K_4-free Subgraphs of Random Graphs Revisited

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July 31, 2002

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Abstract

In Combinatorica 17(2), 1997, Kohayakawa, Łuczak and Rödl state a conjecture which, if true, would permit the application of Szemerédi's regularity lemma for the estimation of $\mathrm{ex}(G_{n,p}, G)$, where $\mathrm{ex}(G_{n,p}, G)$ denotes the maximum number of edges in a G-free subgraph of a random graph $G_{n,p}$. In this paper we verify their conjecture for $G = K_4$, thereby providing a conceptually simple proof for the main result in the paper cited above.

Mathematics Subject Classification: 05C80, 05C35

1 Introduction

The Erdős-Stone Theorem [ES46], a classical result in extremal graph theory, states that for an arbitrary graph G

$$\mathrm{ex}(n, G) = \left(1 - \frac{1}{\chi(G) - 1} + o(1)\right)\binom{n}{2},$$

where $\mathrm{ex}(n, G)$ denotes the maximum number of edges that a G-free graph may have, and $\chi(G)$ is the chromatic number of G. In general, the quantity $\mathrm{ex}(F, G) := \max\{e(H) \mid G \not\subseteq H \subseteq F\}$ can be studied for an arbitrary graph F. Observe that $\mathrm{ex}(n, G) = \mathrm{ex}(K_n, G)$ for the complete graph K_n on n vertices.

A lot of effort has been made to estimate $\mathrm{ex}(G_{n,p}, G)$ for a binomial random graph $G_{n,p}$, and to prove an analogous result to the Erdős-Stone Theorem. It has been conjectured (cf. [HKŁ95], [HKŁ96], [KŁR97]) that $\mathrm{ex}(G_{n,p}, G)$ is approximately equal to $\mathrm{ex}(K_n, G)$ whenever the number of (not necessarily induced) copies of G in $G_{n,p}$ is substantially larger than the number of edges $e(G_{n,p})$. This should happen if $p \gg n^{-1/d_2(G)}$, where

$$d_2(G) := \max\left\{\frac{|E_H| - 1}{|V_H| - 2} \mid H = (V_H, E_H) \subseteq G, |V_H| \geq 3\right\}$$

denotes the *2-density* of G. For a precise statement of the conjecture the reader is referred to [KŁR97] or the monograph [JŁR00], which provides an overview of extremal properties of random graphs.

*Supported by DFG-Grant Ste 464/3-1

As far as complete graphs $G = K_\ell$ are concerned, nothing is known for the case $\ell \geq 5$. In fact, apart from the case $G = K_4$, which is treated in [KŁR97], only the case that G is a cycle has been verified. The corresponding results can be found in [Für94, HKŁ95, HKŁ96, KŁR96, KK97, KKS98, Kre97, KW82].

Recently, significant progress has been made towards a proof of the conjecture for arbitrary complete graphs $G = K_\ell$. In [KRS02] a result for arbitrary ℓ has been presented. However, this result assumes that $p \gg n^{-1/\ell}$, which is larger than the conjectured threshold $n^{-1/d_2(K_\ell)} = n^{-2/(\ell+1)}$.

The approach used in [KŁR97] for the estimate of $\mathrm{ex}(G_{n,p}, K_4)$ is technically quite involved and the proof is rather lengthy. As basis for an alternative proof strategy, the authors of [KŁR97] also state a conjecture which, if true, would imply the desired results on the behaviour of $\mathrm{ex}(G_{n,p}, G)$ (see Conjecture 2.3 below) for arbitrary G by a standard application of Szemerédi's regularity lemma for sparse graphs (see [Koh97, Łuc00, JŁR00, KR01]). In this paper we prove this conjecture for $G = K_4$, thereby providing a conceptually simple proof for the result in [KŁR97].

Outline of the paper In Section 2 we introduce our main result. Section 3 gives an outline of our proof strategy. In Section 4 we introduce some notation and prove a few technical lemmas. The following two sections then introduce two basic concepts which play a central rôle in our proof. Section 5 presents the concept of covers, multicovers and triangle covers, and Section 6 discusses a counting lemma which is used repeatedly thereafter. Section 7 contains the proof of the main theorem, and Section 8 concludes the paper with some remarks concerning the generalisation of our result to arbitrary complete graphs K_ℓ.

