On the Evolution of Triangle-Free Graphs

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August 21, 2003

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Abstract

Let $\mathcal{T}(n,m)$ denote the set of all labeled triangle-free graphs with and n vertices and exactly m edges. In this paper we give a short self-contained proof for the fact that there exists a constant $C > 0$ such that for all $m \geq Cn^{3/2}\sqrt{\log n}$ a graph chosen uniformly at random from $\mathcal{T}(n,m)$ is with probability $1 - o(1)$ bipartite.

1 Introduction

Let $\mathcal{T}(n,m)$ denote the set of all labeled triangle-free graphs with vertex set $[n] = \{1,\ldots,n\}$ and exactly m edges and let $T_{n,m}$ denote a graph chosen uniformly at random from $\mathcal{T}(n,m)$. In this paper we are concerned with the probability that the graph $T_{n,m}$ is bipartite. Note that this probability is equal to

$$\Pr[T_{n,m} \text{ is bipartite}] = \frac{|\mathcal{S}(n,m)|}{|\mathcal{T}(n,m)|},$$

where $\mathcal{S}(n,m)$ denote the set of all labeled bipartite graphs with vertex set $[n]$ and exactly m edges.

In 1906 Mantel proved that every triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges and that *every* triangle-free graph with that many edges is bipartite. That is, for $m = \lfloor n^2/4 \rfloor$ we have $\Pr[T_{n,m} \text{ is bipartite}] = 1$. Moreover, Erdős, Kleitman, and Rothschild [1] proved that a random triangle-free graph T_n (with no restriction on the number of edges) is almost surely bipartite and has about $m = \frac{1}{8}n^2$ edges. This implies that $\Pr[T_{n,m} \text{ is bipartite}] = 1 - o(1)$ whenever $m \geq \frac{1}{8}n^2$. In [7] Prömel and Steger observed that within the evolution from $m = 0$ to $m = \lfloor \frac{1}{4}n^2 \rfloor$ there might be two thresholds. More precisely, they showed:

Theorem 1 [7] *There exist constants c_1, c_2, c_3 such that*

$$\Pr[T_{n,m} \textit{ is bipartite}] \to \begin{cases} 1 & \textit{if } m = o(n) \\ 0 & \textit{if } c_1 m \leq c_2 n^{3/2} \\ 1 & \textit{if } m \geq c_3 n^{7/4} \log n. \end{cases}$$

They also conjectured that the second threshold should in fact be located at $m = n^{3/2+o(1)}$. This conjecture was proven independently by Steger and Osthus, Prömel, Taraz. The later were in fact able to show that this threshold is located precisely at $t(n) = \frac{\sqrt{3}}{4} n^{3/2} \sqrt{\log n}$.

Theorem 2 [5] *For any $\varepsilon > 0$,*

$$\Pr[T_{n,m} \text{ is bipartite}] \;\to\; \begin{cases} 1 & \text{if } m = o(n) \\ 0 & \text{if } \frac{1}{2}nm \le (1-\varepsilon)\frac{\sqrt{3}}{4} n^{3/2}\sqrt{\log n} \\ 1 & \text{if } m \ge (1+\varepsilon)\frac{\sqrt{3}}{4} n^{3/2}\sqrt{\log n}. \end{cases}$$

The main difficulty of this theorem is the proof of the second 1-statement. Here the proof of Osthus, Prömel, and Taraz is based on ingenious counting in combination with the correlation inequalities from [2], which bound the probabilities for slightly dependent events. They also rely on a result of Łuczak [4] who proved that for $m \ge Cn^{3/2}$ almost all triangle-free graphs are "almost" bipartite. The proof of this later result is based on the sparse version of Szemeredi's Regularity Lemma [3].

In this paper we give a short, self-contained proof of the following slightly weaker statement of the second 1-statement of Theorem 2.

Theorem 3 *There exists a constant $C > 0$ such that*

$$\Pr[T_{n,m} \text{ is bipartite}] \;\to\; 1 \qquad \text{if } m \ge Cn^{3/2}\sqrt{\log n}.$$

Notations and Definitions. To simplify notation we will usually omit the explicit use of the Gauß-brackets $\lfloor\ \rfloor$ in denoting the cardinality of vertex sets, but instead tacitly assume them. In this paper $\log n$ denotes the logarithm to base 2. The natural logarithm to base $e = 2.71..$ is denoted by $\ln n$.

The following inequalities are easily seen to be true and will be used in the sequel without further reference.

i) For all $a, b, x \in \mathbb{N}$: $\binom{a}{b} \le \binom{a+x}{b}$, $\binom{a-x}{b-x} \le \binom{a}{b}$, and $\binom{a}{b} \le \left(\frac{ea}{b}\right)^b$.

ii) For all $a, b, i, k \in \mathbb{N}$: $\binom{a}{i} \cdot \binom{b}{k} \le \binom{a+b}{i+k}$.

iii) For all $a, b, x \in \mathbb{N}$: $\binom{a-x}{b} \cdot e^{\frac{xb}{a}} \le \binom{a}{b} \le \binom{a-x}{b} \cdot e^{\frac{xb}{a-b-x}}$ and $\binom{a+x}{b} \le \binom{a}{b} \cdot e^{x \ln a}$.

iv) For all $a, b, x \in \mathbb{N}$: $\binom{a}{b-x} \le \binom{a}{b} \cdot \left(\frac{b-x}{a-b}\right)^x \le \binom{a}{b} \cdot 2^{x \log b}$.

2 Proof of Theorem 3

In [6] Prömel and Steger obtained precise bounds for the number of bipartite graphs on n vertices with m edges. For the purpose of this paper, however, the following weaker estimates suffice. See [7] for a short proof.

Lemma 4 [7] *For all m with $m \gg n \log n$ and n sufficiently large*

$$2^{n-2\log n} \cdot \binom{\frac{1}{4}n^2}{m} \le |\mathcal{S}(n,m)| \le 2^n \cdot \binom{\frac{1}{4}n^2}{m}.$$

The following simple lemma was also proven in [7].

