is smaller than one half, this lemma implies Szemerédi's uniformity lemma – and also Theorem 8.2.7. □

An ϵ-uniform partition of a bipartite graph $G = (A \cup B, E)$ is a partition $A = A_0 \cup A_1 \cup \ldots \cup A_k$ and $B = B_0 \cup B_1 \cup \ldots \cup B_k$ such that $|A_0| < \epsilon|A|$, $|B_0| < \epsilon|B|$, $|A_1| = \ldots = |A_k|$, $|B_1| = \ldots = |B_k|$, and at most ϵk^2 pairs (A_i,B_j), $1 \leq i,j \leq k$, are ϵ–irregular. Observe that Theorem 8.2.7 immediately implies that every sufficiently large bipartite graph has an ϵ-uniform partition.

Notation and definitions. If $G = (V, E)$ is a graph and $A, B \subseteq V_n$ are two disjoint sets of vertices, then the pair (A, B) is called *(ϵ, δ)-regular (-dense, -sparse)* iff (A, B) is ϵ-regular and $\delta \leq d(A, B) \leq 1-\delta$ ($d(A, B) > 1-\delta$, $d(A, B) < \delta$). Furthermore, a graph $G = (V, E)$ is said to contain an (ϵ, δ)-regular (-dense, -sparse) pair of size N iff there exist disjoint sets $A, B \subseteq V$ of size at least N such that (A, B) is (ϵ, δ)-regular (-dense, -sparse).
Let H be a finite graph. A graph $G = (V, E)$ is said to contain an (ϵ, δ, H)-subgraph of size N iff there exist pairwise disjoint sets $A_1, \ldots, A_{|V(H)|}$ each of size N such that
 i) (A_i, A_j) is (ϵ, δ)-dense if and only if $\{i, j\} \in E(H)$ and
 ii) (A_i, A_j) is (ϵ, δ)-sparse if and only if $\{i, j\} \notin E(H)$.
In particular we will consider (ϵ, δ, H)-subgraphs for $H = P_2$, a path consisting of exactly two edges, and for $H = \overline{P}_2$, the complement of such a path.

Let $q_0(n), q_1(n), q_2(n)$ and $q_3(n)$ be four functions such that $\log n = o(q_0(n))$, $q_0(n) = o(q_1(n))$, $q_1(n) = o(q_2(n))$, $q_2(n) = o(q_3(n))$ and $q_3(n) = o(\sqrt{n})$. We call a subset $Q \subseteq V_n$ of the vertex set of a graph $G = (V_n, E)$ a *q-set*, iff $q_1(n) \leq |Q| \leq q_3(n)$. Also, we say that the graph has an (ϵ, δ, H)-subgraph if it has an (ϵ, δ, H)-subgraph of size q for some $q_1(n) \leq q \leq q_3(n)$.

In the rest of this section $m(\epsilon)$ denotes the constant of Theorem 8.2.7 and an ϵ-uniform partition of a graph (or a pair (A, B)) will henceforth always mean an ϵ-uniform partition into at most $m(\epsilon) + 1$ classes (or $2(m(\epsilon) + 1)$ classes, respectively).

Furthermore, we define for every $0 < \epsilon < 1$ a constant $f(\epsilon) = -(\epsilon \log \epsilon + (1 - \epsilon)\log(1 - \epsilon)) + 10^{-10}$. Observe that this implies that $n\binom{n}{\epsilon n} < 2^{f(\epsilon)n}$, for n sufficiently large. With $|\mathcal{F}orb_n^*(C_5)|$ we denote the size of the class $\mathcal{F}orb_n^*(C_5)$, i.e. the number of C_5-free graphs on n vertices.

8.2.3 Proof of the theorem

Demonstratio est omnis divisa in partes tres. In the first part (Section 8.2.3.1) we shall show that almost all C_5-free graphs have a bipartite structure, i.e. that there exist two disjoint vertex sets of size approximately $\frac{n}{2}$ which may be connected arbitrarily, but the subgraphs induced by each of the two sets satisfy quite restrictive conditions. More precisely, we shall prove the following lemma.

Lemma 8.2.8 *Let $0 < \alpha, \delta, \epsilon < 1$ be constants. If $\frac{5}{2}\delta + f(\delta) \leq \frac{\alpha}{8}$ and ϵ is sufficiently small, then there exist constants $0 < \mu_1, \mu_2, \nu < 1$ and sets $\mathcal{Y}_i(n, q')$, $q_0(n) \leq q' \leq q_2(n)$ and $1 \leq i \leq 10$, that fulfill condition ii) and iii) of Theorem 8.2.6 such that for all sufficiently large n all graphs $G = (V_n, E)$ in $\mathcal{F}orb_n^*(C_5) \setminus \left(\bigcup_{i=1}^{10} \bigcup_{q'=q_0(n)}^{q_3(n)} \mathcal{Y}_i(n, q')\right)$ satisfy:*

> *If (Q_1, Q_2) is a (μ_1, ν)-regular pair of q-sets and $(Q_{i0}, Q_{i1}, \ldots, Q_{ik_i}; \Gamma_{i0}, \Gamma_{i1}, \ldots, \Gamma_{ik_i})$ is a μ_2-uniform partition of $(Q_i, \Gamma(Q_i) \setminus Q_{3-i})$, $i = 1, 2$ and I_1 and I_2 denote the sets*
>
> $$I_i = \{1 \leq j \leq k_i \mid \exists 1 \leq j' \leq k \text{ such that } (\Gamma_{ij}, Q_{ij'}) \text{ is } (\mu_2, \nu)\text{-regular}\}$$

then

$$|\bigcup_{j\in I_i} \Gamma_{ij}| \le (\frac{1}{2}+\alpha)n \qquad (i=1,2)$$

and there exist sets

$$S_i \subseteq \bigcup_{j\in I_i} \Gamma_{ij} \qquad (i=1,2)$$

such that

i) $(\frac{1}{2}-8\alpha)n \le |S_i| \le (\frac{1}{2}+\alpha)n$ *for* $i=1,2$,

ii) $S_1 \cap S_2 = \emptyset$, *and*

iii) neither S_1 *nor* S_2 *contains an* (ϵ,δ)*-regular pair* (P_1,P_2) *of* q*-sets.*

In the second part (Section 8.2.3.2) we shall show that in addition for almost all C_5-free graphs the two parts of this bipartite structure are "close" to being a union of cliques and a clique, respectively.

Lemma 8.2.9 *Let* $0<\alpha,\delta,\epsilon<1$ *be constants such that* $6\delta<\frac{1}{200}$, $\frac{5}{2}\delta+f(\delta)\le\frac{\alpha}{8}$ *and such that* ϵ *is sufficiently small and let* μ_1,μ_2 *and* ν *be the constants of Lemma 8.2.8. Then there exist sets* $\mathcal{Y}_{11}(n,q')$, $q_0(n)\le q'\le q_3(n)$ *that fulfill condition ii) and iii) of Theorem 8.2.6 such that for all sufficiently large* n *all graphs* $G=(V_n,E)$ *in* $\mathcal{F}orb_n^*(C_5)\setminus\left(\bigcup_{i=1}^{11}\bigcup_{q'=q_0(n)}^{q_3(n)}\mathcal{Y}_i(n,q')\right)$ *satisfy:*

If (Q_1,Q_2) *is a* (μ_1,ν)*-regular pair of* q*-sets and* S_1 *and* S_2 *denote the sets of Lemma 8.2.8, then - possibly after renaming* Q_1 *and* Q_2 *- either* G *or* $\bar{G}$ *has the following property:*

i) The subgraph induced by S_1 *does not contain an* $(\epsilon,2\delta,P_2)$*-subgraph.*

ii) If B_{ij}, $i=1,2, j=1,\dots,l$ *are pairwise disjoint* q*-sets in* S_2 *such that* $|B_{1j}|=|B_{2j}|$ *and* (B_{1j},B_{2j}) *is* (ϵ,δ)*-sparse for all* $j=1,\dots,l$ *then* $\sum_{i,j}|B_{ij}| \le 300\alpha n$.

In the final part (sections 8.2.3.3 and 8.2.3.4) we then shall show how to define some additional exceptional sets $\mathcal{Y}_i$ so that the remaining graphs in $\mathcal{F}orb_n^*(C_5)$ are indeed generalized split graphs.

8.2.3.1 Proof of Lemma 8.2.8

The following lemma essentially states that no graph without induced C_5 may contain a sufficiently large and sufficiently regular tripartite subgraph. In fact this lemma is a special case of Lemma 7.2.6. Nevertheless, we here also give a short, direct proof.

Lemma 8.2.10 *Let* $0<\delta<\frac{1}{2}$ *be a constant. Then there exist constants* $0<\epsilon_0(\delta)\le\delta$ *and* $n_0=n_0(\delta)$ *such that for all* $\epsilon\le\epsilon_0(\delta)$ *every graph* $G=(V,E)$ *has the following property: If* $V=A_1\cup A_2\cup A_3$ *is a partition of the vertex set such that all parts have size at least* n_0 *and all pairs* (A_i,A_j), $1\le i<j\le 3$ *are* (ϵ,δ)*-regular, then* G *contains an induced* C_5.

Proof. Choose the constant $\epsilon_0 = \epsilon_0(\delta)$ and a constant μ in such a way that

$$\frac{\delta}{2} > \sqrt{\epsilon_0}, \quad \frac{1}{2} - 2\mu > \delta, \quad \frac{1-\mu}{m(\mu)} \geq \sqrt{\epsilon_0} \quad \text{and} \quad (\delta - 2\sqrt{\epsilon_0})^4 - \mu - 3\sqrt{\epsilon_0} > 0$$

and choose $n_0 = n_0(\delta)$ such that

$$n_0 > m(\mu) \qquad \text{and} \qquad (\delta - 2\sqrt{\epsilon_0})^4 \cdot n_0 \geq 1.$$

For the remaining part of the proof of Lemma 8.2.10 let now ϵ be a fixed constant less than or equal to ϵ_0 and let A, B and C be three sets of size $|A|, |B|, |C| \geq n_0(\delta)$ such that the three pairs (A, B), (B, C) and (C, A) are all (ϵ, δ)-regular. We will construct an induced C_5 in $A \cup B \cup C$.

By our assumption on n_0 each of the subgraphs induced by A, B and C, respectively, contains a μ-uniform partition. Since C_5 is self-complementary we may therefore assume without loss of generality that there exist subsets $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ such that $|A_1| = |A_2| \geq \frac{1-\mu}{m(\mu)}|A|$, $|B_1| = |B_2| \geq \frac{1-\mu}{m(\mu)}|B|$ and the pairs (A_1, A_2) and (B_1, B_2) are both $(\mu, \frac{1}{2})$-dense. (Observe also that by the definition of μ the pairs (A_i, B_j), (A_i, C) and (B_i, C) are all $(\sqrt{\epsilon}, \delta - \epsilon)$-regular.)

For the construction of the C_5 the following observation is essential: If (X, Y) is an α-regular pair and $X' \subseteq X$ and $Y' \subseteq Y$ are two subsets such that $|Y'| \geq \alpha|Y|$ then the regularity implies that

$$|\{v \in X' \mid |\Gamma(v) \cap Y'| < (d(X,Y) - \alpha)|Y'|\}| < \alpha|X|$$

and

$$|\{v \in X' \mid |\Gamma(v) \cap Y'| > (d(X,Y) + \alpha)|Y'|\}| < \alpha|X|.$$

Since by choice of ϵ_0 and μ we have $|A_1| - \mu|A_1| - 3\sqrt{\epsilon}|A_1| \geq 1$ this implies that there exists a vertex $v_1 \in A_1$ such that the sets $A_2^{(1)} = \Gamma(v_1) \cap A_2$, $B_1^{(1)} = \Gamma(v_1) \cap B_1$, $B_2^{(1)} = B_2 \setminus \Gamma(v_1)$ and $C^{(1)} = C \setminus \Gamma(v_1)$ satisfy $|A_2^{(1)}| \geq (\frac{1}{2} - \mu)|A_2|$, $|B_i^{(1)}| \geq (\delta - 2\sqrt{\epsilon})|B_i|$ and $|C^{(1)}| \geq (\delta - 2\sqrt{\epsilon})|C|$. Repeating this argument we then obtain a vertex $v_2 \in A_2^{(1)}$ such that the sets $B_1^{(2)} = B_1^{(1)} \setminus \Gamma(v_2)$, $B_2^{(2)} = B_2^{(1)} \setminus \Gamma(v_2)$ and $C^{(2)} = C^{(1)} \cap \Gamma(v_2)$ satisfy $|B_i^{(2)}| \geq (\delta - 2\sqrt{\epsilon})|B_i^{(1)}| \geq (\delta - 2\sqrt{\epsilon})^2|B_i|$ and $|C^{(2)}| \geq (\delta - 2\sqrt{\epsilon})|C^{(1)}| \geq (\delta - 2\sqrt{\epsilon})^2|C|$ and a vertex $v_3 \in B_1^{(2)}$ such that the sets $B_2^{(3)} = B_2^{(2)} \cap \Gamma(v_3)$ and $C^{(3)} = C^{(2)} \setminus \Gamma(v_3)$ satisfy $|B_2^{(3)}| \geq (\frac{1}{2} - 2\mu)|B_i^{(2)}| \geq (\delta - 2\sqrt{\epsilon})^3|B_i|$ and $|C^{(3)}| \geq (\delta - 2\sqrt{\epsilon})|C^{(2)}| \geq (\delta - 2\sqrt{\epsilon})^3|C|$. Finally, we obtain a vertex $v_4 \in B_2^{(3)}$ such that the set $C^{(4)} = C^{(3)} \cap \Gamma(v_3)$ satisfies $|C^{(4)}| \geq (\delta - 2\sqrt{\epsilon})|C^{(3)}| \geq (\delta - 2\sqrt{\epsilon})^4|C|$. By the assumption on n_0 this implies that $C^{(4)} \neq \emptyset$ which concludes the proof of Lemma 8.2.10. □

In the rest of this section $\epsilon_0(\delta)$ will denote the function of Lemma 8.2.10. Without loss of generality we may assume that $\epsilon_0(\delta) \leq \delta$ for all δ and that $\epsilon_0(\delta_1) \leq \epsilon_0(\delta_2)$ whenever $\delta_1 \leq \delta_2$.