2 Main Result

We introduce the important concept of regularity and state the aforementioned conjecture.

Definition 2.1 (ε-regularity) *A bipartite graph $B = (U \dot\cup W, E)$ with $|U| = |W| = n$ and $|E| = m$ is called (ε, n, m)-regular if for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon n$ and $|W'| \geq \varepsilon n$,*

$$\left| |E(U', W')| - m \cdot \frac{|U'| \cdot |W'|}{n^2} \right| \leq \varepsilon m \cdot \frac{|U'| \cdot |W'|}{n^2}.$$

Definition 2.2 *Let H be a graph on ℓ vertices. An ℓ-partite graph $G = (V_1 \dot\cup \ldots \dot\cup V_\ell, E)$ is called $(H, n, m; \varepsilon)$*-regular *if $G[V_i, V_j]$ is (ε, n, m)-regular for all $\{i, j\} \in E(H)$, and there are no edges between V_i and V_j otherwise.*

Let $\mathcal{S}(H, n, m; \varepsilon)$ denote the set of $(H, n, m; \varepsilon)$-regular ℓ-partite graphs, and let

$$\mathcal{F}(H, n, m; \varepsilon) := \{G \in \mathcal{S}(H, n, m; \varepsilon) \mid H \not\subseteq G\}.$$

Although the following conjecture was first stated in [KŁR97] we cite a simplified version from [KR01].

Conjecture 2.3 *[KR01] Let H denote an arbitrary graph. For any $\beta > 0$ there exist constants $\varepsilon_0 > 0, C > 0, n_0 > 0$ such that*

$$|\mathcal{F}(H, n, m; \varepsilon)| \leq \beta^m \binom{n^2}{m}^{e(H)}$$

for all $m \geq Cn^{2-1/d_2(H)}$, $n \geq n_0$ and $0 < \varepsilon \leq \varepsilon_0$.

Since we concentrate on the case when H is a complete graph, we write $\mathcal{S}_\ell(n, m; \varepsilon) := \mathcal{S}(K_\ell, n, m; \varepsilon)$ and $\mathcal{F}_\ell(n, m; \varepsilon) := \mathcal{F}(K_\ell, n, m; \varepsilon)$. Using this notation we now state the main result of this paper.

Theorem 2.4 *For any $\beta > 0$ there exist constants $\varepsilon_0 > 0, C > 0, n_0 > 0$ such that*

$$|\mathcal{F}_4(n, m; \varepsilon)| \leq \beta^m \binom{n^2}{m}^6$$

for all $m \geq Cn^{8/5}$, $n \geq n_0$ and $0 < \varepsilon \leq \varepsilon_0$.

3 Outline of the Proof

Let $G = G(n, m, 4)$ be a random subgraph of the complete 4-partite graph $K_{n,n,n,n}$ with n vertices in each partition and with m edges between each pair of partitions. As an alternative we may consider a binomial random subgraph $G = G(n, p, 4)$ of $K_{n,n,n,n}$ in which all edges are present independently with probability $p := m/n^2$. It is easy to see that for $m = \omega(n)$ such random graphs are ε-regular with high probability. Thus, random graphs provide a good approximation of ε-regular graph which helps us to gain some intuition on the structure of the latter.

Let $q := m/n$. Note that in a random graph $G(n, p, 4)$, the neighbourhood of a vertex v in one of the partitions has expected size q, since $pn = (m/n^2)n = q$. Intuitively, conditioning on the ε-regularity of G should even increase the probability that the size of a neighbourhood is close to its expectation because the edges in ε-regular graphs are very evenly spread, i.e., the ε-regularity helps to avoid untypical cases which may occur in random graphs (although with small probability). Based on this intuition, which can clearly be transferred to other sets than neighbourhoods, e.g., to the common neighbourhood of two vertices, we will now introduce the crucial ideas behind our proof strategy.