Lemma 5 [7] *For all $0 \le m \le \lfloor \frac{1}{8}n^2 \rfloor$*

$$|\mathcal{F}orb(n,m)| \le e^{2m} \cdot |\mathcal{S}(n,m)|.$$

Let $b \ge 1$ and $0 < \epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}, \epsilon_{c}, \epsilon_{\mathcal{X}} < \frac{1}{100}$ be positive constants such that

$$\begin{array}{c} \frac{40}{\log b} + \log(12e\epsilon_{\mathcal{A}}^{-1})\epsilon_{\mathcal{A}} \ < \ \frac{1}{2}, \qquad \frac{1}{2}\epsilon_{\mathcal{A}}\epsilon_{\mathcal{B}} \ > \ \frac{1700}{\log b} + \frac{10}{\sqrt{b}}, \qquad 2(\epsilon_{\mathcal{X}} - \epsilon_{\mathcal{B}}) \ > \ \epsilon_{\mathcal{B}}, \\ \frac{1}{2}\epsilon_{\mathcal{A}}(\epsilon_c - 3\epsilon_{\mathcal{B}}) \ \ge \ \frac{1700}{\log b} + \frac{4}{\sqrt{b}}, \qquad \sqrt{b} \ge 8800 \end{array} \tag{1}$$

Note that we may choose these constants such that $\varepsilon := \max\{\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}, \epsilon_c, \epsilon_{\mathcal{X}}\}$ is arbitrarily small. Furthermore we set

$$a \ := \ \frac{8\sqrt{b}}{\log b}, \qquad q = q(n,m) \ := \ \frac{1}{2\sqrt{b}} \frac{m}{n \log n}.$$

Define sets $\mathcal{A}_i(n,m)$, $\mathcal{B}_i(n,m)$, $\mathcal{C}(n,m)$, and $\mathcal{D}(n,m)$ as follows:

$\mathcal{A}_1(n,m) :=$ set of all graphs G in $\mathcal{F}(n,m)$ which contain a vertex v of degree less than $\epsilon_{\mathcal{A}}\frac{m}{n}$.

$\mathcal{A}_2(n,m) :=$ set of all graphs G in $\mathcal{F}(n,m)$ which contain a vertex v of degree greater than $70\frac{m}{n}$.

$\mathcal{B}_1(n,m) :=$ set of all graphs G in $\mathcal{F}(n,m) \setminus \bigcup_{i=1,2} \mathcal{A}_i(n,m)$ which contain a vertex v and a set $Q \subseteq \Gamma(v)$ of size $|Q| = 2q$ such that $|\Gamma(Q)| \ge (\frac{1}{2} + \epsilon_{\mathcal{B}})n$.

$\mathcal{B}_2(n,m) :=$ set of all graphs G in $\mathcal{F}(n,m) \setminus \bigcup_{i=1,2} \mathcal{A}_i(n,m)$ which contain a set $Q \subseteq V$ of size $|Q| = q$ such that at least one of the following conditions is satisfied:

i) $|\Gamma(Q)| \le (\frac{1}{2} - \epsilon_{\mathcal{B}})n$,

ii) there exists a set $X \subseteq \Gamma(Q)$ of size $|X| = \epsilon_{\mathcal{X}} n$ such that all vertices in of Q have at most q neighbors in X.

$\mathcal{C}(n,m) :=$ set of all graphs G in $\mathcal{F}(n,m) \setminus (\bigcup_{i=1,2}(\mathcal{A}_i(n,m) \cup \mathcal{B}_i(n,m)))$ which contain a vertex v and sets $Q, Q' \subseteq \Gamma(v)$ of size $|Q| = |Q'| = q$ such that for every vertex $w \in Q'$ there exists a set $Q_w \subseteq \Gamma(w)$ of size $|Q_w| = q$ such that $|\Gamma(Q_w) \cap \Gamma(Q)| \ge \epsilon_c n$.

$\mathcal{D}(n,m) :=$ set of all graphs G in $\mathcal{F}(n,m) \setminus (\bigcup_{i=1,2}(\mathcal{A}_i(n,m) \cup \mathcal{B}_i(n,m)) \cup \mathcal{C}(n,m))$ that are not bipartite but for which there exists a bipartition of the vertex set V into two sets V_1 and V_2 such that $|V_1|, |V_2| \le (\frac{1}{2} - \frac{1}{100})n$ and such that the bipartite graph induced by this partition has the following two properties:

i) it contains at least $m - nq$ edges, and

ii) every vertex v has at most q neighbors in the same class as v.

Lemma 6 *If ε is sufficiently small, then for all sufficiently large n and all $m \geq \sqrt{b}n^{3/2}\sqrt{\log n}$:*

$$\mathcal{F}(n,m) \setminus \left(\bigcup_{i=1,2} (\mathcal{A}_i(n,m) \cup \mathcal{B}_i(n,m)) \cup \mathcal{C}(n,m) \cup \mathcal{D}(n,m) \right) \subseteq \mathcal{S}(n,m).$$

Proof: Let $G = (V,E)$ be an arbitrary but fixed graph in $\mathcal{F}(n,m) \setminus (\bigcup_{i=1,2}(\mathcal{A}_i(n,m) \cup \mathcal{B}_i(n,m)) \cup \mathcal{C}(n,m) \cup \mathcal{D}(n,m))$. In order to concentrate on the main ideas of the proof instead of technical details we will use the following two notations for subsets $X, Y \subseteq V$:

$$|X| \cong \frac{1}{2}n \;\; :\Longleftrightarrow \;\; \exists c_X \in \mathbb{N} \;\; s.t. \;\; (\frac{1}{2} - c_X\varepsilon)n \leq |X| \leq (\frac{1}{2} + c_X\varepsilon)n, \;\; \text{and}$$
$$X \cong Y \;\; :\Longleftrightarrow \;\; |X| \cong \frac{1}{2}n \;\wedge\; |Y| \cong \frac{1}{2}n \;\wedge\; |X \cap Y| \cong \frac{1}{2}n.$$

In the following we assume that ε is sufficiently small so that $c_X\varepsilon \leq \frac{1}{100}$ for all constants c_X which occur in the proof. The reader is invited to observe that in the sequel of the proof the constants c_X can always be chosen independently of $\varepsilon = \max\{\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}, \epsilon_{\mathcal{C}}, \epsilon_{\mathcal{X}}\}$.

With this notation at hand we are ready to collect some properties of G which follow immediately from the definition of the sets $\mathcal{A}_1(n,m), \ldots, \mathcal{C}(n,m)$:

1. For every vertex $v \in V$ all sets $Q_v \subseteq \Gamma(v)$ of size $|Q_v| = q$ are such that

   · $|\Gamma(Q_v)| \cong \frac{1}{2}n$, and

   · every set $X \subseteq \Gamma(v) \setminus Q_v$ of size $|X| = q$ satisfies $\Gamma(X) \cong \Gamma(Q_v)$.