The next lemma bounds the number of bipartite subgraphs which are not "sufficiently" regular.

Lemma 8.2.11 *Let $0 < \mu, \nu < \frac{1}{2}$ and $1 \leq k \leq m(\mu)$ be fixed. For positive integers a, b and M let $N = N(a, b, M)$ denote the number of bipartite graphs with parts A and B, $|A| = a$ and $|B| = b$, that have a μ-uniform partition $(A_0, A_1, \ldots, A_k; B_0, B_1, \ldots, B_k)$ into $2(k+1)$ classes such that at most M pairs (A_i, B_j), $1 \leq i, j \leq k$ are (μ, ν)-regular. Then*

$$N \leq 2^{(4\mu + f(\nu))ab + ab\frac{M}{k^2}},$$

whenever a and b are sufficiently large.

Proof. We construct all such bipartite graphs as follows. First choose an appropriate partition of A and B (at most $(k+1)^{a+b}$ ways) and connect A_0 and B_0, respectively (at most $2^{2\mu ab}$ ways). Then decide for every pair (A_i, B_j), $1 \leq i, j \leq k$ whether it is μ-irregular, (μ, ν)-regular, (μ, ν)-sparse or (μ, ν)-dense (less than 4^{k^2} ways). Next choose the edges in μ-irregular pairs (at most $\left[2^{|A_1||B_1|}\right]^{\mu k^2} \leq 2^{\mu ab}$ ways), in (μ, ν)-regular pairs (at most $\left[2^{|A_1||B_1|}\right]^{M} \leq 2^{ab\frac{M}{k^2}}$ ways), and in the remaining pairs (at most $\left[\sum_{i=0}^{\nu|A_1||B_1|} \binom{|A_1||B_1|}{i}\right]^{k^2} \leq 2^{f(\nu)ab}$, for a and b sufficiently large). Together this gives

$$N \leq (k+1)^{a+b} \cdot 2^{2\mu ab} \cdot 4^{k^2} \cdot 2^{\mu ab} \cdot 2^{ab\frac{M}{k^2}} \cdot 2^{f(\nu)ab} \leq 2^{(4\mu + f(\nu))ab + ab\frac{M}{k^2}},$$

for a, b sufficiently large. □

As a first step towards the proof of Lemma 8.2.8 we show that almost all graphs in $\mathcal{F}orb_n^*(C_5)$ have the property that the neighborhood $\Gamma(Q)$ of every sufficiently large set Q contains a set of size approximately $\frac{n}{2}$ to which Q is connected in a "regular" fashion. More precisely, given constants $0 < \mu, \nu < 1$ and a set Q we call a subset $R \subseteq \Gamma(Q)$ an *$r(\mu, \nu)$-set* of Q, if there exists a partition $R = \bigcup_{i=1}^{l_R} R_i$ such that

i) $q_1(n) \leq |R_i|$ for all $1 \leq i \leq l_R$, and

ii) there exist subsets $Q_1, \ldots, Q_{l_R}$ of Q such that for all $i = 1, \ldots, l_R$ $Q_i \geq \frac{1}{2m(\nu)}|Q|$ and (Q_i, R_i) is (μ, ν)-regular.

We now show that almost all graphs in $\mathcal{F}orb_n^*(C_5)$ have the property that all q-sets have a large $r(\mu, \nu)$-set and prove some structural properties of such $r(\mu, \nu)$-sets.

Lemma 8.2.12 *Let $0 < \alpha, \mu, \nu < 1$ be constants such that $5\mu + f(\nu) \leq \frac{\alpha}{8}$. For $q_0(n) \leq q' \leq q_3(n)$ let $\mathcal{A}(n, q', \alpha, \mu, \nu)$ denote the set of all graphs $G = (V_n, E)$ in $\mathcal{F}orb_n^*(C_5)$ that contain a set Q of size $|Q| = q'$ for which there exists a partition $A_1 \cup \ldots \cup A_l$ of $\Gamma(Q)$ and for every A_i a μ-uniform partition $(A_{i0}, A_{i1}, \ldots, A_{ik_i}; Q_{i0}, Q_{i1}, \ldots, Q_{ik_i})$ of (A_i, Q) such that the sets*

$$I_i = \{1 \leq j \leq k_i \mid \exists 1 \leq j' \leq k_i \textit{ such that } (A_{ij}, Q_{ij'}) \textit{ is } (\mu, \nu)\textit{-regular}\}$$

satisfy

$$\sum_{i=1}^{l} |I_i| \frac{|A_i|}{k_i} \leq (\frac{1}{2} - \frac{\alpha}{2})n.$$

Then for n sufficiently large

$$\log \frac{|\mathcal{A}(n,q',\alpha,\mu,\nu)|}{|\mathcal{F}orb^*_{n-q'}(C_5)|} \leq (\frac{1}{2}-\frac{3}{8}\alpha)q'n \qquad \text{for all } q_0(n) \leq q' \leq q_3(n).$$

Proof. Construct all graphs in $\mathcal{A}(n,q',\alpha,\mu,\nu)$ as follows. First choose the set Q and a graph without induced C_5 on $V_n \setminus Q$ (at most $\binom{n}{q'}|\mathcal{F}orb^*_{n-q'}(C_5)|$ ways). Then choose the set $\Gamma(Q)$ and the partition $\Gamma(Q) = \bigcup_{i=1}^{l} A_i$ (less than n^n ways) and choose the numbers k_i and $|I_i|$ for each set A_i (less than $m(\mu)^{2n}$ ways). Finally, choose edges inside Q (less than $2^{q'q'}$ ways) and connect Q to the sets A_i (at most

$$2^{\sum_{i=1}^{l}((4\mu+f(\nu))q'|A_i| + q'|A_i|\frac{|I_i|k_i}{k_i^2})} \leq 2^{(4\mu+f(\nu))q'n+q'\sum_{i=1}^{l}\frac{|A_i||I_i|}{k_i}}$$

ways, cf. Lemma 8.2.11).
Together this gives

$$\begin{aligned}
\log \frac{|\mathcal{A}(n,q',\alpha,\mu,\nu)|}{|\mathcal{F}orb^*_{n-q'}(C_5)|} &\leq q'\log n + n\log n + 2n\log m(\mu) + q'q' \\
&\quad + (4\mu+f(\nu))q'n + q'\sum_{i=1}^{l}\frac{|A_i||I_i|}{k_i} \\
&\leq (5\mu+f(\nu))q'n + q'(\frac{1}{2}-\frac{\alpha}{2})n \\
&\leq (\frac{1}{2}-\frac{3}{8}\alpha)q'n,
\end{aligned}$$

for n sufficiently large, independently of $q_0(n) \leq q' \leq q_3(n)$. □

Corollary 8.2.13 *Let $0 < \alpha,\mu,\nu < 1$ be constants such that $5\mu + f(\nu) \leq \frac{\alpha}{8}$. For n sufficiently large every graph $G=(V_n,E)$ in $\mathcal{F}orb^*_n(C_5) \setminus \bigcup_{q'=q_0(n)}^{q_3(n)} \mathcal{A}(n,q',\alpha,\mu,\nu)$ has the following property: Every q-set Q has an $r(\mu,\nu)$-set of size at least $(\frac{1}{2}-\frac{\alpha}{2})n$.* □

Lemma 8.2.14 *Let $0 < \alpha < 1$ and $0 < \mu_1 \leq \mu_2 \leq \frac{\nu}{2} < \frac{1}{2}$ be constants such that $5\epsilon_0(\frac{\nu}{2}) + f(\nu) \leq \frac{\alpha}{2}$, $\frac{\epsilon_0(\frac{\nu}{2})}{2m(\mu_2)} \geq \mu_1$ and $\frac{\epsilon_0(\frac{\nu}{2})}{2m(\epsilon_0(\frac{\nu}{2}))} \geq \mu_2$. For $q_0(n) \leq q' \leq q_3(n)$ let $\mathcal{B}(n,q',\alpha,\mu_1,\mu_2,\nu)$ denote the set of all graphs $G=(V_n,E)$ in $\mathcal{F}orb^*_n(C_5)$ that contain a (μ_1,ν)-regular pair (Q_1,Q_2) of disjoint sets Q_1 and Q_2 such that $q_0(n) \leq |Q_1| \leq q_3(n)$ and $|Q_2| = q'$ for which there exists a μ_2-uniform partition $(Q_{10},Q_{11},\ldots,Q_{1k};\Gamma_0,\Gamma_1,\ldots,\Gamma_k)$ of $(Q_1,\Gamma(Q_1)\setminus Q_2)$ such that the set*

$$I = \{1 \leq j \leq k \mid \exists 1 \leq j' \leq k \text{ such that } (\Gamma_j, Q_{1j'}) \text{ is } (\mu_2,\nu)\text{-regular}\}$$

satisfies

$$\sum_{j\in I}|\Gamma_j| \geq (\frac{1}{2}+\alpha)n.$$

Then for n sufficiently large

$$\log \frac{|\mathcal{B}(n, q', \alpha, \mu_1, \mu_2, \nu)|}{|\mathcal{F}orb^*_{n-q'}(C_5)|} \leq (\frac{1}{2} - \frac{\alpha}{2})q'n \qquad \textit{for all } q_0(n) \leq q' \leq q_3(n).$$

Proof. Construct all graphs in $\mathcal{B}(n, q', \alpha, \mu_1, \mu_2, \nu)$ as follows. First choose the set Q_2 and a graph without induced C_5 on $V_n \setminus Q_2$ (at most $\binom{n}{q'}|\mathcal{F}orb^*_{n-q'}(C_5)|$ ways). Then choose edges inside Q_2, the set Q_1 and an appropriate partition of $\Gamma(Q_1) \setminus Q_2$ (less than $2^{q'q'} \cdot n^{q_3(n)} \cdot m(\mu_2)^n$ ways). Finally, connect Q_2 to the remaining vertices. To do this observe that Lemma 8.2.10 implies that for $j \in I$ no $\epsilon_0(\frac{\nu}{2})$-uniform partition of (Γ_j, Q_2) may contain a $(\epsilon_0(\frac{\nu}{2}), \nu)$-regular pair $(Q_{2i}, \Gamma_{ji'})$. (Otherwise the three pairs induced by Q_{2i}, $\Gamma_{ji'}$ and $Q_{1j'}$ are all $(\epsilon_0(\frac{\nu}{2}), \frac{\nu}{2})$-regular, which is a contradiction.) Hence, the number of ways to connect Q_2 to $V_n \setminus Q_2$ is bounded by (cf. Lemma 8.2.11)

$$2^{\sum_{j\in I}((4\epsilon_0(\frac{\nu}{2}) + f(\nu))q'|\Gamma_j|} \cdot 2^{(n - \sum_{j\in I}|\Gamma_j|)q'}.$$