Finding K_4-candidates For every vertex $v \in V_1$, we want to find subgraphs which can be turned into a K_4 by adding a single edge from v to some vertex $v' \in V_4$. We call such a vertex v' a *clique candidate* of v. Note that we do not insist on a clique candidate v' being adjacent to v. We show that there are $\Omega(n)$ vertices $v \in V_1$ each of which has almost n clique candidates in V_4. Since it is very unlikely that the neighbourhood of v does not intersect the big set of clique candidates, this will suffice to prove Theorem 2.4.

Figure 1 illustrates how the clique candidates in V_4 are found. As discussed earlier, a vertex $v \in V_1$ should have approximately q neighbours in V_2 and V_3. We denote the neighbourhood of v in V_i by $\Gamma_i(v)$. Consider a vertex $w \in \Gamma_2(v)$. We expect that $|\Gamma_3(v) \cap \Gamma_3(w)| \approx pq = (m/n^2)(m/n) = m^2/n^3$. The same argument shows that for every vertex $u \in \Gamma_3(v) \cap \Gamma_3(w)$ there are approximately m^2/n^3 vertices in $\Gamma_4(u) \cap \Gamma_4(w)$. These m^2/n^3 vertices are clique candidates of v. Note that the same arguments still hold, as far as the order of magnitude of the results is concerned, if we forbid at most, say, $(1-\xi)n$ vertices in V_4, provided that ξ is not too tiny. Hence, we may find m^2/n^3 clique candidates for every vertex in $\Gamma_3(v) \cap \Gamma_3(w)$, yielding a total of $(m^2/n^3)^2$ clique candidates. Finally, we repeat this process for the q vertices in $\Gamma_2(v)$, and get $(m^2/n^3)^2 q = (m^4/n^6)(m/n) = m^5/n^7$ clique candidates in V_4. If $m \geq Cn^{8/5}$ for C sufficiently large we may thus expect that almost all vertices in V_4 are clique candidates of v.

Introducing bad sets From a more technical point of view our proof strategy proceeds as follows. We define families of 'bad' graphs, i.e., graphs with 'unusual' properties. By estimating the number of such graphs we show that the overwhelming majority of graphs does not possess these properties. Hence we collect more and more information on structural properties of typical ε-regular graphs. Based on this knowledge we identify certain substructures in the graph which will, in the end, lead to K_4-candidates. As a red thread which may guide the reader through the proof we briefly sketch the structural properties that we use.

Firstly, as an immediate (and well-known) consequence of ε-regularity, the *degrees* of single vertices are approximately equal to their expectation q.

Secondly, the neighbourhood of sets with cardinality $\Theta(n^2/m)$ and $\Theta(m/n) = \Theta(q)$ is examined by introducing the notions of *covers* and *multicovers*. In particular, it turns out the neighbourhood of a single vertex in a set of cardinality $\Theta(q)$ has size $\Theta(q \cdot m/n^2) = \Theta(m^2/n^3)$, as one may expect due to the affinity with random graphs $G(n, m, 4)$.

Thirdly, we prove that a set of size $\Theta(m^2/n^3)$ and a single vertex have a common neighbourhood of size $t = \Omega(n^2/m)$. Note that the expectation in a random graph $G(n, m, 4)$ amounts to $\Theta(m^2/n^3 \cdot m/n^2) = \Theta(m^3/n^5)$ and for $m = \omega(n^{8/5})$ this is larger than t. The results of this step correspond to the auxiliary notion of a *triangle candidate cover*.

In a fourth step, we then combine multicovers and triangle candidate covers. Consider again a vertex $v \in V_1$ and its neighbourhood $\Gamma_3(v)$ of size $\Theta(q)$. We prove that a single vertex $w \in V_2$ indeed has a neighbourhood of size $\Theta(m^2/n^3)$ inside $\Gamma_3(v)$. This follows from the properties of a multicover. Moreover, these $\Theta(m^2/n^3)$ vertices close $\Omega(n^2/m)$ triangles (between the partitions V_2, V_3, and V_4), as implied by the properties of a triangle candidate cover. Together, this will enable us to introduce *triangle covers*, which show that almost all vertices in V_2 close many triangles.