   (Proof: This follows immediately from the definition of the sets $\mathcal{B}_1(n,m)$ and $\mathcal{B}_2(n,m)$; for the second part observe that the set Q in the definition of $\mathcal{B}_1(n,m)$ has size $2q$.)

   In the following we use Q_v to denote an arbitrary but fixed set $Q_v \subseteq \Gamma(v)$ of size $|Q_v| = q$.

2. For every vertex v there exist at most q many vertices $v_1, \ldots, v_q \in \Gamma(v)$ that satisfy $|\Gamma(Q_{v_1}) \cap \Gamma(Q_v)| \geq \epsilon_{\mathcal{C}}n$.
   (Proof: This follows immediately from the definition of the set $\mathcal{C}(n,m)$.)

   In the following we use B_v to denote these at most q "bad" vertices in $\Gamma(v)$.

3. For every vertex v there exists a set $Y_v \subseteq \Gamma(Q_v)$ such that $|Y_v| \cong |\Gamma(Q_v)| \cong \frac{1}{2}n$ and such that
   $$\Gamma(Q_y) \cong \Gamma(Q_v) \qquad \text{for all } y \in Y_v.$$

   (Proof: Let $\widetilde{Q}_v$ denote a set of size q within $\Gamma(v) \setminus B_v$. Recall that we know from the first property that $\Gamma(\widetilde{Q}_v) \cong \Gamma(Q_v)$. Consider a vertex w in $\widetilde{Q}_v$. Then, according to the second property, $\Gamma(Q_w) \cong V \setminus \Gamma(Q_v)$. Repeating this argument we deduce that all vertices y in $\Gamma(w) \setminus B_w$ satisfy $\Gamma(Q_y) \cong V \setminus \Gamma(Q_w) \cong \Gamma(Q_v)$. Now consider the set $X := \{x \in \Gamma(\widetilde{Q}_v) \mid \forall w \in \Gamma(x) \cap \widetilde{Q}_v : x \in B_w\}$. Then the definition of X implies that every vertex $w \in \widetilde{Q}_v$ has at most $|B_w| \leq q$ neighbors in X. From the definition of the set $\mathcal{B}_2(n,m)$ it therefore follows that $|X| < \epsilon_{\mathcal{X}}n$. The claim follows as the definition of X also implies that for every vertex $y \in Y := \Gamma(\widetilde{Q}_v) \setminus X$ there exists a $w \in \widetilde{Q}_v$ such that $y \in \Gamma(w) \setminus B_w$ and hence $\Gamma(Q_y) \cong \Gamma(Q_v)$.)

4. For every pair of vertices x, y we have

$$\Gamma(Q_x) \cong \Gamma(Q_y) \qquad \text{or} \qquad \Gamma(Q_x) \cong V \setminus \Gamma(Q_y).$$

(Proof: This follows easily from the previous property, as the assumption that $|\Gamma(Q_x) \cap \Gamma(Q_v)| > \mu n$ for an appropriately defined constant $\mu > 0$ implies that there exists a vertex $z \in \Gamma(Q_x) \cap \Gamma(Q_v)$ such that $\Gamma(Q_z) \cong \Gamma(Q_x)$ and $\Gamma(Q_z) \cong \Gamma(Q_y)$.)

Choose an arbitrary vertex v_0. Let $A := \Gamma(Q_{v_0})$ and $B := V \setminus A$. Recall that $|A|, |B| \cong \frac{1}{2}n$. We use these two sets to define a bipartition of the vertex set:

$$\begin{aligned} V_1 &:= \{v \in V \mid \Gamma(Q_v) \cong A\}, \\ V_2 &:= \{v \in V \mid \Gamma(Q_v) \cong B\} \end{aligned}$$

Observe that according to property 4 this defines indeed a *partition* of the vertex set V. As a last step observe that property 2 implies that for $i = 1, 2$ every vertex $v \in V_i$ has at most q neighbors in V_i. That is, (V_1, V_2) induces a bipartite graph that contains at least $m - qn$ edges and in which every vertex has at most q neighbors inside its class. As by assumption the graph G is not contained in $\mathcal{D}(n, m)$, this implies that G is indeed bipartite. □

Theorem 7 *There exists a constant ξ_0 such that for all $n \in \mathbb{N}$*

$$F_{n,m} \le (1 + 2^{\xi(n,m)} + 2^{-\frac{1}{800}m^2/n^3}) \cdot S_{n,m} \qquad \text{for all } 0 \le m \le \binom{n}{2},$$

where

$$\xi(n, m) = an^{3/2}\sqrt{\log n} \cdot \log \frac{bn^{3/2}\sqrt{\log n}}{m} + \xi_0.$$

Observe that Theorem 7 implies Theorem 3 for, e.g., $C = 2b$.

Proof: Choose n_0 in such a way that all inequalities below hold for all $n \ge n_0$. Subsequently choose ξ_0 in such a way that the claim of the theorem holds for all $n \le n_0$. (The reader is invited to observe that the inequalities in the sequel which are used to define n_0 will not depend on ξ_0.)

We proceed by induction on n. So assume the theorem holds for all $n' < n$ for some $n \in \mathbb{N}$. We will show that it then also holds for n. Let m be an arbitrary integer between 1 and $\binom{n}{2}$. Observe that if

$$an^{3/2}\sqrt{\log n} \cdot \log \frac{bn^{3/2}\sqrt{\log n}}{m} > 2\log e \cdot m$$

then by Lemma 5 there is nothing to show. So we may assume without loss of generality that

$$\sqrt{b}n^{3/2}\sqrt{\log n} \le m < \frac{1}{4}n^2. \tag{2}$$

(cf. the definition of a and the fact at every triangle-free graph with $\frac{1}{4}n^2$ edges is bipartite).