Together this gives

$$\begin{aligned}
\log \frac{|\mathcal{B}(n, q', \alpha, \mu_1, \mu_2, \nu)|}{|\mathcal{F}orb^*_{n-q'}(C_5)|} &\leq q' \log n + q'q' + q_3(n) \log n + n \log m(\mu_2) \\
&\quad + nq' - (1 - (4\epsilon_0(\frac{\nu}{2}) + f(\nu))q' \sum_{j\in I} |\Gamma_j| \\
&\leq (\frac{1}{2} - \frac{\alpha}{2})q'n,
\end{aligned}$$

for n sufficiently large, independently of $q_0(n) \leq q' \leq q_3(n)$. □

Lemma 8.2.15 *Let $0 < \alpha < 1$ and $0 < \mu_1 \leq \mu_2, \mu_2' \leq \frac{\nu}{2} < \frac{1}{2}$ be constants such that $5\epsilon_0(\frac{\nu}{2}) + f(\nu) \leq \frac{\alpha}{4}$, $\frac{\epsilon_0(\frac{\nu}{2})}{2m(\mu_2)} \geq \mu_1$, $\frac{\epsilon_0(\frac{\nu}{2})}{2m(\mu_2')} \geq \mu_1$ and $\frac{\epsilon_0(\frac{\nu}{2})}{2m(\epsilon_0(\frac{\nu}{2}))} \geq \max\{\mu_2, \mu_2'\}$. For $q_0(n) \leq q' \leq q_3(n)$ let $\mathcal{C}(n, q', \alpha, \mu_1, \mu_2, \mu_2', \nu)$ denote the set of all graphs $G = (V_n, E)$ in $\mathcal{F}orb^*_n(C_5)$ that contain a (μ_1, ν)-regular pair (Q_1, Q_2) of disjoint sets Q_1 and Q_2 such that $q_0(n) \leq |Q_1| \leq q_3(n)$ and $|Q_2| = q'$ for which there exists an $r(\mu_2, \nu)$-set R and an $r(\mu_2', \nu)$-set R' of Q_1 such that*

$$|R| \geq (\frac{1}{2} - \alpha)n, \quad |R'| \geq (\frac{1}{2} - \alpha)n \quad \textit{and} \quad |R \cap R'| \leq (\frac{1}{2} - 3\alpha)n.$$

Then for n sufficiently large

$$\log \frac{|\mathcal{C}(n, q', \alpha, \mu_1, \mu_2, \mu_2', \nu)|}{|\mathcal{F}orb^*_{n-q'}(C_5)|} \leq (\frac{1}{2} - \frac{\alpha}{2})q'n \qquad \textit{for all } q_0(n) \leq q' \leq q_3(n).$$

Proof. Construct all graphs in $\mathcal{C}(n, q', \alpha, \mu_1, \mu_2, \mu_2', \nu)$ as follows. First choose the set Q_2 and a graph without induced C_5 on $V_n \setminus Q_2$ (at most $\binom{n}{q'}|\mathcal{F}orb^*_{n-q'}(C_5)|$ ways). Then choose edges inside Q_2, the set Q_1 and appropriate r-sets $R = \bigcup R_i$ and $R' =$

$\bigcup R_i'$ (less than $2^{q'q'} \cdot n^{q_3(n)} \cdot n^{2n}$ ways). As in the proof of Lemma 8.2.14 we observe that no $\epsilon_0(\frac{\nu}{2})$-uniform partition of a set R_i or R_i' with Q_2 may contain a $(\epsilon_0(\frac{\nu}{2}), \nu)$-regular pair. Hence, the number of ways to connect Q_2 to $R \cup R'$ is bounded by (cf. Lemma 8.2.11) $2^{(4\epsilon_0(\frac{\nu}{2})+f(\nu))q'(|R|+|R'|)}$. Finally, connect Q_2 to $V_n \setminus (Q_2 \cup R \cup R')$ (at most $2^{q'(n-|R|-|R'|+|R\cap R'|)}$ ways).
Together this gives

$$\begin{aligned}\log \frac{|\mathcal{C}(n,q',\alpha,\mu_1,\mu_2,\mu_2',\nu)|}{|\mathcal{F}orb^*_{n-q'}(C_5)|} &\leq q'\log n + q'q' + q_3(n)\log n + 2n\log n \\ + (4\epsilon_0(\frac{\nu}{2}) + f(\nu))q'(|R|+|R'|) &+ q'(n-|R|-|R'|+|R\cap R'|) \\ &\leq (\frac{1}{2} - \frac{\alpha}{2})q'n,\end{aligned}$$

for n sufficiently large, independently of $q_0(n) \leq q' \leq q_3(n)$. □

Proof of Lemma 8.2.8. Let $\nu = \delta$ and choose constants $\mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4 \leq \mu_5 \leq \frac{\nu}{4}$ such that the following inequalities hold simultaneously:

$$\begin{gathered}\tfrac{\mu_i}{2m(\mu_i)} \geq \mu_{i-1} \quad \text{for } i = 2,\ldots,5, \\ \mu_1 \leq \tfrac{\epsilon_0(\frac{\nu}{2})}{2m(\mu_2)} \leq \tfrac{\epsilon_0(\frac{\nu}{2})}{2m(\mu_3)} \quad \text{and} \quad \mu_3 \leq \tfrac{\epsilon_0(\frac{\nu}{2})}{2m(\mu_4)} \leq \tfrac{\epsilon_0(\frac{\nu}{2})}{2m(\mu_5)}, \\ \mu_2 \leq \mu_3 \leq \tfrac{\epsilon_0(\frac{\nu}{2})}{2m(\epsilon_0(\frac{\nu}{2}))} \quad \text{and} \quad \mu_4 \leq \mu_5 \leq \tfrac{\epsilon_0(\frac{\nu}{4})}{2m(\epsilon_0(\frac{\nu}{4}))}, \\ \epsilon \leq \mu_2.\end{gathered}$$

Obviously, this can be done, if ϵ is sufficiently small compared to $\delta = \nu$.
Define the $\mathcal{Y}_i$-sets as follows:

$$\begin{aligned}\mathcal{Y}_i(n,q') &= \mathcal{A}(n,q',\alpha,\mu_{1+i},\nu) \quad \text{for } i = 1,2, \\ \mathcal{Y}_i(n,q') &= \mathcal{A}(n,q',\alpha,\mu_{1+i},\frac{\nu}{2}) \quad \text{for } i = 3,4, \\ \mathcal{Y}_5(n,q') &= \mathcal{B}(n,q',\alpha,\mu_1,\mu_2,\nu), \\ \mathcal{Y}_6(n,q') &= \mathcal{B}(n,q',\alpha,\mu_3,\mu_4,\frac{\nu}{2}), \\ \mathcal{Y}_i(n,q') &= \mathcal{C}(n,q',\alpha,\mu_1,\mu_2,\mu_{i-5},\nu) \quad \text{for } i = 7,8, \text{ and} \\ \mathcal{Y}_i(n,q') &= \mathcal{C}(n,q',\alpha,\mu_3,\mu_4,\mu_{i-5},\nu) \quad \text{for } i = 9,10.\end{aligned}$$

It is not difficult to check that the above assumptions on the constants imply that the conditions of Lemma 8.2.12, 8.2.14 and 8.2.15 hold. (Recall that $\epsilon_0(\delta) \leq \delta$ for all δ and observe that the assumption of the lemma on δ implies that $5\epsilon_0(\frac{\nu}{2}) + f(\nu) \leq \frac{\alpha}{8}$ and $5\epsilon_0(\frac{\nu}{4}) + f(\frac{\nu}{2}) \leq \frac{\alpha}{8}$.) This in turn shows that the $\mathcal{Y}_i$-sets satisfy condition ii) and iii) of Theorem 8.2.6.

To see the structural result let $G = (V_n, E)$ be an arbitrary graph in $\mathcal{F}orb^*_n(C_5) \setminus \bigcup_{i=1}^{10} \bigcup_{q'=q_0(n)}^{q_3(n)} \mathcal{Y}_i(n,q')$, let (Q_1, Q_2) be a (μ_1,ν)-regular pair of q-sets $Q_1, Q_2 \subseteq V_n$ such that $q_1(n) \leq |Q_1|, |Q_2| \leq q_3(n)$ and let $R_i = \bigcup_{j\in I_i} \Gamma_{ij}$ denote the sets defined in the lemma. Since $G \notin \mathcal{Y}_5(n,q')$ Lemma 8.2.14 implies that $|R_i| \leq (\frac{1}{2}+\alpha)n$. To prove the existence of the sets S_i we show as a first step:

($*$) There exist sets $S_i \subseteq R_i$, $i = 1, 2$ such that

- $|S_i| \geq (\frac{1}{2} - 7\alpha)n$, and
- if $\tilde{S} \subseteq S_i$ with $q_0(n) \leq |\tilde{S}| \leq q_3(n)$, then every μ_2-uniform partition of $(Q_i, \tilde{S})$ contains a (μ_2, ν)-regular pair and every μ_3-uniform partition of $(Q_i, \tilde{S})$ contains a (μ_3, ν)-regular pair.

And secondly,

($**$) Every such pair of sets S_1 and S_2 as defined in ($*$) satisfies $|S_1 \cap S_2| \leq q_1(n)$.

To see ($*$) it obviously suffices to prove the existence of S_1. Observe that by definition R_1 is an $r(\mu_2, \nu)$-set of Q_1 and that $|R_1| \geq (1 - \mu_2)|I_i|\frac{|\Gamma(Q_1)| - |Q_2|}{k_i}$ and hence by Lemma 8.2.12 $|R_1| \geq (\frac{1}{2} - \alpha)n$. Call a subset A of R_1 "bad" of type I (of type II), if there exists a μ_2-uniform partition (a μ_3-uniform partition) of (Q_1, A) which contains no (μ_2, ν)-regular pair (no (μ_3, ν)-regular pair). Let $A_1, \ldots, A_l$ be a collection of pairwise disjoint subsets of R_1 which are bad of type I such that $|\bigcup_{i=1}^{l} A_i|$ is maximal. Apply Lemma 8.2.12 to the partition $A_1, \ldots, A_l, R_1 \setminus \bigcup_{i=1}^{l} A_i, \Gamma(Q_1) \setminus R_1$ of $\Gamma(Q_1)$: there exists an $r(\mu_2, \nu)$-set R_1' of size at least $(\frac{1}{2} - \alpha)n$ such that $R_1' \cap \bigcup_{i=1}^{l} A_i = \emptyset$. By Lemma 8.2.15 we know in addition that $|R_1 \cap R_1'| \geq (\frac{1}{2} - 3\alpha)n$. Similarly, if $B_1, \ldots, B_{l'}$ is a collection of pairwise disjoint subsets of R_1 which are bad of type II such that $|\bigcup_{i=1}^{l'} B_i|$ is maximal, then there exists an $r(\mu_3, \nu)$-set R_1'' of size at least $(\frac{1}{2} - \alpha)n$ such that $R_1'' \cap \bigcup_{i=1}^{l'} B_i = \emptyset$ and $|R_1 \cap R_1''| \geq (\frac{1}{2} - 3\alpha)n$. Hence, $|R_1| \leq (\frac{1}{2} + \alpha)n$ (cf. Lemma 8.2.14) implies that $|R_1 \setminus \left(\bigcup_{i=1}^{l} A_i \cup \bigcup_{i=1}^{l'} B_i\right)| \geq (\frac{1}{2} - 7\alpha)n$ and that therefore $R_1 \setminus \left(\bigcup_{i=1}^{l} A_i \cup \bigcup_{i=1}^{l'} B_i\right)$ is the required set S_1.

To see ($**$) assume the contrary, i.e. assume that there exists a set $\tilde{S} \subseteq S_1 \cap S_2$ of size $q_1(n) \leq |\tilde{S}| \leq q_3(n)$. Since $\tilde{S} \subseteq S_1$ every μ_2-uniform partition of $(Q_1, \tilde{S})$ contains a (μ_2, ν)-regular pair. Let $(A, \tilde{A})$ be such a pair. Since $\tilde{A} \subseteq S_2$ every μ_3-uniform partition of $(Q_2, \tilde{A})$ contains a (μ_3, ν)-regular pair. Let $(B, \tilde{B})$ be such a pair. Then by choice of the constants μ_i the three pairs $(A, \tilde{B})$, $(\tilde{B}, B)$, and (A, B) are all $(\epsilon_0(\frac{\nu}{2}), \frac{\nu}{2})$-regular, which is by Lemma 8.2.10 impossible.

Combining ($*$) and ($**$) we conclude that there exist sets $S_i \subseteq R_i$ which fulfill conditions i) and ii) of the lemma and the second condition of ($*$). It remains to be shown that they also satisfy condition iii) of the lemma. Therefore, let us assume that (P_1, P_2) is an (ϵ, δ)-regular pair, i.e. by choice of μ_2 in particular also a (μ_2, ν)-regular pair, of q-sets $P_1, P_2 \subseteq S_1$. By choice of S_1 every μ_3-uniform partition of (P_i, Q_1) contains a (μ_3, ν)-regular pair, say (A_i, B_i). Observe now that the five pairs (A_1, A_2), (A_1, B_1), (A_2, B_2), (B_1, Q_2) and (B_2, Q_2) are all $(\mu_3, \frac{\nu}{2})$-regular. At this point note that the proof of ($*$) and ($**$) implies that ($*$) and ($**$) hold also if we replace μ_1 by μ_3, μ_2 by μ_4, μ_3 by μ_5, ν by $\frac{\nu}{2}$, and the set R_1 by the analogously defined set. However, this then shows that the sets A_1, A_2, B_1, B_2 and Q_2 induce sets $T_1, \ldots, T_5$ each of size approximately $\frac{n}{2}$ such that $|T_1 \cap T_5|$ and $|T_i \cap T_{i+1}|$ are $o(n)$ for all $i = 1, \ldots, 5$, which is an obvious contradiction. $\square$

8.2.3.2 Proof of Lemma 8.2.9

The main tool in the proof of Lemma 8.2.9 is the following lemma which bounds the number of ways in which two graphs G_1 and G_2 of special structure can be connected without creating an induced C_5.