Now the fifth step is straightforward. We prove that the neighbourhood $\Gamma_2(v)$ and the many vertices in the triangle cover intersect. This finally yields the desired K_4*-candidates*, since only the edge between V_1 and V_4 is missing from the structure. By iterating the process for $\Theta(q)$ vertices in $\Gamma_2(v)$ we get $\Theta(q \cdot t) = \Theta(m/n \cdot n^2/m) = \Theta(n)$ K_4-candidates.

As indicated above, a simple sixth step of the proof shows that this implies the existence of subgraphs K_4 for all but a few 'bad' graphs.

Due to technical reasons, i.e., as to obtain a sufficiently small probability, one major deviation from the scheme sketched above is necessary. When showing the existence of triangle covers we have to examine the neighbourhoods of many vertices in V_1 at once. This makes the proof a bit more hairy than the rather clear strategy might suggest.

Technical difficulties Of course, this outline of the proof is very rough and neglects several important aspects. Firstly, we must deal with random ε*-regular* graphs. This rules out many tools from probability theory which are formulated for product spaces. Hence, most of the time we will directly count the number of graphs with given properties. Secondly, a little deviation from the expectation has to be tolerated and corresponding error terms must be taken into account. Thirdly, so far we have only discussed the order of magnitude of the results, whereas the choice of suitable constants represents a delicate part of the proof. Finally, the order in which the neighbourhoods and intersections of neighbourhoods are considered is crucial because dependencies would spoil our proof strategy.

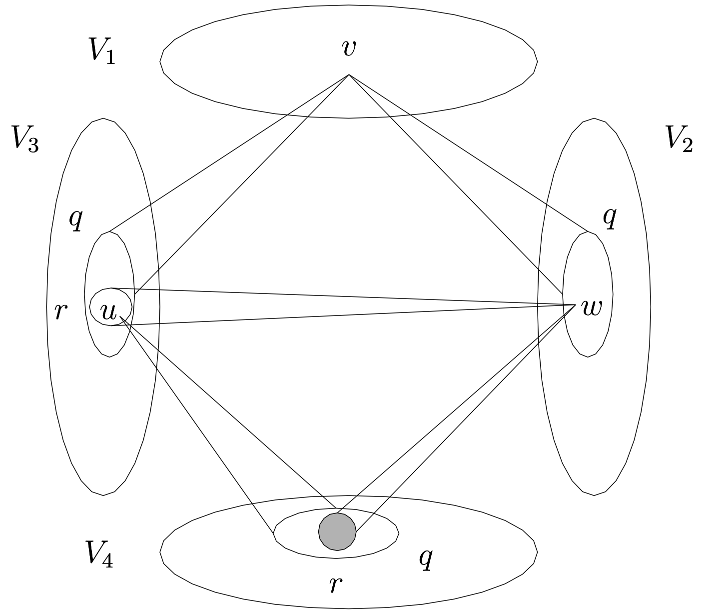

Figure 1: Idea for construction of K_4-candidates.

4 Preliminaries

4.1 Conventions and notation

In order to increase the clarity of the presentation we make use of the following conventions:

All constants in the paper which are denoted by greek letters are tacitly assumed to be smaller than, say, 10^{-3}. Furthermore, ε will always be smaller than all other constants.

We will not introduce floors and ceilings when we are talking about integral terms, e.g., cardinalities of sets. Since we are only interested in the asymptotic behaviour of those quantities, this would merely introduce lower order error terms which complicate the exposition. However, it would be a standard but laborious task to modify the proofs such that the integrality of all terms is respected.

The neighbourhood of a vertex v is denoted by $\Gamma(v)$. We will use the abbreviations $\Gamma_i(v) := \Gamma(v) \cap V_i$ and $d_i(v) := |\Gamma_i(v)|$.

For technical reasons, i.e., as to gain control over the size of certain sets, we will often need to deterministically fix a subset of a given cardinality in a larger set of vertices. In order to do so we assume that the vertices in V_i are ordered in an arbitrary but unique way (e.g. we may assume that $V_i = \{1, \ldots, n\}$). By $[A]_x$ we denote the set $B \subseteq A$ of size $|B| = x$ that contains the x smallest elements in A. If $|A| < x$, we define $[A]_x := A$.