As a first step we convert the bounds on $F_{n-x,m-y}$ provided by the induction hypothesis into a somewhat nicer form. First observe that for all $n, x \in \mathbb{N}$ with $x < n$ we have

$$(n - x)^{3/2} \cdot \sqrt{\log(n - x)} \le (n - x)\sqrt{n}\sqrt{\log n} = n^{3/2}\sqrt{\log n} - x\sqrt{n \log n}. \tag{3}$$

and that for all $1 \le x \le q$ and $0 \le y \le \frac{1}{2}m$ and n sufficiently large:

$$\begin{aligned} S_{n-x,m-y} &\le 2^n \cdot \binom{\frac{1}{4}(n-x)^2}{m-y} \le 2^n \cdot \binom{\frac{1}{4}n^2 - \frac{1}{2}nx}{m-y} \cdot e^{\frac{1}{4}x^2 \ln(\frac{1}{4}n^2)} \\ &\le 2^{n+x^2\log n} \cdot \binom{\frac{1}{4}n^2 - \frac{1}{2}nx}{m-y} \le 2^{n+\frac{x}{\log b}\frac{m}{n}} \cdot \binom{\frac{1}{4}n^2 - \frac{1}{2}nx}{m-y} \end{aligned} \tag{4}$$

(cf. Lemma 4 and the definition of q) and that

$$\begin{aligned} \xi(n-x,m-y) &\le \xi(n,m) + ax\sqrt{n\log n} \cdot \log \frac{m}{bn^{3/2}\sqrt{\log n}} + \frac{3ayn^{3/2}\sqrt{\log n}}{m} \\ &\le \xi(n,m) + \frac{8x}{\log b}\frac{m}{n} + \frac{24y}{\log b}. \end{aligned}$$

This follows from equations (2) and (3), the definition of a, and the observation that for all $y \le \frac{1}{2}m$

$$\begin{aligned} \log \frac{b(n-x)^{3/2}\sqrt{\log(n-x)}}{m-y} &\le \log(\frac{bn^{3/2}\sqrt{\log n}}{m} \cdot (1 + \frac{y}{m-y})) \\ &\le \log(\frac{bn^{3/2}\sqrt{\log n}}{m} \cdot (1 + \frac{2y}{m})) \\ &= \log \frac{bn^{3/2}\sqrt{\log n}}{m} + \log(1 + \frac{2y}{m}) \\ &\le \log \frac{bn^{3/2}\sqrt{\log n}}{m} + \frac{3y}{m} \end{aligned}$$

(to see the last inequality use $\log(1+x) \le \frac{x}{\ln 2} \le \frac{3}{2}x$). We therefore conclude that the induction hypothesis and (4) imply that for all $1 \le x \le q$ and $0 \le y \le \frac{1}{2}m$:

$$\begin{aligned} F_{n-x,m-y} &\le (1 + 2^{\xi(n,m)+\frac{8x}{\log b}\frac{m}{n}+\frac{24y}{\log b}} + 2^{-\frac{1}{800}\frac{(m-y)^2}{(n-x)^3}}) \cdot 2^{n+\frac{x}{\log b}\frac{m}{n}} \binom{\frac{1}{4}n^2 - \frac{1}{2}xn}{m-y} \\ &\le 2^{\xi(n,m)+\frac{10x}{\log b}\frac{m}{n}+\frac{24y}{\log b}} \cdot 2^n \cdot \binom{\frac{1}{4}n^2 - \frac{1}{2}nx}{m-y}. \end{aligned} \tag{5}$$

We also note that whenever $y \le 70x\frac{m}{n}$, also the following slightly weaker, but shorter bound holds:

$$F_{n-x,m-y} \le 2^{\xi(n,m)+\frac{1700}{\log b}x\frac{m}{n}} \cdot 2^n \cdot \binom{\frac{1}{4}n^2 - \frac{1}{2}nx}{m-y}. \tag{6}$$

These last two inequalities will be heavily used throughout the upcoming parts of the proof. Before we do so we first recall that by Lemma 6

$$F_{n,m} \le \sum_{i=1,2}(|\mathcal{A}_i(n,m)| + |\mathcal{B}_i(n,m)|) + |\mathcal{C}(n,m)| + |\mathcal{D}(n,m)| + S_{n,m}.$$

It hence suffices to show that all summands on the right hand side except the last one are smaller or equal than

$$\frac{1}{6} \cdot \left(2^{\xi(n,m)} + 2^{-\frac{1}{800}\frac{m^2}{n^3}}\right) \cdot S_{n,m}. \tag{7}$$

Moreover, by Lemma 4 it also suffices to show that each of these summands is smaller or equal than

$$\frac{1}{6} \cdot 2^{\xi(n,m)} \cdot 2^{n-2\log n} \cdot \binom{\frac{1}{4}n^2}{m}. \tag{8}$$

This is what we will do now. (Recall that we assumed that n is sufficiently large, so that all inequalities below do hold.)

The Sets $\mathcal{A}(n,m)$. All graphs in $\mathcal{A}_1(n,m) \cup \mathcal{A}_2(n,m)$ can be constructed as follows. First choose the vertex v (at most n choices). For all integers e such that $0 \le e < \epsilon_{\mathcal{A}} \frac{m}{n}$ or $70\frac{m}{n} < e < n$ choose then a triangle-free graph with $m-e$ edges on the remaining vertices ($F_{n-1,m-e}$ ways). Finally, choose the e neighbors of v (at most $\binom{n-1}{e}$ ways). Together, this gives

$$|\mathcal{A}_1(n,m)| \;\le\; 2^{\log n} \cdot \sum_{e=0}^{\epsilon_{\mathcal{A}}\frac{m}{n}} \binom{n-1}{e} F_{n-1,m-e} \;\le\; 2^{2\log n} \cdot \max_{e \le \epsilon_{\mathcal{A}}\frac{m}{n}} \binom{n}{e} F_{n-1,m-e},$$

respectively

$$|\mathcal{A}_2(n,m)| \;\le\; 2^{\log n} \cdot \sum_{e=70\frac{m}{n}}^{n-1} \binom{n-1}{e} F_{n-1,m-e} \;\le\; 2^{2\log n} \cdot \max_{e \ge 70\frac{m}{n}} \binom{n}{e} F_{n-1,m-e}.$$

Using the bound on $F_{n-1,m-e}$ from (5) we deduce

$$|\mathcal{A}_i(n,m)| \;\le\; 2^{2\log n} \cdot \max_{y<\epsilon_{\mathcal{A}}\frac{m}{n} | y > 70\frac{m}{n}} \binom{n}{y}\binom{\frac{1}{4}n^2 - \frac{1}{2}n}{m-y} \cdot 2^n \cdot 2^{\xi(n,m) + \frac{10}{\log b}\frac{m}{n} + \frac{24}{\log b} y}.$$