Lemma 8.2.16 *Let $0 < \epsilon < \delta < \frac{1}{4}$ and $q, s \in \mathbb{N}$ be constants such that $6\delta + f(3\epsilon) < \frac{1}{100}$. Furthermore let $G_1 = (A \cup B \cup C, E_1)$ and $G_2 = (S \cup T, E_2)$ be two graphs without induced C_5 such that $|A| = |B| = |C| = q$, $|S| = |T| = s$, (B, C) is $(\epsilon, 2\delta)$-sparse, the pairs (A, B), (A, C) are $(\epsilon, 2\delta)$-dense and (S, T) is (ϵ, δ)-dense. Then the number of ways to connect G_1 and G_2 such that the resulting graph contains no induced C_5 is bounded from above by $2^{(6-\frac{1}{6})qs}$, for q, s sufficiently large.*

Proof. We distinguish two types of connections. First assume that there exist $\epsilon|S|$ many vertices $v \in S$ such that

$$(*) \qquad |\Gamma(v) \cap B| > \frac{1}{4}|B|, \quad |A \setminus \Gamma(v)| > \frac{1}{4}|A| \quad \text{and} \quad |C \setminus \Gamma(v)| > \frac{1}{4}|C|.$$

Then at least one of these vertices, say v_0, must have a neighborhood T_0 of size at least $(1 - \delta - \epsilon)|T|$ in T. Observe now that no vertex x in T_0 may fulfill the following three conditions simultaneously:

- $|(A \setminus \Gamma(v_0)) \setminus \Gamma(x)| \geq 3\epsilon|A|$,
- $|(B \cap \Gamma(v_0)) \setminus \Gamma(x)| \geq 3\epsilon|B|$,
- $|(C \setminus \Gamma(v_0)) \cap \Gamma(x)| \geq 3\epsilon|C|$.

(Otherwise the regularity and density assumptions on the pairs (A, B), (A, C) and (B, C) would imply the existence of a vertex $v_A \in (A \setminus \Gamma(v_0)) \setminus \Gamma(x)$ such that the intersections of $\Gamma(v_A)$ with the sets $B' = (B \cap \Gamma(v_0)) \setminus \Gamma(x)$ and $C' = (C \setminus \Gamma(v_0)) \cap \Gamma(x)$ would have size at least $(1 - 2\delta - \epsilon)|B'| \geq \epsilon|B|$ and $(1 - 2\delta - \epsilon)|C'| \geq \epsilon|C|$, respectively, which in turn would guarantee the existence of an induced C_5.) Hence, this case occurs for q, s sufficiently large at most

$$\begin{aligned} 2^{3q|S|} \cdot 2^{3q(\delta+\epsilon)|T|} \cdot \left[3 \left(\sum_{i=0}^{3\epsilon q} \binom{q}{i} \right) 2^{\frac{11}{4}q} \right]^{(1-\delta-\epsilon)|T|} &\leq 2^{(6-\frac{1}{4}+3(\delta+\epsilon)+f(3\epsilon))qs} \\ &< 2^{(6-\frac{1}{6})qs} \end{aligned}$$

times. Secondly, if $\epsilon|S|$ many vertices which satisfy $(*)$ do not exist in S, the number of ways to connect G_1 and G_2 is for q, s sufficiently large obviously bounded by

$$\begin{aligned} \left(\sum_{i=0}^{\epsilon|S|} \binom{|S|}{i} \right) \cdot 2^{3q\epsilon|S|} \cdot \left[3 \left(\sum_{i=0}^{\frac{1}{4}q} \binom{q}{i} \right) 2^{2q} \right]^{|S|} \cdot 2^{3q|T|} &\leq 2^{(5+3\epsilon+f(\epsilon)+f(\frac{1}{4}))qs} \\ &< 2^{(6-\frac{1}{6})qs}. \end{aligned}$$

(To see the last inequality, recall that $f(\frac{1}{4}) = -(\frac{1}{4}\log\frac{1}{4} + \frac{3}{4}\log\frac{3}{4}) + 10^{-10} = 0.811\ldots$.)

□

Lemma 8.2.17 *Let $0 < \alpha < \frac{1}{2}$ and $0 < \epsilon < \delta < \frac{1}{4}$ be constants such that $6\delta + f(3\epsilon) < \frac{1}{100}$ and $4\epsilon + f(\delta) \leq \frac{\alpha}{15}$. For $q_0(n) \leq q' \leq q_3(n)$ let $\mathcal{D}(n, q', \alpha, \epsilon, \delta)$ denote the set of all graphs $G = (V_n, E)$ in $\mathcal{F}orb_n^*(C_5)$ that contain a set S_1 with $|S_1| \geq (\frac{1}{2} - 8\alpha)n$ such that either G or $\bar{G}$ satisfies the following properties:*

i) S_1 contains no (ϵ, δ)-regular pair of q-sets,

ii) there exist pairwise disjoint sets $A_1, A_2, A_3 \subseteq S_1$ such that $|A_1| = |A_2| = |A_3| = q'$, (A_1, A_2) and (A_1, A_3) are $(\epsilon, 2\delta)$-dense, and (A_2, A_3) is $(\epsilon, 2\delta)$-sparse, and

iii) there exist pairwise disjoint q-sets B_{ij} in $V_n \setminus S_1$, $i = 1, 2$, $j = 1, \dots, l$ such that $|B_{1j}| = |B_{2j}|$ for all $j = 1, \dots, l$, $\sum_{i,j} |B_{ij}| \geq 300\alpha n$ and all pairs (B_{1j}, B_{2j}) are (ϵ, δ)-dense.

Then for n sufficiently large

$$\log \frac{|\mathcal{D}(n, q', \alpha, \epsilon, \delta)|}{|\mathcal{F}orb_{n-3q'}^*(C_5)|} \leq 3(\frac{1}{2} - \frac{\alpha}{5})q'n \qquad \text{for all } q_0(n) \leq q' \leq q_3(n).$$

Proof. Construct all graphs in $\mathcal{D}(n, q', \alpha, \epsilon, \delta)$ as follows. First decide whether to construct G or $\bar{G}$ (2 ways), choose the sets A_1, A_2, A_3 (less than $\binom{n}{q'}^3$ ways), choose edges inside $A_1 \cup A_2 \cup A_3$ (less than $2^{9q'q'}$ ways) and an appropriate graph on $V_n \setminus (A_1 \cup A_2 \cup A_3)$ (at most $|\mathcal{F}orb_{n-3q'}^*(C_5)|$ ways). Then choose the set S_1 and the sets B_{ij} (less than $2^n \cdot n^n$ ways). Finally, connect $A_1 \cup A_2 \cup A_3$ to the remaining vertices. To do this observe that if $\tilde{A}$ is an arbitrary subset of size $|\tilde{A}| = q'$ of $S_1 \setminus (A_1 \cup A_2 \cup A_3)$ the number of ways to connect a set A_i with $\tilde{A}$ is by Lemma 8.2.11 bounded by $2^{(4\epsilon + f(\delta))q'q'}$ — since no ϵ-uniform partition of $(A_i, \tilde{A})$ may contain a (ϵ, δ)-regular pair. Hence, the number of ways to connect $A_1 \cup A_2 \cup A_3$ to S_1 is bounded by

$$2^{3(4\epsilon + f(\delta))q'n}.$$

By Lemma 8.2.16 the number of ways to connect $A_1 \cup A_2 \cup A_3$ to $\bigcup_{i,j} B_{ij}$ is bounded by

$$2^{(6 - \frac{1}{6})q' \frac{1}{2} \sum_{i,j} |B_{ij}|} \leq 2^{875\alpha q'n}.$$

Finally, the number of ways to connect $A_1 \cup A_2 \cup A_3$ to $V_n \setminus (S_1 \cup \bigcup_{i,j} B_{ij})$ is bounded by

$$2^{3q'(\frac{1}{2} + 8\alpha - 300\alpha)n} \leq 2^{\frac{3}{2}q'n - 876\alpha q'n}.$$

Together this gives

$$\begin{aligned}
\log \frac{|\mathcal{D}(n, q', \alpha, \epsilon, \delta)|}{|\mathcal{F}orb_{n-3q'}^*(C_5)|} &\leq 1 + 3q' \log n + 9q'q' + n + n \log n \\
&\quad + 3(4\epsilon + f(\delta))q'n + 875\alpha q'n + \frac{3}{2}q'n - 876\alpha q'n \\
&\leq 3(\frac{1}{2} - \frac{\alpha}{5})q'n,
\end{aligned}$$

for n sufficiently large, independently of $q_0(n) \leq q' \leq q_3(n)$. $\square$

Proof of Lemma 8.2.9. Without loss of generality assume that ϵ is sufficiently small so that Lemma 8.2.8 holds and that $6\delta + f(3\epsilon) < \frac{1}{100}$ and $4\epsilon + f(\delta) \leq \frac{\alpha}{15}$. Set $\mathcal{Y}_{11}(n, q') = \mathcal{D}(n, q', \alpha, \epsilon, \delta)$. Lemma 8.2.9 is then an immediate corollary of Lemma 8.2.8 and Lemma 8.2.17. (To see this observe that if S_i contains a (μ, δ)-dense pair (A_1, A_2) and a (μ, δ)-sparse pair (B_1, B_2), where μ is some constant such that $\frac{1-\epsilon}{m(\epsilon)} \cdot \epsilon \geq \mu$ and $\mu \leq \delta$, then every ϵ-uniform refinement of $(A_1, A_2; B_1, B_2)$ gives rise to a $(\epsilon, 2\delta, P_2)$- or a $(\epsilon, 2\delta, \bar{P}_2)$-subgraph.) □

8.2.3.3 Some auxiliary lemmas

Proposition 8.2.18 *Let $\epsilon, \delta, \epsilon' > 0$ be constants such that $(1-\epsilon)^3(1-\delta) > 1-\epsilon'$ and let (A, B) be a bipartite graph such that $d(A, B) < 1-\epsilon'$. Then every ϵ-uniform partition of (A, B) contains an (ϵ, δ)-regular pair or an (ϵ, δ)-sparse pair.*

Proof. Assume that there exists an ϵ-uniform partition of (A, B) into $2(k+1)$ parts such that all ϵ-regular pairs are (ϵ, δ)-dense. Then obviously

$$e(A, B) > (1-\epsilon)k^2 \left((1-\delta)\frac{(1-\epsilon)|A|}{k}\frac{(1-\epsilon)|B|}{k}\right) > (1-\epsilon')|A||B|,$$

which is a contradiction. □

A similar result holds for bipartite graphs which are not too sparse.

Proposition 8.2.19 *Let $\epsilon, \delta, \epsilon' > 0$ be constants such that $3\epsilon + \delta < \epsilon'$ and let (A, B) be a bipartite graph such that $d(A, B) > \epsilon'$. Then every ϵ-uniform partition of (A, B) contains an (ϵ, δ)-regular pair or an (ϵ, δ)-dense pair.*

□

Lemma 8.2.20 *Let $0 < \nu < \epsilon' < 1$ be constants. There exist constants $\epsilon, \mu > 0$ and $m_0, n_0 \in \mathbb{N}$ such that the following is true: If $G = (V, E)$ is a graph and $V = A \cup B \cup C$ is a partition of the vertex set with $n_0 \leq n = |A| \leq |B|, |C|$ such that G contains neither a (μ, ν)-regular pair of size at least $\frac{n}{m_0}$ nor a $(\mu, 2\nu, P_2)$-subgraph of size at least $\frac{n}{m_0}$ and*

$$|\Gamma(v) \cap A| \geq (1-\epsilon)n \qquad \textit{for all } v \in B \cup C$$

then

$$d(B, C) > 1-\epsilon'.$$

Proof. Choose constants $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3, \mu > 0$ such that the following inequalities hold:

$$1-\epsilon' < (1-\epsilon_1)^3(1-\nu) \quad \text{and} \quad \epsilon_1 \leq \mu \tag{8.2.1}$$

$$3\epsilon_2 + \nu < 1 - \epsilon \cdot \frac{m(\epsilon_1)}{1-\epsilon_1} \quad \text{and} \quad \epsilon_2 \leq \mu \tag{8.2.2}$$

$$3\epsilon_3 + \nu < 1 - \epsilon \cdot \frac{m(\epsilon_1)}{1-\epsilon_1} \cdot \frac{m(\epsilon_2)}{1-\epsilon_2} \quad \text{and} \quad \epsilon_3 \leq \mu \tag{8.2.3}$$

$$\frac{1-\epsilon_2}{m(\epsilon_2)} \cdot \frac{1-\epsilon_3}{m(\epsilon_3)} \cdot \mu > \epsilon_1 \quad \text{and} \quad 2\nu \geq \nu + \epsilon_1 \tag{8.2.4}$$

$$\frac{1-\epsilon_3}{m(\epsilon_3)} \cdot \mu > \epsilon_2 \quad \text{and} \quad 1 - 2\nu \leq 1 - \nu - \epsilon_2 \tag{8.2.5}$$