4.2 Technical lemmas

This section contains a collection of auxiliary results with a rather technical flavour. Hence, it may be skipped at first reading and then be consulted on demand.

Lemma 4.1 (Technical inequalities for binomial coefficients)

(i)

$$\binom{n}{a}\binom{n-a}{b-a} = \binom{n}{b}\binom{b}{a} \leq \binom{n}{b}2^b.$$

(ii) If $0 \le a+b \le 0.9n$ *then*

$$\binom{n}{b} \le \binom{n-a}{b} e^{10b}.$$

(iii) If $0 \le x \le 1$ *and* $b(1+1/x) \le 0.9a$ *then*

$$\binom{a}{b} \cdot (x \cdot e^{-10})^b \le \binom{xa}{b} \le \binom{a}{b} \cdot x^b.$$

Proof (i) is clear due to combinatorial arguments. Instead of choosing first a elements and then $b-a$ elements, we can choose b elements and then mark a elements of the b elements.

Using $1 - z \ge e^{-10z}$ for $z \le 0.9$ we conclude that

$$\frac{\binom{n}{b}}{\binom{n-a}{b}} = \frac{n^{\underline{b}}}{(n-a)^{\underline{b}}} \le \frac{n^b}{(n-a-b)^b} = \left(1 - \frac{a+b}{n}\right)^{-b} \le e^{10(a+b)b/n} \le e^{10b},$$

proving (ii) ($n^{\underline{z}}$ denotes the z-th falling factorial of n).

For the proof of (iii) we observe that

$$\begin{aligned} \binom{xa}{b} &= \frac{(xa)^{\underline{b}}}{b!} = x^b \cdot \frac{a(a-\frac{1}{x})\dots(a-\frac{b-1}{x})}{b!} \ge x^b \cdot \frac{(a-b/x)^{\underline{b}}}{b!} \\ &= x^b \binom{a-b/x}{b} \overset{(ii)}{\ge} \binom{a}{b} \cdot x^b e^{-10b} \end{aligned}$$

This proves the first inequality. The second inequality is obvious. $\square$

The following lemma is inspired by a very simple observation: If in a bipartite graph $G = (U \dot\cup W, E)$ all vertices in U have large degree, then there must also be many vertices in W with large degree. This is proved by an easy counting argument.

Lemma 4.2 (Overlap lemma) *Let* $\alpha > 0$*, and let* $G = (U \dot\cup W, E)$ *be a bipartite graph with* $d(u) \ge \alpha|W|$ *for all* $u \in U$*. Then for all* $\beta > 0$*,*

$$|\{w \in W \mid d(w) \ge \beta|U|\}| \ge \frac{\alpha-\beta}{1-\beta}|W|.$$

Proof Assume for a contradiction that there are less than $\frac{\alpha-\beta}{1-\beta}|W|$ vertices in W with degree at least $\beta|U|$. It follows that

$$|E| < \frac{\alpha-\beta}{1-\beta}|W| \cdot |U| + \left(1 - \frac{\alpha-\beta}{1-\beta}\right)|W| \cdot \beta|U| = \alpha|W| \cdot |U|.$$

On the other hand it is clear by the lower bound on the degree of the vertices in U that $|E| \ge \alpha|W| \cdot |U|$ and we get a contradiction. $\square$

Although the idea behind Lemma 4.2 is quite simple, the following corollary will become a key stone in the proof of our main result.

Corollary 4.3 *Let* $\varepsilon > 0$*. If a bipartite graph* $G = (U \dot\cup W, E)$ *satisfies* $d(u) \ge (1-\varepsilon)|W|$ *for all* $u \in U$*, then*

$$|\{w \in W \mid d(w) \ge (1-\sqrt{\varepsilon})|U|\}| \ge (1-\sqrt{\varepsilon})|W|.$$

Proof Set $\alpha = 1 - \varepsilon$ and $\beta = 1 - \sqrt{\varepsilon}$ in Lemma 4.2. $\square$

The following lemma is a simple consequence of the definition of ε-regularity.