First we consider the case $m \le \frac{1}{6}n^2$. As we assumed that $m \ge \sqrt{b} n^{3/2}\sqrt{\log n}$ we know that, with room to spare, $y < n \le \frac{1}{3}m$ and the binomial coefficients can therefore be bounded as follows:

$$\begin{aligned}
\binom{n}{y}\binom{\frac{1}{4}n^2 - \frac{1}{2}n}{m-y} &\le \left(\frac{en}{y}\right)^y \cdot \binom{\frac{1}{4}n^2}{m-y} \cdot e^{-\frac{1}{2}n \cdot \frac{m-y}{\frac{1}{4}n^2}} \\
&\le \left(\frac{en}{y}\right)^y \cdot \binom{\frac{1}{4}n^2}{m} \left(\frac{m-y}{\frac{1}{4}n^2 - m}\right)^y \cdot e^{-\frac{m}{n}} \\
&\le \binom{\frac{1}{4}n^2}{m} \cdot \left(\frac{12em}{yn}\right)^y \cdot e^{-\frac{m}{n}}.
\end{aligned}$$

For $y \ge 70\frac{m}{n}$ we have

$$\left(\frac{12em}{yn}\right)^y \le 2^{-y}.$$

For $y \le \epsilon_{\mathcal{A}}\frac{m}{n}$ we observe that the maximum is attained at $y = \epsilon_{\mathcal{A}}\frac{m}{n}$, and that therefore

$$\left(\frac{12em}{yn}\right)^y \le 2^{\log\frac{12e}{\epsilon_{\mathcal{A}}} \cdot \epsilon_{\mathcal{A}}\frac{m}{n}}.$$

In both cases one easily checks that the definitions in (1) imply that we indeed have that the cardinalities of the sets $\mathcal{A}_i(n,m)$ are smaller than the term in (8).

Now consider the case $m > \frac{1}{6}n^2$. For $y < \epsilon_{\mathcal{A}}\frac{m}{n} \leq \frac{1}{4}\epsilon_{\mathcal{A}} n$ (as $m \leq \frac{1}{4}n^2$ for triangle-free graphs) we bound the binomial coefficients as follows:

$$\begin{aligned}\binom{n}{y}\binom{\frac{1}{4}n^2 - \frac{1}{2}n}{m-y} &\leq \left(\frac{en}{y}\right)^y \cdot \binom{\frac{1}{4}n^2 - (\frac{1}{2}n - y)}{m} \\ &\leq \left(\frac{en}{y}\right)^y \cdot \binom{\frac{1}{4}n^2}{m} e^{-(\frac{1}{2}n-y)\frac{m}{\frac{1}{4}n^2}} \\ &\leq \binom{\frac{1}{4}n^2}{m} \cdot \left(\frac{en}{y}\right)^y \cdot e^{-\frac{m}{n}}.\end{aligned}$$

One again easily checks that the maximum is attained at $y = \epsilon_{\mathcal{A}}\frac{m}{n} > \frac{1}{6}\epsilon_{\mathcal{A}} n$ (recall that we assumed $m > \frac{1}{6}n^2$), and that therefore

$$\left(\frac{en}{y}\right)^y \leq (6e/\epsilon_{\mathcal{A}})^{\epsilon_{\mathcal{A}}\frac{m}{n}} \leq 2^{\log\frac{6e}{\epsilon_{\mathcal{A}}}\cdot \epsilon_{\mathcal{A}}\frac{m}{n}}.$$

Similarly as above, the definitions in (1) therefore imply that the cardinality of the set $\mathcal{A}_1(n,m)$ is smaller than the term in (8). Finally, observe that for $m > \frac{1}{6}n^2$ we have that $70\frac{m}{n} > n$ and hence $|\mathcal{A}_2(n,m)| = 0$.

The Set $\mathcal{B}_1(n,m)$. All graphs in $\mathcal{B}_1(n,m)$ can be constructed as follows. First choose the vertex v (at most n ways). Let e denote the degree of v. Note that $\epsilon_{\mathcal{A}}\frac{m}{n} \leq e \leq 70\frac{m}{n}$. For all possible values choose a triangle-free graph with $m - e$ edges on the remaining vertices ($F_{n-1,m-e}$ ways). Next choose an appropriate set Q of size $2q$ (at most $\binom{n-1}{2q} \leq 2^{2q\log n}$ ways). As v cannot be connected to any vertex in $\Gamma(Q)$ (this would close a cycle of length 3) there are at most $\binom{n-|\Gamma(Q)|}{e}$ ways to choose the remaining neighbors of v. Together we obtain

$$\begin{aligned}B_1(n,m) &\leq 2^{\log n + 2q\log n} \cdot \sum_{\epsilon_{\mathcal{A}}\frac{m}{n} \leq e \leq 70\frac{m}{n}} \binom{n - |\Gamma(Q)|}{e} F_{n-1,m-e} \\ &\leq 2^{3q\log n} \cdot \max_{\epsilon_{\mathcal{A}}\frac{m}{n} \leq e \leq 70\frac{m}{n}} \binom{(\frac{1}{2}-\epsilon_{\mathcal{B}})n}{e} F_{n-1,m-e}.\end{aligned}$$

We now use the bound on $F_{n-x,m-y}$ from (6) for $x = 1$ and $y = e$ to obtain

$$|\mathcal{B}_1(n,m)| \leq 2^{3q\log n} \cdot \max_{\epsilon_{\mathcal{A}}\frac{m}{n} \leq y \leq 70\frac{m}{n}} \binom{(\frac{1}{2}-\epsilon_{\mathcal{B}})n}{y} \cdot \binom{\frac{1}{4}n^2 - \frac{1}{2}n}{m-y} \cdot 2^n \cdot 2^{\xi(n,m) + \frac{1700}{\log b}\frac{m}{n}}.$$

We bound the binomial coefficients:

$$\begin{aligned}\binom{(\frac{1}{2}-\epsilon_{\mathcal{B}})n}{y}\binom{\frac{1}{4}n^2 - \frac{1}{2}n}{m-y} &\leq \binom{\frac{1}{4}n^2 - \epsilon_{\mathcal{B}} n}{m} \\ &\leq \binom{\frac{1}{4}n^2}{m} \cdot e^{-\epsilon_{\mathcal{B}} n \cdot \frac{4m}{n^2}} = \binom{\frac{1}{4}n^2}{m} \cdot e^{-4\epsilon_{\mathcal{B}}\frac{m}{n}}.\end{aligned}$$

Using the definitions (1) one easily deduces that (8) is an upper bound for $|\mathcal{B}_1(n,m)|$.