$$\mu \geq \epsilon_3. \tag{8.2.6}$$

To see that this can be done, choose for example $\mu = \epsilon_3$ with $3\epsilon_3 + \nu < 1$ arbitrarily. Then choose ϵ_2 and ϵ_1 such that (8.2.5), (8.2.4) and (8.2.1) hold. Next choose ϵ such that (8.2.2) and (8.2.3) hold. We claim that for these ϵ, ϵ', μ and ν together with $\frac{1}{m_0} = \frac{1-\epsilon_1}{m(\epsilon_1)} \frac{1-\epsilon_2}{m(\epsilon_2)} \frac{1-\epsilon_3}{m(\epsilon_3)}$ and n_0 sufficiently large the lemma holds. Assume the contrary, i.e. assume that $G = (V, E)$ and $V = A \cup B \cup C$ satisfy the assumptions of the lemma, but $d(B, C) \leq 1 - \epsilon'$. Without loss of generality we may assume that $n = |A| = |B| = |C|$. By Proposition 8.2.18 and assumption (8.2.1) every ϵ_1-uniform partition of (B, C) contains a (ϵ_1, ν)-sparse pair (B', C'). Note that $|B'| = |C'| \geq \frac{1-\epsilon_1}{m(\epsilon_1)} n$. Consider an arbitrary subset $A' \subseteq A$ of size $|B'|$. By assumption we know that $d(B', A') \geq \frac{|B'|(|A'|-\epsilon n)}{|B'||A'|} \geq 1 - \epsilon \frac{m(\epsilon_1)}{1-\epsilon_1}$. Hence Proposition 8.2.19 and assumption (8.2.2) guarantee the existence of subsets $A'' \subseteq A'$ and $B'' \subseteq B'$ of size $|A''| = |B''| \geq \frac{1-\epsilon_2}{m(\epsilon_2)} |A'|$ such that (A'', B'') is (ϵ_2, ν)-dense. Let C'' be an arbitrary subset of C' of size $|C''| = |A''|$. Then $d(C'', A'') \geq \frac{|C''|(|A''|-\epsilon n)}{|C''||A''|} \geq 1 - \epsilon \frac{m(\epsilon_1)}{1-\epsilon_1} \frac{m(\epsilon_2)}{1-\epsilon_2}$ and hence again by Proposition 8.2.19 and assumption (8.2.3) there exist subsets $A''' \subseteq A''$ and $C''' \subseteq C''$ of size $|A'''| = |C'''| \geq \frac{1-\epsilon_3}{m(\epsilon_3)} |A''|$ such that (A''', C''') is (ϵ_3, ν)-dense. Now observe that (8.2.4)-(8.2.6) imply that for every subset B''' of B'' of size $|B'''| = |A'''|$ the sets A''', B''' and C''' form a $(\mu, 2\nu, P_2)$-subgraph, which is a contradiction. □

Corollary 8.2.21 *Let $0 < \nu < \epsilon' < 1$ be constants. There exist constants $\epsilon, \mu > 0$ and $m_0, n_0 \in \mathbb{N}$ such that the following is true: If $G = (V, E)$ is a graph and $V = A \cup B \cup C$ is a partition of the vertex set with $n_0 \leq n = |A| \leq |B|, |C|$ such that G contains neither a (μ, ν)-regular pair of size at least $\frac{n}{m_0}$ nor a $(\mu, 2\nu, P_2)$-subgraph of size at least $\frac{n}{m_0}$ then*

i) $|\Gamma(v) \cap A| \geq (1 - \epsilon)n$ for all $v \in B \cup C$ implies $d(B, C) > 1 - \epsilon'$, and

ii) $|\Gamma(v) \cap A| \geq (1 - \epsilon)n$ for all $v \in B$ and $|\Gamma(v) \cap A| \leq \epsilon n$ for all $v \in C$ implies $d(B, C) < \epsilon'$.

Proof. As in the proof of Lemma 8.2.20 it can be shown that every system of constants that satisfy (8.2.4)–(8.2.6) and

$$3\epsilon_1 + \nu < \epsilon' \quad \text{and} \quad \epsilon_1 \leq \mu \tag{8.2.7}$$

$$3\epsilon_2 + \nu < 1 - \epsilon \cdot \frac{m(\epsilon_1)}{1-\epsilon_1} \quad \text{and} \quad \epsilon_2 \leq \mu \tag{8.2.8}$$

$$\epsilon \cdot \frac{m(\epsilon_1)}{1-\epsilon_1} \cdot \frac{m(\epsilon_2)}{1-\epsilon_2} < (1-\epsilon_3)^3(1-\nu) \quad \text{and} \quad \epsilon_3 \leq \mu \tag{8.2.9}$$

implies the validity of ii) for every graph G satisfying the assumptions of the corollary. To conclude the proof it suffices to observe that (8.2.1)–(8.2.9) can be fulfilled simultaneously. □

8.2.3.4 Proof of Theorem 8.2.2

Roughly speaking, the remaining part of the proof of Theorem 8.2.2 runs as follows. First we choose appropriate constants $\alpha, \delta_1, \delta_2$ and ϵ and define $\mathcal{Z}(n)$ to be the set of all those C_5-free graphs which do not lie in any of the 22 $\mathcal{Y}_i$-sets given by Lemma 8.2.9 with respect to α, δ_1 and ϵ and α, δ_2 and ϵ, respectively. Secondly we define additional exceptional sets $\mathcal{E}(n)$, $\mathcal{F}(n)$ $\mathcal{G}(n)$, $\mathcal{H}(n)$, and $\mathcal{I}(n)$ and show that they are asymptotically negligible, i.e. satisfy condition iii) of Theorem 8.2.6. In a final step we then show that all graphs in $\mathcal{Z}(n)\setminus(\mathcal{E}(n)\cup\mathcal{F}(n)\cup\mathcal{G}(n)\cup\mathcal{H}(n)\cup\mathcal{I}(n))$ are generalized split graphs, i.e. that condition i) of Theorem 8.2.6 is fulfilled with respect to the 22 $\mathcal{Y}_i$-sets and the $\mathcal{E}$-, $\mathcal{F}$-, $\mathcal{G}$-, $\mathcal{H}$-, and $\mathcal{I}$-sets. This then clearly concludes the proof of Theorem 8.2.2.

Let $\alpha = \frac{1}{17}10^{-10}, \epsilon' = 10^{-9}$ and ν be constants such that $\nu < \epsilon'$ and $\frac{5}{2}\nu + f(\nu) \le \frac{\alpha}{15}$. Let ϵ_0 and μ_0 be the constants guaranteed by Corollary 8.2.21 with respect to ν and ϵ'. Choose constants $\mu \le \epsilon \le \delta_2 \le \delta_1 \le \frac{1}{2}$ such that Lemma 8.2.9 can be applied to α, δ_1 and ϵ and to α, δ_2 and ϵ, respectively and such that the following inequalities hold simultaneously (notice that some of these inequalities, which will be used further on, are redundant):

$$\delta_1 = \nu \tag{8.2.10}$$
$$\epsilon \le \mu_0 \tag{8.2.11}$$
$$4\epsilon + 2\delta_2 \le \tfrac{1}{2}\epsilon' \tag{8.2.12}$$
$$6\mu + 2\delta_2 < \tfrac{1}{2}10^{-10}\epsilon' \tag{8.2.13}$$
$$3\epsilon + \delta_2 \le \tfrac{1}{2}10^{-10}\epsilon' \tag{8.2.14}$$
$$\epsilon + \delta_2 < \epsilon_0 \tag{8.2.15}$$
$$\mu \le \tfrac{1-\epsilon}{m(\epsilon)}\epsilon. \tag{8.2.16}$$

Define

$$\mathcal{Z}(n) = \mathcal{F}orb_n^*(C_5) \setminus \left(\bigcup_{i=1}^{11} \bigcup_{q=q_0(n)}^{q_3(n)} \mathcal{Y}_i(n,q) \cup \mathcal{Y}_i'(n,q) \right),$$

where the $\mathcal{Y}_i$- and $\mathcal{Y}_i'$-sets are the sets of Lemma 8.2.9 with respect to α, δ_1, ϵ and $\alpha, \delta_2, \epsilon$, respectively.

By Lemma 8.2.9 there exist constants μ_1', μ_2', ν' with respect to α, δ_1 and ϵ and constants μ_1'', μ_2'', ν'' with respect to α, δ_2 and ϵ. Let $\kappa = \min\{\mu_1', \mu_1''\}$ and $\lambda = \max\{\nu', \nu''\}$. Then the following is true: If $G = (V, E)$ is a graph in $\mathcal{Z}(n)$ then either in G or in $\bar{G}$ every (κ, λ)-regular pair of q-sets (Q_1, Q_2) has disjoint s-sets S_1 and S_2 such that

- $|S_1|, |S_2| \ge (\frac{1}{2} - 17\alpha)n = (\frac{1}{2} - 10^{-10})n$,
- S_1 and S_2 do not contain an (ϵ, δ_2)-regular pair (and thus no (ϵ, δ_1)-regular pair),
- the number of ways to choose an S_i set is bounded from above by $\sum_{i=0}^{18\alpha n} \binom{(\frac{1}{2}+\alpha)n}{i} \le 2^{f(36\alpha)(\frac{1}{2}+\alpha)n} \le 2^{4\cdot 10^{-9}n}$, for n sufficiently large,
- S_1 does not contain an $(\epsilon, 2\delta_1, P_2)$-subgraph (and thus no $(\epsilon, 2\delta_2, P_2)$-subgraph),

· If B_{ij}, $i = 1, 2, j = 1, \ldots, l$ are pairwise disjoint q-sets in S_2 such that $|B_{1j}| = |B_{2j}|$ and (B_{1j}, B_{2j}) is (ϵ, δ_1)-sparse for all $j = 1, \ldots, l$ then $\sum_{i,j} |B_{ij}| \leq 300\alpha n \leq \frac{300}{17} 10^{-10} n \leq 2 \cdot 10^{-9} n$.

In the following we call such sets S_1 and S_2 s_1-set and s_2-set, respectively.

Now partition $\mathcal{Z}(n)$ into parts $\mathcal{W}(n)$ and $\mathcal{Z}(n) \setminus \mathcal{W}(n)$ such that $\mathcal{W}(n)$ consists of those graphs G in $\mathcal{Z}(n)$ where G has the above properties. Thus for all graphs G in $\mathcal{Z}(n) \setminus \mathcal{W}(n)$ the complement $\bar{G}$ has the above properties. In the following we consider without loss of generality only the class $\mathcal{W}(n)$.

The next two lemmas show that in S_1 every (μ, δ_2)-dense pair of q-sets (A, B) induces a set $C'(A, B)$ which is in some sense "close" to a clique.

Lemma 8.2.22 *Let $G = (V_n, E)$ be a graph in $\mathcal{W}(n)$, let $S_1 \subseteq V_n$ be an s_1-set in G and let (A, B) be a (μ, δ_2)-dense pair of q-sets with $|A| = |B|$ contained in S_1. Then for n sufficiently large the following is true:*

The sets

$$C = A \cup B \cup \{v \in S_1 \setminus (A \cup B) \mid |\Gamma(v) \cap (A \cup B)| \geq (2 - 4\epsilon - 2\delta_2)|A|\}$$

and

$$C' = C'(A, B) = \{v \in C \mid |C \setminus \Gamma(v)| \leq \epsilon' n \text{ and } |\Gamma(v) \cap (S_1 \setminus C)| \leq \epsilon' n\}$$

satisfy

$$|(A \cup B) \cap C'| \geq \frac{1}{4}|A|.$$

Proof. Let $q = |A| = |B|$ and let X denote the set $X = \{v \in S_1 \mid (4\epsilon + 2\delta_2)q \leq |\Gamma(v) \cap (A \cup B)| \leq (2 - 4\epsilon - 2\delta_2)q\}$. We will show

$(*)$ $|X| \leq q$.

By bounding $e(A \cup B, C \setminus (A \cup B))$ from above and from below, and by making use of (8.2.12), we see that the definition of C together with $(*)$ implies that

$$|\{v \in A \cup B \mid |C \setminus \Gamma(v)| \geq \epsilon'(|C| - 2q)\}| \leq \frac{4\epsilon + 2\delta_2}{\epsilon'} q \leq \frac{1}{2} q$$

and

$$|\{v \in A \cup B \mid |\Gamma(v) \cap (S_1 \setminus (C \cup X))| \geq \frac{\epsilon'}{2} |S_1 \setminus (C \cup X)|\}| \leq \frac{4\epsilon + 2\delta_2}{\frac{\epsilon'}{2}} q \leq q,$$

which completes the proof of the lemma.