Lemma 4.4 (Degree lemma) *Consider a graph $B = (U \dot\cup W, E) \subseteq \mathcal{S}_2(n, m; \varepsilon)$ and a set $W' \subseteq W$ with $|W'| \geq \varepsilon n$. For $q := m/n$ we define*

$$X^< := \left\{ u \in U \mid |\Gamma(u) \cap W'| < (1-\varepsilon) q \cdot \frac{|W'|}{n} \right\}.$$

and $X^>$ analogously. Then $|X^<|, |X^>| < \varepsilon n$.

Proof We only consider the case that $|X^<| \geq \varepsilon n$, since the case $|X^>| \geq \varepsilon n$ is almost identical. By the definition of $X^<$ we know that

$$|E(X^<, W')| < (1-\varepsilon) q \cdot \frac{|W'|}{n} \cdot |X^<| = (1-\varepsilon) \frac{|X^<| \cdot |W'|}{n^2} \cdot m.$$

This obviously contradicts the (ε, n, m)-regularity of B. $\square$

5 Covering sets

5.1 Abbreviations

In this section we will need quite a few abbreviations, which we collect here for easier reference. The following quantities are defined as functions of n, m and a parameter $\lambda > 0$. This parameter will later be replaced by one of several constants.

$$q := \frac{m}{n}, \tag{1}$$

$$p_\lambda := \frac{2}{\lambda} \frac{n^2}{m}, \tag{2}$$

$$q_\lambda := (1-\lambda)\lambda q, \tag{3}$$

$$t_\lambda := \frac{1}{\lambda} \frac{n^2}{m}, \tag{4}$$

$$r_\lambda(x) := \frac{x}{2p_\lambda}. \tag{5}$$

The following quantities are also functions of one or two parameters $\nu, \mu > 0$. In order to avoid clumsy notation we will often drop these dependencies when it will be clear from the context where the values for ν and μ come from.

$$\tilde{r}_1 = \tilde{r}_1(\nu, \mu) := r_\nu(\mu^2 q) = \frac{\mu^2 \nu}{4} \frac{m^2}{n^3}, \qquad \tilde{r}_2 = \tilde{r}_2(\mu) := r_\mu(\mu^2 q) = \frac{\mu^3}{4} \frac{m^2}{n^3}, \tag{6}$$

$$\tilde{p} = \tilde{p}(\mu) := \frac{p_\mu}{4} = \frac{1}{2\mu} \frac{n^2}{m} \tag{7}$$

Note the redundancy in this notation, as for example $p_\lambda = 2t(\lambda)$. However, this is just a coincidence as we will use these two abbreviations for completely different purposes. Therefore, this redundant notation is intended to remind the reader where the various quantities come from.

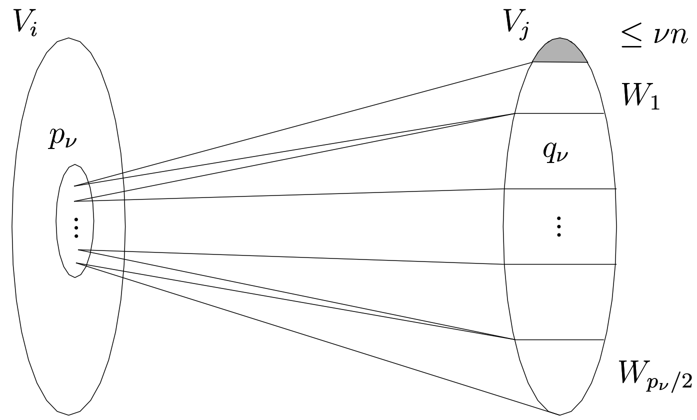

Figure 2: A ν-cover.

5.2 Simple covers

In a random graph with edge probability m/n^2 a set of size n^2/m has $(n^2/m)\cdot n\cdot(m/n^2) = n$ incident edges on the average. Hence, we may expect that a set P of size $\Theta(n^2/m)$ in partition V_i covers a partition V_j for $j \neq i$, i.e., $\Gamma_j(P) \approx V_j$. Such covering sets will play a crucial rôle in our proof.