The Set $\mathcal{B}_2(n,m)$. All graphs in $\mathcal{B}_2(n,m)$ can be constructed as follows. First choose the set Q (less than 2^n ways). Let e_q denote the number of edges inside Q and let e_r denote the number of edges connecting Q with $\Gamma(Q)$. As the degree of every vertex in Q is between $\epsilon_{\mathcal{A}}\frac{m}{n}$

and $70\frac{m}{n}$, we know that $e_r \geq q \cdot (\epsilon_{\mathcal{A}}\frac{m}{n} - q) \geq \frac{1}{2}\epsilon_{\mathcal{A}} q\frac{m}{n}$ and $e_q + e_r \leq 70q\frac{m}{n}$. For all possible values choose a triangle-free graph with $m - e_q - e_r$ edges on the remaining vertices ($F_{n-q,m-e_q-e_r}$ ways). Next choose the set $\Gamma(Q)$ (less than 2^n ways) and the edges inside the set Q (less than $\binom{\frac{1}{2}q^2}{e_q} \leq 2^{q^2}$ ways).

In order to choose the edges between Q and $\Gamma(Q)$ we distinguish between the two properties in the definition of the set $\mathcal{B}_2(n,m)$.

If property *i)* is satisfied then there are clearly only

$$\begin{aligned}\binom{q \cdot (\frac{1}{2} - \epsilon_{\mathcal{B}})n}{e_r} &\leq e^{-\epsilon_{\mathcal{B}} qn \frac{e_r}{\frac{1}{2}qn}} \cdot \binom{\frac{1}{2}qn}{e_r} = e^{-2\epsilon_{\mathcal{B}} e_r} \cdot \binom{\frac{1}{2}qn}{e_r} \leq e^{-\epsilon_{\mathcal{A}}\epsilon_{\mathcal{B}} q\frac{m}{n}} \cdot \binom{\frac{1}{2}qn}{e_r} \\ &\leq e^{-\epsilon_{\mathcal{A}}\epsilon_{\mathcal{B}} q\frac{m}{n}} \cdot \binom{\frac{1}{2}qn}{e_q + e_r} \cdot 2^{2e_q \log n} \leq e^{-\epsilon_{\mathcal{A}}\epsilon_{\mathcal{B}} q\frac{m}{n}} \cdot 2^{q^2 \log n} \cdot \binom{\frac{1}{2}qn}{e_q + e_r}\end{aligned}$$

ways to choose the edges between Q and $\Gamma(Q)$.

If property *ii)* is satisfied we argue as follows. First we choose the set X (2^n ways). Observe that there are at most

$$\left(\sum_{i=0}^{q} \binom{|X|}{i}\right)^{|Q|} \leq 2^{q|Q|\log n} = 2^{q^2 \log n}$$

ways to connect Q to X. By the definition of $\mathcal{B}_1(n,m)$ we know that $|\Gamma(Q)| \leq (\frac{1}{2} + \epsilon_{\mathcal{B}})n$. Hence, the number of ways to connect Q to $\Gamma(Q) \setminus X$ can be bounded by

$$\begin{aligned}\sum_{e=e_r-q^2}^{e_r} \binom{|Q| \cdot |\Gamma(Q) \setminus X|}{e} &\leq \binom{|Q| \cdot |\Gamma(Q) \setminus X|}{e_r} \cdot 2^{q^2 \log(qn)} \leq \binom{\frac{1}{2}qn - (\epsilon_{\mathcal{X}} - \epsilon_{\mathcal{B}})qn}{e_r} \cdot 2^{2q^2 \log n} \\ &\leq e^{-2(\epsilon_{\mathcal{X}} - \epsilon_{\mathcal{B}})e_r} \cdot 2^{2q^2 \log n} \cdot \binom{\frac{1}{2}qn}{e_r} \leq e^{-2(\epsilon_{\mathcal{X}} - \epsilon_{\mathcal{B}})e_r} \cdot 2^{3q^2 \log n} \cdot \binom{\frac{1}{2}qn}{e_q + e_r}.\end{aligned}$$

Summarizing, we conclude that in case property *ii)* is satisfied the number of ways to choose the edges between Q and $\Gamma(Q)$ is bounded by

$$\begin{aligned}2^{n+3q^2 \log n} \cdot e^{-2(\epsilon_{\mathcal{X}} - \epsilon_{\mathcal{B}})e_r} \cdot \binom{\frac{1}{2}qn}{e_q + e_r} &\leq 2^{n+3q^2 \log n} \cdot e^{-2\epsilon_{\mathcal{B}} e_r} \cdot \binom{\frac{1}{2}qn}{e_q + e_r} \\ &\leq 2^{n+3q^2 \log n} \cdot e^{-\epsilon_{\mathcal{A}}\epsilon_{\mathcal{B}} q\frac{m}{n}} \cdot \binom{\frac{1}{2}qn}{e_q + e_r}.\end{aligned}$$

(Here we used (1) and the fact that $e_r \geq \frac{1}{2}\epsilon_{\mathcal{A}} q\frac{m}{n}$.)

Together we obtain

$$\begin{aligned}B_2(n,m) &\leq 2^{2n+q^2} \cdot \sum_{\frac{1}{2}\epsilon_{\mathcal{A}} q\frac{m}{n} \leq e_q + e_r \leq 70q\frac{m}{n}} 2 \cdot 2^{n+3q^2 \log n} \cdot e^{-\epsilon_{\mathcal{A}}\epsilon_{\mathcal{B}} q\frac{m}{n}} \cdot \binom{\frac{1}{2}qn}{e_q + e_r} F_{n-q,m-e_q-e_r} \\ &\leq 2^{20q^2 \log n} \cdot \max_{\frac{1}{2}\epsilon_{\mathcal{A}} q\frac{m}{n} \leq \tilde{e} \leq 70q\frac{m}{n}} e^{-\epsilon_{\mathcal{A}}\epsilon_{\mathcal{B}} q\frac{m}{n}} \cdot \binom{\frac{1}{2}qn}{\tilde{e}} F_{n-q,m-\tilde{e}}.\end{aligned}$$

(Note that $m \geq \sqrt{b}n^{3/2}\sqrt{\log n}$ and the definition of q imply that $q^2 \log n \geq \frac{1}{4}n$). We again use the bound on $F_{n-x,m-y}$ from (6), this time for $x = q$ and $y = \tilde{e}$, to obtain

$$|\mathcal{B}_2(n,m)| \leq 2^{20x^2 \log n} \cdot \max_{\frac{1}{2}\epsilon_{\mathcal{A}} x\frac{m}{n} \leq y \leq 70x\frac{m}{n}} e^{-\epsilon_{\mathcal{A}}\epsilon_{\mathcal{B}} x\frac{m}{n}} \cdot \binom{\frac{1}{2}xn}{y} \binom{\frac{1}{4}n^2 - \frac{1}{2}nx}{m-y} \cdot 2^n \cdot 2^{\xi(n,m) + \frac{1700}{\log b} x\frac{m}{n}}$$

$$\leq \quad 2^{20x^2\log n}\cdot e^{-\epsilon_{\mathcal{A}}\epsilon_{\mathcal{B}}x\frac{m}{n}}\cdot\binom{\frac{1}{4}n^2}{m}\cdot 2^n\cdot 2^{\xi(n,m)+\frac{1700}{\log b}x\frac{m}{n}}.$$

Using the definitions from (1) one again easily checks that (8) is an upper bound for $|\mathcal{B}_2(n,m)|$.