Proof of $(*)$. Let n be sufficiently large and assume there exists a subset $X' \subseteq X$ of size q. Consider an ϵ-uniform partition of $(A \cup B, X')$ into $3(k+1)$ parts, which is an ϵ-uniform refinement of $(A, B; X')$. There certainly exists a X_{i_0} which is contained in at most $2\epsilon k$ many ϵ-irregular pairs. Since by definition of $\mathcal{W}(n)$ none of the pairs is (ϵ, δ_2)-regular, there exists A_{j_0} such that (A_{j_0}, X_{i_0}) is either (ϵ, δ_2)-dense or (ϵ, δ_2)-sparse. We consider only the case that (A_{j_0}, X_{i_0}) is (ϵ, δ_2)-dense. The other case can be treated similarly. If there exists a B_{j_1} such that (B_{j_1}, X_{i_0}) is (ϵ, δ_2)-sparse, A_{j_0}, B_{j_1} and X_{i_0} form a $(\epsilon, 2\delta_2, P_2)$-subgraph (cf. assumption (8.2.16)), which is a contradiction. Otherwise all pairs (B_j, X_{i_0}) are (ϵ, δ_2)-dense or ϵ-irregular. In particular at least one of the pairs, say (B_{j_2}, X_{i_0}), would be (ϵ, δ_2)-dense. Repeating the argument from above we conclude that all pairs (A_j, X_{i_0}) and (B_j, X_{i_0}) have to be (ϵ, δ_2)-dense or ϵ-irregular. Hence,

$$|\Gamma(X_{i_0}) \cap (A \cup B)| \geq (2k - 2\epsilon k)(1 - \delta_2)|X_{i_0}||A_j| \geq 2(1-\epsilon)^2(1-\delta_2)|X_{i_0}|q,$$

contradicting the definition of $X' \subseteq X$. □

Lemma 8.2.23 *Let $G = (V_n, E)$ be a graph in $\mathcal{W}(n)$, let $S_1 \subseteq V_n$ be an s_1-set in G and let (A_i, B_i) be (μ, δ_2)-dense pairs of q-sets of size at least $q_2(n)$ contained in S_1 in such a way that the set $\bigcup_i C'(A_i, B_i)$ is maximal. (Here $C'(A_i, B_i)$ denotes the set defined in Lemma 8.2.22.) Then the sets $S_1^C := \bigcup_i C'(A_i, B_i)$ and $S_1^R := S_1 \setminus S_1^C$ satisfy:*

All but at most $10^{-10}n$ vertices of S_1^R have degree at most $\epsilon' n$ in S_1.

Proof. Without loss of generality we may assume that $|S_1^R| \geq 10^{-10}n$. We will show:

i) all but at most $\frac{1}{2}10^{-10}n$ vertices of S_1^R have degree at most $\frac{\epsilon'}{2}|S_1^R|$ in S_1^R, and

ii) if $|S_1^C| \geq q_3(n)$ then all but at most $\frac{1}{2}10^{-10}n$ vertices of S_1^R have degree at most $\epsilon'|S_1^C|$ in S_1^C.

Clearly, i) and ii) imply the statement of the lemma.

To see i) observe that S_1^R may neither contain a (μ, δ_2)-dense pair of size at least $q_2(n)$ (cf. the assumption on the maximality of $\bigcup_i C'(A_i, B_i)$ and Lemma 8.2.22) nor a (μ, δ_2)-regular pair (cf. the definition of S_1). Therefore for n sufficiently large we can easily bound the number of edges in S_1^R:

$$|E(G[S_1^R])| \leq (3\mu + \delta_2)\binom{|S_1^R|}{2}.$$

This now immediately implies that the number of vertices in S_1^R with degree at least $\frac{\epsilon'}{2}|S_1^R|$ in S_1^R is bounded from above by $\frac{3\mu+\delta_2}{\frac{\epsilon'}{2}}|S_1^R|$, which implies i) by the choice of the constants (cf. assumption (8.2.13)).

To see ii) observe that if $|S_1^C| \geq q_3(n)$ then for all sufficiently large n there either exists a set $C \subseteq S_1^C$ of size $\frac{1}{4}\tilde{k}q_2(n)$ and a subset $D \subseteq S_1^R$ of size $\tilde{k}q_2(n)$ for some constant $\tilde{k}$ which will be determined later such that

$$d(C, D) > \frac{1}{2}10^{-10}\epsilon'$$

or we have

$$e(S_1^R, S_1^C) \quad \leq \quad \frac{1}{2}10^{-10}\epsilon'|S_1^R||S_1^C|$$

and therefore

$$|\{v \in S_1^R \mid |\Gamma(v) \cap S_1^C| > \epsilon'|S_1^C|\}| \leq \frac{\frac{1}{2}10^{-10}\epsilon'|S_1^R||S_1^C|}{\epsilon'|S_1^C|} = \frac{1}{2}10^{-10}|S_1^R|.$$

So the only thing left is to show that the existence of a set C and a set D leads to a contradiction. Apply Proposition 8.2.19 (recall that by assumption (8.2.14) $3\epsilon + \delta_2 \leq \frac{1}{2}10^{-10}\epsilon'$) to (C, D) to obtain an (ϵ, δ_2)-dense pair $(\tilde{C}, \tilde{D})$ with $|\tilde{C}| \geq \frac{1-\epsilon}{m(\epsilon)}|C|$ and $|\tilde{D}| \geq \frac{1-\epsilon}{m(\epsilon)}|D|$. Hence (cf. assumption (8.2.15))

$$D' = \{v \in \tilde{D} \mid |\Gamma(v) \cap \tilde{C}| \geq (1 - \epsilon_0)|\tilde{C}|\}$$

satisfies

$$|D'| \geq (1 - \epsilon)|\tilde{D}| \geq \frac{1}{2}|\tilde{D}|.$$

Split D' into two equally large sets D'_1 and D'_2. By Corollary 8.2.21 we have $d(D'_1, D'_2) > 1 - \epsilon'$. Hence (D'_1, D'_2) contains by Proposition 8.2.19 a (μ, δ_2)-dense pair (D''_1, D''_2) of size

$$|D''_i| \geq \frac{1-\mu}{m(\mu)} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1-\epsilon}{m(\epsilon)}\tilde{k}q_2(n) \geq q_2(n),$$

for an appropriate chosen constant $\tilde{k}$. This concludes the proof of ii). $\square$

The next lemma essentially shows that almost all C_5-free graphs have the property that any four vertices that are contained in some "almost clique" have a large neighborhood or nonneighborhood in S_2.

Lemma 8.2.24 *Let $\mathcal{E}(n)$ denote the set of all graphs $G = (V_n, E)$ in $\mathcal{W}(n)$ that contain a (κ, λ)-regular pair of q-sets (Q_1, Q_2) and associated s-sets S_1 and S_2 such that there exist a vertex $v \in V_n$, a set $X \subseteq V$, four vertices $v_1, v_2, v_3, v_4 \in S_1$ and associated (μ, δ_2)-dense pairs (A_i, B_i) of q-sets so that the following conditions hold:*

i) $|X| \geq 10^{-5}n$ *and* $X = S_2$, $X = \Gamma(v) \cap S_2$ *or* $X = S_2 \setminus \Gamma(v)$,

ii) *for all* $1 \leq i \leq 4$ *one has that* $v_i \in C'(A_i, B_i)$ *or* $|\Gamma(v_i) \cap S_1| \leq \epsilon' n$,

iii) *the set*

$$N = \{u \in X \mid u \in \Gamma(v_1) \cap \Gamma(v_2) \text{ and } u \notin \Gamma(v_3) \cup \Gamma(v_4)\}$$

satisfies

$$|N| \leq 10^{-8}n.$$

Then for n sufficiently large

$$\log \frac{|\mathcal{E}(n)|}{|\mathcal{F}orb^*_{n-4}(C_5)|} \leq 4(\frac{1}{2} - 10^{-8})n.$$

Proof. Construct all graphs in $\mathcal{E}(n)$ as follows. First choose the vertices v_1, v_2, v_3, v_4 and v (less than n^5 ways), choose an appropriate graph on $V_n \setminus \{v_1, v_2, v_3, v_4\}$ (at most $|\mathcal{F}orb^*_{n-4}(C_5)|$ ways). Then choose appropriate q-sets Q_1 and Q_2 and s-sets S_1 and S_2 (less than $\binom{n}{q_3(n)}^2 \cdot 2^{8\cdot 10^{-9}n}$ ways). Finally, connect the v_i to the remaining vertices. To do this choose first the sets A_i and B_i (at most $\binom{n}{q_3(n)}^8$ ways) and connect the v_i to S_1 (at most $2^{8f(\epsilon')n} \leq 2^{3\cdot 10^{-7}n}$ ways, cf. the definition of the sets $C'(A_i, B_i)$). Next decide whether X equals S_2, $S_2 \cap \Gamma(v)$ or $S_2 \setminus \Gamma(v)$ (3 choices), choose the set N and connect the v_i to $X \setminus N$ (at most $10^{-8}n\binom{n}{10^{-8}n} \cdot 15^{|X|-|N|} \leq 2^{3\cdot 10^{-7}n+(4-\frac{1}{11})|X|}$ ways). Finally, connect the v_i to $V_n \setminus (S_1 \cup X)$ (at most $2^{4(n-|S_1|-|X|)} \leq 2^{4((\frac{1}{2}+10^{-10})n-|X|)}$ ways). Together this gives

$$\begin{aligned}\log \frac{|\mathcal{E}(n)|}{|\mathcal{F}orb^*_{n-4}(C_5)|} &\leq 5\log n + 2q_3(n)\log n + 8\cdot 10^{-9}n + 8q_3(n)\log n + \log 3 \\ &\quad + 6\cdot 10^{-7}n + (4-\frac{1}{11})|X| + 4((\frac{1}{2}+10^{-10})n - |X|) \\ &\leq 4(\frac{1}{2} - 10^{-8})n,\end{aligned}$$

for n sufficiently large. □

Lemma 8.2.25 *Let $\mathcal{F}(n)$ denote the set of all graphs $G = (V_n, E)$ in $\mathcal{W}(n)$ that contain a (κ, λ)-regular pair of q-sets (Q_1, Q_2) and associated s-sets S_1 and S_2 such that the following properties are fulfilled. There exist three vertices $v_1, v_2, v_3 \in V_n$, a set $W \subseteq S_1$ of size $|W| = 10^{-10}n$ such that $G[S_1 \setminus W]$ forms a partition into pairwise disjoint cliques and for all $j = 1, 2, 3$ either $\Gamma(v_j) \cap (S_1 \setminus W) = \emptyset$ or $\Gamma(v_j)$ induces a maximal clique in $S_1 \setminus W$, and such that the set*

$$N = \{w \in S_2 \mid w \in \Gamma(v_1) \text{ and } w \notin \Gamma(v_2) \cup \Gamma(v_3)\}$$

satisfies

$$|N| \leq 10^{-5}n.$$

Then for n sufficiently large

$$\log \frac{|\mathcal{F}(n)|}{|\mathcal{F}orb^*_{n-3}(C_5)|} \leq 3(\frac{1}{2} - 10^{-4})n.$$

Proof. Construct all graphs in $\mathcal{F}(n)$ as follows. First choose the vertices v_1, v_2, v_3 (less than n^3 ways), choose an appropriate graph on $V_n \setminus \{v_1, v_2, v_3\}$ (at most $|\mathcal{F}orb^*_{n-3}(C_5)|$ ways). Then choose appropriate q-sets Q_1 and Q_2 and s-sets S_1 and S_2 (less than $\binom{n}{q_3(n)}^2 \cdot 2^{8\cdot 10^{-9}n}$ ways). Finally, connect the v_i to the remaining vertices. To do this choose first the set W (at most $2^{f(10^{-10})n} \leq 2^{4\cdot 10^{-9}}$ ways) and connect the v_i to $S_1 \setminus W$ (at most n^3 ways). Then connect the v_i to $W \cup (V_n \setminus (S_1 \cup S_2))$ (at most $2^{3(3\cdot 10^{-10})n} \leq 2^{10^{-9}n}$ ways). Next choose the set N and connect the v_i to $S_2 \setminus N$ (at most $10^{-5}n\binom{n}{10^{-5}n} \cdot 7^{|S_2|-|N|} \leq 2^{2\cdot 10^{-4}n+(3-\frac{1}{10})|S_2|}$ ways).

Together this gives

$$\begin{aligned}\log\frac{|\mathcal{F}(n)|}{|\mathcal{F}orb^*_{n-3}(C_5)|} &\leq 3\log n + 2q_3(n)\log n + 13\cdot 10^{-9}n + 2\cdot 10^{-4}n + (3-\frac{1}{10})|S_2| \\ &\leq 3(\frac{1}{2}-10^{-2})n,\end{aligned}$$

for n sufficiently large. □

Lemma 8.2.26 *Let $\mathcal{G}(n)$ denote the set of all graphs $G=(V_n,E)$ in $\mathcal{W}(n)$ that contain a (κ,λ)-regular pair of q-sets (Q_1,Q_2) and associated s-sets S_1 and S_2 such that there exist three vertices $a_1,a_2,a_3\in V_n$ with*

$$|\Gamma(a_j)\cap S_2|\leq 10^{-5}|S_2| \quad or \quad |\Gamma(a_j)\cap S_2|\geq (1-10^{-5})|S_2| \qquad for\ all\ j=1,2,3$$

so that at least one of the following two conditions is satisfied:

i) $\{a_1,a_2\},\{a_1,a_3\}\in E$, $\{a_2,a_3\}\notin E$ and $V_n\setminus S_2$ contains $\frac{1}{26}n$ pairwise disjoint edges,

ii) $\{a_1,a_2\},\{a_1,a_3\}\notin E$, $\{a_2,a_3\}\in E$ and $V_n\setminus S_2$ contains $\frac{1}{26}n$ pairwise disjoint non-edges.