The Set $\mathcal{C}(n,m)$. All graphs in $\mathcal{C}(n,m)$ can be constructed as follows. First choose the set Q' (less than n^q ways). Let e_q denote the number of edges inside Q' and let e_r denote the number of edges between Q' and $\Gamma(Q')$. Note that we may assume that $e_r \geq q\cdot(\epsilon_{\mathcal{A}}\frac{m}{n}-q) \geq \frac{1}{2}\epsilon_{\mathcal{A}}q\frac{m}{n}$ and $e_q+e_r \leq 70q\frac{m}{n}$. For all possible values choose a triangle-free graph with $m-e_q-e_r$ edges on the remaining vertices ($F_{n-q,m-e_q-e_r}$ ways). Next choose an appropriate vertex v and an appropriate set $Q \subseteq \Gamma(v)$ of size q (less than $n\cdot n^q = n^{q+1}$ ways).

The number of ways to choose the edges inside the set Q' is bounded by 2^{q^2}. In order to bound the number of ways to connect Q' to $\Gamma(Q')$ we first choose the set $\Gamma(Q')$ (in at most 2^n ways). Note that by the definition of $\mathcal{B}_1(n,m)$ and $\mathcal{B}_2(n,m)$ we only have to consider sets $\Gamma(Q')$ such that

$$|\Gamma(Q')| \leq (\frac{1}{2}+\epsilon_{\mathcal{B}})n \qquad \text{and} \qquad |\Gamma(Q)\setminus\Gamma(Q')| \leq 2\epsilon_{\mathcal{B}}n.$$

For every vertex $w \in Q'$ we first choose the degree d_w and the set Q_w (in at most n^{q+1} ways) such that $|\Gamma(Q_w)\cap\Gamma(Q)| \geq \epsilon_c n$ and hence $|\Gamma(Q')\setminus\Gamma(Q_w)| \leq (\frac{1}{2}-\epsilon_c+3\epsilon_{\mathcal{B}})n$. Next we connect w to $\Gamma(Q')$. Note that w may not be connected to a vertex in $\Gamma(Q_w)$. Hence, the number of choices is bounded by

$$\binom{|\Gamma(Q')\setminus\Gamma(Q_w)|}{d_w} \leq \binom{(\frac{1}{2}-\epsilon_c+3\epsilon_{\mathcal{B}})n}{d_w}.$$

Hence, the number of ways to connect Q' to $\Gamma(Q')$ is bounded by

$$\begin{aligned}\prod_{w\in Q'} n^{q+1}\binom{(\frac{1}{2}-\epsilon_c+3\epsilon_{\mathcal{B}})n}{d_w} &\leq n^{q(q+1)}\cdot\binom{(\frac{1}{2}-\epsilon_c+3\epsilon_{\mathcal{B}})qn}{e_r}\\ &\leq n^{q(q+1)}\cdot e^{-2(\epsilon_c-3\epsilon_{\mathcal{B}})e_r}\cdot\binom{\frac{1}{2}qn}{e_r}\\ &\leq n^{q(q+1)}\cdot e^{-\epsilon_{\mathcal{A}}(\epsilon_c-3\epsilon_{\mathcal{B}})q\frac{m}{n}}\cdot 2^{q^2\log n}\binom{\frac{1}{2}qn}{e_q+e_r},\end{aligned}$$

as $e_q \leq \frac{1}{2}q^2$. That is, we obtain

$$\begin{aligned}|\mathcal{C}(n,m)| &\leq \sum_{\frac{1}{2}\epsilon_{\mathcal{A}}q\frac{m}{n}\leq e_q+e_r\leq 70q\frac{m}{n}} n^{q^2+3q+1}\cdot 2^{q^2+n+q^2\log n}\cdot e^{-\epsilon_{\mathcal{A}}(\epsilon_c-3\epsilon_{\mathcal{B}})q\frac{m}{n}}\cdot\binom{\frac{1}{2}qn}{e_q+e_r}\cdot F_{n-q,m-e_q-e_r}\\ &\leq \max_{\frac{1}{2}\epsilon_{\mathcal{A}}q\frac{m}{n}\leq e_q+e_r\leq 70q\frac{m}{n}} \cdot 2^{8q^2\log n}\cdot e^{-\epsilon_{\mathcal{A}}(\epsilon_c-3\epsilon_{\mathcal{B}})q\frac{m}{n}}\cdot\binom{\frac{1}{2}qn}{e_q+e_r}\cdot F_{n-q,m-e_q-e_r}.\end{aligned}$$

(Here we used again that $q^2\log n \geq \frac{1}{4}n$ for $m \geq \sqrt{b}n^{3/2}\sqrt{\log n}$.)

We again use the bound on $F_{n-x,m-y}$, this time for $x=q$ and $y=e_q+e_r$ to obtain

$$|\mathcal{C}(n,m)| \leq 2^{8x^2\log n}\cdot\max_{\frac{1}{2}\epsilon_{\mathcal{A}}x\frac{m}{n}\leq y<70x\frac{m}{n}} e^{-\epsilon_{\mathcal{A}}(\epsilon_c-3\epsilon_{\mathcal{B}})x\frac{m}{n}}\cdot\binom{\frac{1}{2}xn}{y}\binom{\frac{1}{4}n^2-\frac{1}{2}nx}{m-y}\cdot 2^n\cdot 2^{\xi(n,m)+\frac{1700x}{\log b}\frac{m}{n}}$$

Using the definitions (1) one again easily observes that (8) is an upper bound for $|\mathcal{C}(n,m)|$.

The Set $\boldsymbol{\mathcal{D}(n,m)}$. All graphs in $\mathcal{D}(n,m)$ can be constructed as follows. First we choose an appropriate bipartition $V = V_1 \cup V_2$ (less than 2^n ways). Let e_v denote the number of edges inside the sets V_i. For the moment we will consider e_v as fixed. At the end we will sum over all possible values $1 \le e_v \le nq$. There are at most

$$\binom{\binom{|V_1|}{2} + \binom{|V_2|}{2}}{e_v} \le \binom{\frac{1}{2}n^2}{e_v} \le e^{2e_v \ln n}$$

ways to choose these edges. For further use we also note that we may assume without loss of generality that at least $\frac{1}{2}e_v$ of these edges are contained inside the set V_1.

We now have to choose a bipartite graph with $m - e_v$ vertices on the vertex set $V_1 \cup V_2$ — in such a way that no triangle is generated.

In order to do that we first consider a random bipartite graph on $V_1 \cup V_2$ with edge probability $p = (m - e_v)/(|V_1||V_2|)$. Note that the assumptions on the size of e_v and $|V_i|$ imply that $\frac{1}{5}\frac{m}{n^2} \le p \le \frac{1}{3}\frac{m}{n^2}$. For every combination of an edge $\{u, v\}$ inside V_1 and a vertex $w \in V_2$ (i.e., for every possible triangle with an edge inside V_1) we construct an indicator random variable I_α, $\alpha = \{\,\{u,w\},\{v,w\}\,\}$, which indicates whether the triangle is present. Let $\mathcal{S}$ denote the set of all such events. Note that

$$\mathbb{E}[I_\alpha] = \mathbb{P}[I_\alpha = 1] = p^2 \qquad \text{for all } \alpha \in \mathcal{S}$$

and that

$$\mathbb{E}[I_\alpha I_{\alpha'}] = \mathbb{P}[I_\alpha = 1 \cap I_{\alpha'} = 1] = p^3 \qquad \text{for all } \alpha, \alpha' \in \mathcal{S} \text{ s.t. } \alpha \cap \alpha' \neq \emptyset.$$

We intend to bound the probability that $X = \sum_\alpha I_\alpha$ is equal to zero. Using the large deviation result from Janson, Łuczak, Ruciński [2] we obtain

$$\mathbb{P}[X = 0] \le \begin{cases} e^{-\frac{1}{2}\mu} & \text{, if } \mu \ge \Delta \\ e^{-\mu^2/(2\Delta)} & \text{, if } \mu \le \Delta \end{cases} \le e^{-\min\{\frac{1}{2}\mu, \frac{\mu^2}{2\Delta}\}}$$

where

$$\mu = \mathbb{E}[X] \qquad \text{and} \qquad \Delta = \sum_{\substack{\alpha,\alpha' \\ \alpha\cap\alpha'\neq\emptyset}} \mathbb{E}[I_\alpha I_{\alpha'}].$$

We now bound μ and Δ.

$$\mu = \mathbb{E}[X] = |\mathcal{S}| \cdot p^2 \ge \frac{1}{2}e_v \cdot (\frac{1}{2} - \frac{1}{100})n \cdot p^2 \ge \frac{1}{200}e_v \cdot \frac{m^2}{n^3}$$

and

$$\Delta = \sum_{\substack{\alpha,\alpha' \\ \alpha\cap\alpha'\neq\emptyset}} \mathbb{E}[I_\alpha I_{\alpha'}] \le \sum_{v \in V_1} \binom{\deg_{V_1}(v)}{2} \cdot |V_2| \cdot p^3 \le \frac{1}{2}\min\{e_v^2, e_v q\} \cdot n \cdot p^3 \le \frac{1}{50}e_v \min\{e_v, q\} \cdot \frac{m^3}{n^5}.$$

(Note that we used here that the degree $\deg_{V_1}(v)$ of every vertex $v \in V_1$ inside the set V_1 is at most q and that $\sum_{v \in V_1} \deg_{V_1}(v) \le e_v$.) Hence, we have that

$$\frac{1}{2}\mu \ge \frac{1}{400}e_v\frac{m^2}{n^3} \ge \frac{1}{800}e_v\frac{m^2}{n^3} + \frac{1}{800}\frac{m^2}{n^3} \ge 11e_v \ln n + \frac{1}{800}\frac{m^2}{n^3},$$

(recall that $m \geq \sqrt{b} n^{3/2} \sqrt{\log n}$) and that

$$\frac{\mu^2}{2\Delta} \geq \frac{1}{800} \frac{e_v}{\min\{e_v, q\}} \frac{m}{n} \geq \frac{2\sqrt{b}}{1600} e_v \log n + \frac{1}{1600} \frac{m}{n} \geq 11 e_v \ln n + \frac{1}{800} \frac{m^2}{n^3}.$$

(recall that $q = \frac{1}{2\sqrt{b}} \frac{m}{n \log n}$ and $m \leq n^2/2$). Hence, we obtain

$$\mathbb{P}[X = 0] \leq e^{-\frac{1}{800} \frac{m^2}{n^3} - 11 e_v \ln n}.$$

The probability that such a random bipartite graph G has exactly $m - e_v$ edges is certainly at least $1/n^2$. That is, we have

$$\mathbb{P}[X = 0 \mid E(G) = m - e_v] \leq \frac{\mathbb{P}[X = 0]}{\mathbb{P}[E(G) = m - e_v]} \leq e^{-\frac{1}{800} \frac{m^2}{n^3} - 9 e_v \ln n}.$$

Together, we obtain

$$\begin{aligned} |\mathcal{D}(n, m)| &\leq 2^n \cdot \sum_{1 \leq e_v \leq nq} e^{2 e_v \ln n} \cdot e^{-\frac{1}{800} \frac{m^2}{n^3} - 9 e_v \ln n} \cdot \binom{\frac{1}{4} n^2}{m - e_v} \\ &\leq 2^n \cdot n^2 \cdot \max_{1 \leq e_v \leq nq} e^{-\frac{1}{800} \frac{m^2}{n^3} - 7 e_v \ln n} \cdot \binom{\frac{1}{4} n^2}{m} \cdot e^{2 e_v \ln n} \\ &\leq \max_{1 \leq e_v \leq nq} e^{-e_v \ln n - \frac{1}{800} \frac{m^2}{n^3}} \cdot S_{n,m} \;=\; e^{-\ln n - \frac{1}{800} \frac{m^2}{n^3}} \cdot S_{n,m}. \end{aligned}$$

Using the definitions (1) one easily observes that (7) is an upper bound for $|\mathcal{D}(n, m)|$.

□

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