Then for n sufficiently large

$$\log\frac{|\mathcal{G}(n)|}{|\mathcal{F}orb^*_{n-3}(C_5)|}\leq 3(\frac{1}{2}-10^{-4})n.$$

Proof. Construct all graphs in $\mathcal{G}(n)$ as follows. First choose the vertices a_1,a_2 and a_3 (less than n^3 ways) and choose an appropriate graph on $V_n\setminus\{a_1,a_2,a_3\}$ (at most $|\mathcal{F}orb^*_{n-3}(C_5)|$ ways). Then choose appropriate q-sets Q_1 and Q_2 and s-sets S_1 and S_2 (less than $\binom{n}{q_3(n)}^2\cdot 2^{8\cdot 10^{-9}n}$ ways). Finally, connect a_1,a_2,a_3 to the remaining vertices. To do this observe first that there are by assumption at most $2^{3(1+f(10^{-5})n)}\leq 2^{6\cdot 10^{-4}n}$ ways to connect $\{a_1,a_2,a_3\}$ to S_2. Next observe that there are only $2^6-2\leq 2^{3(2-\frac{1}{70})}$ ways to connect a P_2 (a $\bar{P}_2$) to an edge (nonedge). Hence, if condition i) or ii) is satisfied there are at most $2^{3(2-\frac{1}{70})\frac{1}{26}n}\cdot 2^{3(n-|S_2|-2\frac{1}{26}n)}\leq 2^{(\frac{3}{2}-10^{-3})n}$ ways to connect a_1,a_2,a_3 to $V_n\setminus S_2$.
Together this gives

$$\begin{aligned}\log\frac{|\mathcal{G}(n)|}{|\mathcal{F}orb^*_{n-3}(C_5)|} &\leq 3\log n + 2q_3(n)\log n + 8\cdot 10^{-9}n + 6\cdot 10^{-4}n \\ &\quad + 1 + (\frac{3}{2}-10^{-3})n \\ &\leq 3(\frac{1}{2}-10^{-4})n,\end{aligned}$$

for n sufficiently large. □

Let $G=(V,E)$ be a graph. A set $W\subseteq V$ is called *homogeneous*, iff W induces in G either a clique or a stable set.

Let $\mathcal{H}(n)$ denote the set of all graphs $G = (V_n, E)$ in $\mathcal{F}orb^*_n(C_5)$ for which the vertex set V_n can be partitioned into two sets $V_n = V_1 \cup V_2$ such that V_1 as well as V_2 induce either in G or in $\bar{G}$ a partition into pairwise disjoint cliques and at least one of the following three conditions is satisfied:

(1) $|V_1| \geq (\frac{1}{2} + 10^{-10})n$,

(2) V_1 contains a homogeneous set of size at least $\frac{1}{26}n$ *and* V_2 contains a homogeneous set of size at least $(\frac{1}{2} - 10^{-4})n$,

(3) either in G or in $\bar{G}$ the set V_1 contains $10^{-10}n$ pairwise disjoint edges *and* V_2 contains $10^{-10}n$ pairwise disjoint induced P_2's.

Lemma 8.2.27 *For all sufficiently large n:*

$$\log \frac{|\mathcal{H}(n)|}{|\mathcal{F}orb^*_{n-1}(C_5)|} \leq -\frac{1}{30} n \log n.$$

Proof. Let $\mathcal{H}_1(n)$ denote the set of all graphs $G = (V_n, E)$ in $\mathcal{H}(n)$ for which condition (1) is true, let $\mathcal{H}_2(n)$ denote the set of all graphs $G = (V_n, E)$ in $\mathcal{H}(n)$ for which condition (1) is not true, but condition (2) is true, and let $\mathcal{H}_3(n)$ denote the set $\mathcal{H}_3(n) = \mathcal{H}(n) \setminus (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$.
The number of graphs in $\mathcal{H}_1(n)$ is easily bounded as follows:

$$\begin{aligned} |\mathcal{H}_1(n)| &\leq 4 \sum_{i=0}^{(\frac{1}{2}-10^{-10})n} \binom{n}{i} i^i (n-i)^{n-i} 2^{i\cdot(n-i)} \leq 4n2^n n^{2n} 2^{(\frac{1}{2}-10^{-10})(\frac{1}{2}+10^{-10})n^2} \\ &\leq 2^{\frac{1}{4}n^2}, \end{aligned}$$

for n sufficiently large.
Secondly, construct all graphs in $\mathcal{H}_2(n)$ as follows. First choose a partition $V = V_1 \cup V_2$ (2^n ways) and decide for both V_1 and V_2 whether to construct $G[V_i]$ or $\overline{G}[V_i]$, respectively (4 ways). To construct the partition of V_1 observe that the condition that V_1 should contain a homogeneous set of size at least $\frac{1}{26}n$ implies that we have to partition V_1 in such a way that either one class has size at least $\frac{1}{26}n$ or we have at least $\frac{1}{26}n$ classes. Hence, the number of ways to partition V_1 is for n sufficiently large bounded by (here B_n denotes the n-th Bell number)

$$\sum_{i=\frac{1}{26}n}^{|V_1|} \binom{|V_1|}{i} B_{|V_1|-i} + \sum_{i=\frac{1}{26}n}^{|V_1|} \binom{|V_1|}{i} i^{|V_1|-i} \leq 2n2^n \cdot 2^{(|V_1|-\frac{1}{26}n)\log n} \leq 2^{(\frac{1}{2}-\frac{1}{27})n\log n}.$$

Similarly, the number of ways to partition V_2 is bounded by

$$2^{2\cdot 10^{-4} n \log n}.$$

Finally, connect V_1 and V_2 (at most $2^{\frac{n^2}{4}}$ ways). Hence, for n sufficiently large

$$|\mathcal{H}_2(n)| \leq 2^{\frac{n^2}{4} + \frac{n}{2}\log n - \frac{1}{28}n\log n}.$$

Thirdly, construct all graphs in $\mathcal{H}_3(n)$ as follows. First choose a partition $V = V_1 \cup V_2$ (2^n ways), decide for both V_1 and V_2 whether to construct $G[V_i]$ or $\overline{G}[V_i]$, respectively (4 ways), and partition V_1 and V_2 into cliques (less than n^n ways). Let X_i denote the set of pairwise disjoint edges (induced P_2's, respectively) in V_i. Next choose the edges in $(V_1 \times V_2) \setminus (X_1 \times X_2)$ (less than $2^{\frac{1}{4}n^2 - |X_1||X_2|}$ ways). Finally, choose edges in $X_1 \times X_2$: As there are only $2^6 - 2 \le 2^{6(1-\frac{1}{140})}$ ways to connect an edge with a P_2 without generating an induced C_5 this can be done in at most $2^{(1-\frac{1}{140})|X_1||X_2|}$ ways. Together, this gives for n sufficiently large

$$|\mathcal{H}_3(n)| \le 2^n \cdot 4 \cdot n^n \cdot 2^{\frac{1}{4}n^2 - \frac{1}{140}10^{-20}n^2} \le 2^{\frac{1}{4}n^2}.$$

Therefore we have for n sufficiently large

$$|\mathcal{H}(n)| \le |\mathcal{H}_1(n)| + |\mathcal{H}_2(n)| + |\mathcal{H}_3(n)| \le 2^{\frac{n^2}{4} + \frac{n}{2}\log n - \frac{1}{29}n\log n}.$$

On the other hand, $\mathcal{S}(n) \subseteq \mathcal{F}orb_n^*(C_5)$ and hence by Lemma 8.2.4 for all sufficiently large n

$$|\mathcal{F}orb_{n-1}^*(C_5)| \ge |\mathcal{S}(n-1)| \ge 2^{\frac{n^2}{4} + \frac{n}{2}\log n - n\log\ln n}$$

and therefore

$$\begin{aligned} \log \frac{|\mathcal{H}(n)|}{|\mathcal{F}orb_{n-1}^*(C_5)|} &\le \left(\frac{n^2}{4} + \frac{n}{2}\log n - \frac{1}{29}n\log n\right) - \left(\frac{n^2}{4} + \frac{n}{2}\log n - n\log\ln n\right) \\ &\le -\frac{1}{30}n\log n, \end{aligned}$$

for n sufficiently large. $\square$

As a last step we show that almost all C_5-free graphs contain a (κ, λ)-regular pair of q-sets.

Let $\mathcal{I}(n)$ denote the set of all graphs $G = (V_n, E)$ in $\mathcal{F}orb_n^*(C_5)$ which contain no (κ, λ)-regular pair of size between $q_0(n)$ and $q_3(n)$.

Lemma 8.2.28 *For all sufficiently large n:*

$$\log \frac{|\mathcal{I}(n)|}{|\mathcal{F}orb_{n-1}^*(C_5)|} \le -\frac{1}{3}n\log n.$$

Proof. Similarly as in the proof of Lemma 8.2.27 it certainly suffices to show that $|\mathcal{I}(n)| \le 2^{\frac{n^2}{4}}$. To see this construct all graphs in $\mathcal{I}(n)$ as follows. First partition the vertex set V_n into classes of size a_i with $q_0(n) \le a_i \le q_3(n)$ (at most n^n ways). Then choose the edges contained in one class (less than $2^{q_3(n)^2\frac{n}{q_3(n)}}$ ways). Finally, choose the edges between two classes. By Lemma 8.2.11 there are at most $m(\kappa)^{\binom{\frac{n}{q_0(n)}}{2}} \cdot 2^{(4\kappa + f(\lambda))\sum_{i<j} a_i a_j} \le 2^{\frac{n^2}{q_0(n)}} \cdot 2^{(4\kappa+f(\lambda))\binom{n}{2}}$ ways to do this. This concludes the proof of the Lemma. $\square$

Observe that Lemma 8.2.22 - 8.2.26 can also be proved for the class $\mathcal{Z}(n) \setminus \mathcal{W}(n)$ instead of the class $\mathcal{W}(n)$. Let $\mathcal{E}'(n)$, $\mathcal{F}'(n)$ and $\mathcal{G}'(n)$ denote the corresponding sets.

In order to complete the proof of Theorem 8.2.2 by applying Theorem 8.2.6 we only have to verify that the twentytwo $\mathcal{Y}_i$-families together with the $\mathcal{E}$-, $\mathcal{E}'$-, $\mathcal{F}$-, $\mathcal{F}'$-, $\mathcal{G}$-, $\mathcal{H}$-, and $\mathcal{I}$-sets satisfy the conditions of Theorem 8.2.6. As we have already checked that ii) and iii) hold for all these sets, it only remains to be shown that condition i) is satisfied as well. This is done in our last lemma.

Lemma 8.2.29 *For all sufficiently large n:*

$$\mathcal{Z}(n) \setminus (\mathcal{E}(n) \cup \mathcal{E}'(n) \cup \mathcal{F}(n) \cup \mathcal{F}'(n) \cup \mathcal{G}(n) \cup \mathcal{G}'(n) \cup \mathcal{H}(n) \cup \mathcal{I}(n)) \subseteq \mathcal{S}(n).$$

Proof. Let $G = (V_n, E)$ be an arbitrary graph in $\mathcal{Z}(n) \setminus (\mathcal{E}(n) \cup \mathcal{E}'(n) \cup \mathcal{F}(n) \cup \mathcal{F}'(n) \cup \mathcal{G}(n) \cup \mathcal{G}'(n) \cup \mathcal{H}(n) \cup \mathcal{I}(n))$. Since G is not in $\mathcal{I}(n)$ there exists a (κ, λ)-regular pair of q-sets, say (Q_1, Q_2). Let S_1 and S_2 be the s-sets associated with this pair (Q_1, Q_2). Observe that it may be the case that also S_2 forms an s_1-set and S_1 an s_2-set with respect to $\bar{G}$. If this is the case we assume without loss of generality that $G[S_2]$ contains a larger clique than $\bar{G}[S_1]$. We henceforth also assume that S_1 and S_2 form s-sets with respect to G.
Now choose (μ, δ_2)-dense pairs (A_i, B_i) of q-sets of size at least $q_2(n)$ contained in S_1 in such a way that the set $\bigcup_i C'(A_i, B_i)$ is maximum. Note that this set may be empty. Here $C'(A_i, B_i)$ denotes the set defined in Lemma 8.2.22. With respect to these sets A_i and B_i define three sets R_1, R_2 and R_3 as follows:

$$R_1 := \{v \in S_1 \setminus \bigcup_i C'(A_i, B_i) \quad | \quad |\Gamma(v) \cap S_1| \leq \epsilon' n\},$$

$$\begin{aligned} R_2 := \{v \in V_n \setminus (\bigcup_i C'(A_i, B_i) \cup R_1) \quad | \quad & |\Gamma(v) \cap S_2| \geq 10^{-5}|S_2| \text{ and} \\ & |\Gamma(v) \cap S_2| \leq (1 - 10^{-5})|S_2|\} \end{aligned}$$

and

$$\begin{aligned} R_3 := \{v \in V_n \setminus (\bigcup_i C'(A_i, B_i) \cup R_1) \quad | \quad & |\Gamma(v) \cap S_2| < 10^{-5}|S_2| \text{ or} \\ & |\Gamma(v) \cap S_2| > (1 - 10^{-5})|S_2|\}. \end{aligned}$$

As G is not in $\mathcal{E}(n)$ we easily conclude that $\bigcup_i C'(A_i, B_i) \cup R_1$ is a union of pairwise disjoint cliques: it suffices to show that $\bigcup_i C'(A_i, B_i) \cup R_1$ can not contain three vertices a_1, a_2, a_3 such that $\{a_1, a_2\}, \{a_1, a_3\} \in E(G)$ but $\{a_2, a_3\} \notin E(G)$. This however follows from the fact that $G \notin \mathcal{E}(n)$ as S_2 can not contain two sets of size $10^{-8}n$ which are not connected by an edge.

Similarly, the fact that $G \notin \mathcal{E}(n)$ implies that every vertex $v \in R_2$ can be connected to at most one clique in $\bigcup_i C'(A_i, B_i) \cup R_1$, and if v is connected to some vertex of a clique in $\bigcup_i C'(A_i, B_i) \cup R_1$ it has to be connected to all vertices of this clique. As by Lemma 8.2.23

$$(*) \qquad |S_1 \setminus (\bigcup_i C'(A_i, B_i) \cup R_1)| \leq 10^{-10} n$$

the fact that $G \notin \mathcal{F}(n)$ implies that $\bigcup_i C'(A_i, B_i) \cup R_1 \cup R_2$ is a union of pairwise disjoint cliques.

Observe also that $(*)$ implies that $\bigcup_i C'(A_i, B_i) \cup R_1$ contains $\frac{1}{26}n$ pairwise disjoint edges or $\frac{1}{26}n$ pairwise disjoint non-edges. Furthermore, the fact that $G \notin \mathcal{G}(n)$ implies that if $\bigcup_i C'(A_i, B_i) \cup R_1$ contains $\frac{1}{26}n$ pairwise disjoint edges then R_3 forms a union of pairwise disjoint cliques and if $\bigcup_i C'(A_i, B_i) \cup R_1$ contains $\frac{1}{26}n$ pairwise disjoint non-edges then $\overline{R_3}$ forms a union of pairwise disjoint cliques. As $G \notin \mathcal{H}(n)$, this implies that $|\bigcup_i C'(A_i, B_i) \cup R_1 \cup R_2| \leq (\frac{1}{2} + 10^{-10})n$ and hence $|R_3 \cap S_2| \geq |R_3| - |V_n \setminus (S_1 \cup S_2)| - |R_3 \cap S_1| \geq (\frac{1}{2} - 4 \cdot 10^{-10})n$.
If $\bigcup_i C'(A_i, B_i) \cup R_1$ contains $\frac{1}{26}n$ pairwise disjoint edges *and* $\frac{1}{26}n$ pairwise disjoint non-edges then R_3 induces either a clique or a stable set in G. By the assumption on S_2 the later is impossible. Therefore, in this case we have that $G \in \mathcal{S}(n)$.
For the remaining cases we certainly may assume that S_1 contains a homogeneous set of size at least $\frac{1}{26}n$. First assume that R_3 forms a union of pairwise disjoint cliques. Then $|R_3 \cap S_2| \geq (\frac{1}{2} - 4 \cdot 10^{-10})n$ together with the assumption on S_2 implies that R_3 contains a clique of size $(\frac{1}{2} - 10^{-4})n$ which contradicts the fact that $G \notin \mathcal{H}(n)$. So we may assume now that $\overline{R_3}$ forms a union of pairwise disjoint cliques. If R_3 does not contain $10^{-10}n$ pairwise disjoint induced P_2's then (cf. the assumption on S_2) R_3 would contain a clique of size $(\frac{1}{2} - 10^{-4})n$ – again contradicting the fact that $G \notin \mathcal{H}(n)$. Hence, we may assume that R_3 contains $10^{-10}n$ pairwise disjoint induced P_2's. In this case the fact that $G \notin \mathcal{H}(n)$ implies that $\bigcup_i C'(A_i, B_i) \cup R_1 \cup R_2$ contains less than $10^{-10}n$ pairwise disjoint edges and hence S_1 contains a stable set of size at least $(\frac{1}{2} - 4 \cdot 10^{-10})n$. As $R_2 \cap S_2 = \emptyset$ (cf. the assumption on S_2) we can at this point conclude that S_2 and S_1 form also an s_1-set, respectively an s_2-set with respect to $\bar{G}$. By our general assumption in the beginning of the proof this implies that S_2 contains a clique of size at least $(\frac{1}{2} - 4 \cdot 10^{-10})n$, again contradicting the fact that $G \notin \mathcal{H}(n)$. This completes the proof of Lemma 8.2.29. □

As remarked before Lemma 8.2.29, this completes the proof of Theorem 8.2.2.

Appendix A

Some Useful Inequalities

Cauchy-Schwarz inequality.

$$\left(\sum_{k=1}^{n} |a_k b_k|\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right) \cdot \left(\sum_{k=1}^{n} b_k^2\right).$$

Defect form of the Cauchy-Schwarz inequality. Let $m \leq n$ and $\delta \in \mathbb{R}$ such that $\sum_{k=1}^{m} x_k = \frac{m}{n} \sum_{k=1}^{n} x_k + \delta$. Then

$$\sum_{k=1}^{n} x_k^2 \geq \frac{1}{n} \left(\sum_{k=1}^{n} x_k\right)^2 + \frac{\delta^2 n}{m(n-m)}.$$

Stirling's formula. For all sufficiently large n:

$$1 \leq \frac{n!}{(2\pi n)^{\frac{1}{2}} n^n e^{-n}} \leq e^{\frac{1}{12n}}.$$

Markov's inequality. Let X be a random variable such that $X \geq 0$ and $t > 0$ be a real number. Then

$$\mathrm{Prob}\,[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

Chebyshev's inequality. Let X be a random variable and $t > 0$ be a real number. Then

$$\mathrm{Prob}\,[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}.$$

Power series of the logarithm. For all $-1 < x < 1$:

$$\ln(1-x) = -[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots].$$

We close this section with a collection of inequalities which are applied frequently, partly without further reference, throughout this thesis.

- $\ln(1+x) \leq x$ for all $-1 < x < 1$.
- $-\frac{x}{1-x} \leq \ln(1-x) \leq -x$ for all $0 \leq x < 1$.
- $\sum_{i=0}^{k} \binom{a}{i} \cdot \binom{b}{k-i} = \binom{a+b}{k}$ for all $a, b, k \in \mathbb{N}$.
- $\binom{a}{b-x}\binom{c}{d+x} \leq \binom{a}{b}\binom{c}{d}$ for all integers $1 \leq x \leq b \leq a \leq d \leq c$ such that $bc \leq ad$.
- $\binom{a-x}{b} \cdot \binom{a}{b}^{-1} \leq e^{-\frac{xb}{a}}$ for all $a, b, x \geq 0$.
- $\binom{a+x}{b} \cdot \binom{a}{b}^{-1} \leq e^{\frac{xb}{a-b}}$ and $\binom{a+x}{b} \cdot \binom{a}{b}^{-1} \leq e^{x \ln a}$ for all $a, b, x \geq 0$.

Appendix B

Tails of the Binomial Distribution

In dealing with the probabilistic method, one often needs good bounds on large deviations. Of particular importance are bounds for the tails of a binomial distribution, as such distributions arise very naturally for a large variety of combinatorial problems. For small deviations such estimates are given by the well-known De-Moivre-Laplace limit theorem. Here we concentrate on large deviations and give a self-contained and short proof of the Bernstein inequality, often also called CHERNOFF-INEQUALITY, in remembrance of Chernoff's seminal paper [18].

Theorem B.1 (BERNSTEIN'S INEQUALITIES) *Let X be a random variable with binomial distribution $\mathcal{B}(n,p)$. Then for all $a > 0$:*

$$\mathrm{Prob}\,[X < np - a] \le e^{-\frac{1}{2}\frac{a^2}{np}} \qquad \textit{and} \qquad \mathrm{Prob}\,[X > np + a] \le e^{-\frac{1}{2}\frac{a^2}{np}+\frac{1}{2}\frac{a^3}{n^2p^2}}.$$

An important special case of Theorem B.1 is the case $a = \epsilon np$ which we state separately.

Corollary B.2 *Let X be a random variable with binomial distribution $\mathcal{B}(n,p)$. Then for all $0 < \epsilon < 1$:*

$$\mathrm{Prob}\,[X < (1-\epsilon)np] \le e^{-\frac{1}{2}\epsilon^2 np} \qquad \textit{and} \qquad \mathrm{Prob}\,[X > (1+\epsilon)np] \le e^{-\frac{1}{2}\epsilon^2(1-\epsilon)np}.$$

In particular, for all $0 < \epsilon \le \frac{1}{2}$ such that $\epsilon^2 np > 20$:

$$\mathrm{Prob}\,[|X - np| > \epsilon np] \le e^{-\frac{1}{5}\epsilon^2 np}.$$

□

Proof of Theorem B.1. Following Alon, Spencer (1992) let $Y_1, \ldots, Y_n$ be independent identically distributed random variables with $\mathrm{Prob}\,[Y_i = 1-p] = p$ and $\mathrm{Prob}\,[Y_i = -p] = 1-p$, and let $Y = \sum_{i=1}^n Y_i$. Then $X - np$ and Y have the same distribution. It thus suffices to show that

$$\mathrm{Prob}\,[Y < -a] \le e^{-\frac{1}{2}\frac{a^2}{np}} \qquad \text{and} \qquad \mathrm{Prob}\,[Y > a] \le e^{-\frac{1}{2}\frac{a^2}{np}+\frac{1}{2}\frac{a^3}{n^2p^2}}.$$

Due to the unsymmetry of the binomial distribution we have to treat the two inequalities separately. We first bound $\mathrm{Prob}\,[Y > a]$. By the Markov inequality we obtain for any constant λ

$$\mathrm{Prob}\,[Y > a] \;=\; \mathrm{Prob}\,[e^{\lambda Y} > e^{\lambda a}]$$

$$
\begin{aligned}
&\leq \frac{\mathbb{E}[e^{\lambda Y}]}{e^{\lambda a}} \\
&= e^{-\lambda a} \cdot \prod_{i=1}^{n} \mathbb{E}[e^{\lambda Y_i}] \\
&= e^{-\lambda a} \cdot \left[p e^{\lambda(1-p)} + (1-p) e^{-\lambda p} \right]^n \\
&= e^{-\lambda(a+np)} \cdot \left[p e^{\lambda} + (1-p) \right]^n .
\end{aligned}
$$

Now let $\lambda = \ln(1 + \frac{a}{pn})$ (this is not the best choice for λ, but it keeps the calculations reasonably simple) and recall that $x - \frac{1}{2}x^2 \leq \ln(1+x) \leq x$ for all $x > 0$. Then

$$
\begin{aligned}
\text{Prob}\,[Y > a] &\leq e^{-(a+np)\ln(1+\frac{a}{pn})} \cdot \left[1 + \frac{a}{n}\right]^n \\
&\leq e^{-(a+np)(\frac{a}{pn} - \frac{1}{2}\frac{a^2}{p^2 n^2}) + a} \\
&= e^{-\frac{1}{2}\frac{a^2}{pn} + \frac{1}{2}\frac{a^3}{p^2 n^2}}.
\end{aligned}
$$

The second inequality follows similarly.

$$
\begin{aligned}
\text{Prob}\,[Y < -a] &= \text{Prob}\,[e^{-\lambda Y} > e^{\lambda a}] \\
&\leq \frac{\mathbb{E}[e^{-\lambda Y}]}{e^{\lambda a}} \\
&= e^{\lambda(np-a)} \cdot \left[p e^{-\lambda} + (1-p) \right]^n .
\end{aligned}
$$

As for all $0 < \lambda < 1$ we have

$$
pe^{-\lambda} + (1-p) = 1 + p(e^{-\lambda} - 1) \leq e^{p(e^{-\lambda}-1)} \leq e^{p((1-\lambda+\frac{1}{2}\lambda^2)-1)} = e^{-p\lambda + \frac{1}{2}p\lambda^2}
$$

we obtain for $\lambda = \frac{a}{pn}$:

$$
\text{Prob}\,[Y < -a] \leq e^{\lambda(np-a) - \lambda pn + \frac{1}{2}\lambda^2 pn} = e^{-\frac{1}{2}\frac{a^2}{pn}}.
$$

□

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