For technical reasons it is beneficial to concentrate on covers with a special structure. Informally speaking, we say that $P = \{v_1, \ldots, v_p\}$ covers V_j if we can find disjoint sets $W_1, \ldots, W_{p/2} \subseteq V_j$ of equal size such that $W_i \subseteq \Gamma(v_i)$ and $\bigcup_{i=1}^{p} W_i \approx V_j$. This situation is visualised in Figure 2.

The following definition provides a formalisation of this approach.

Definition 5.1 (Covers) *Consider a graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$, $\{i, j\} \subseteq \{1, \ldots, \ell\}$ and $\nu > 0$. A set $P \subseteq V_i$ is called a* ν-cover *of V_j if there exists a subset $P^* \subseteq P$ such that $|P^*| \geq |P|/2$ and such that one can find pairwise disjoint sets $W_v \subseteq \Gamma_j(v)$ with $|W_v| = q_\nu$ for all $v \in P^*$. These sets are called* covering neighbourhoods.

Mostly we will be interested in ν-covers with cardinality $|P| = p_\nu$. Note that then

$$|\Gamma_j(P)| \geq \left| \bigcup_{v \in P^*} W_v \right| \geq (p_\nu/2) \cdot q_\nu = (1 - \nu)n. \tag{8}$$

Definition 5.2 (Sets of covers) *Consider a graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$ and $\{i, j\} \subseteq \{1, \ldots, \ell\}$. Let $\mathcal{P}_{i,j}(G; \nu) \subseteq \binom{V_i}{p_\nu}$ denote the family of sets in V_i of cardinality p_ν which form a ν-cover of V_j. Furthermore, let $\bar{\mathcal{P}}_{i,j}(G; \nu) := \binom{V_i}{p_\nu} \setminus \mathcal{P}_{i,j}(G; \nu)$.*

The following lemma shows that almost all sets of size p_ν (up to a superexponentially small fraction) are ν-covers. This lemma closely resembles Lemma 11 in [KŁR96].

Lemma 5.3 ($\bar{\mathcal{P}}$ is small) *For $\pi, \nu > 0$, there exists a constant $\varepsilon^P = \varepsilon^P(\pi, \nu)$ such that for any pair $\{i, j\} \subseteq \{1, \ldots, \ell\}$, any graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$ with $\varepsilon \leq \varepsilon^P$, $m \geq 4(\nu\varepsilon)^{-1} n$, and any $V' \subseteq V_i$ with $|V'| \geq \sqrt{\varepsilon} n$,*

$$|\bar{\mathcal{P}}_{i,j}(G; \nu) \cap \binom{V'}{p_\nu}| \leq \pi^{p_\nu} \binom{|V'|}{p_\nu},$$

for n sufficiently large.

Proof It suffices to show that

$$\Pr[P \in \bar{\mathcal{P}}_{i,j}(G;\nu)] \leq \pi^{p_\nu}$$

for a randomly chosen set $P \subseteq V'$ with $|P| = p_\nu$. We will assume that the set P is ordered, although the claim to be proved calls for the probability space of unordered sets. But note that this difference does not matter, since the event '$P \in \bar{\mathcal{P}}_{i,j}(G;\nu)$' does not depend on the ordering of the vertices in P.

We choose the vertices in $\{v_1, \dots, v_{p_\nu}\} = P$ one by one, uniformly at random from V'. While doing this, we try to fix pairwise disjoint sets $W_{v_1}, \dots, W_{v_{p_\nu}}$ with (according to Definition 5.1) $|W_{v_i}| = q_\nu$. Assume that we have already selected the vertices $v_1, \dots, v_k$. Let $N_k := (W_{v_1} \cup \dots \cup W_{v_k})$. Pick a vertex v_{k+1}. If it has at least q_ν neighbours in $V_j \setminus N_k$, we can find enough vertices for $W_{v_{k+1}}$; otherwise, we set $W_{v_{k+1}} = \emptyset$, and call v_{k+1} a bad vertex. Let P^* denote the set of good vertices in V'. If $|P^*| \geq p_\nu/2$, then we have constructed an assignment of W-sets which satisfies Definition 5.1 and the set P is a ν-cover. Thus,

$$\Pr[P \in \bar{\mathcal{P}}_{i,j}(G;\nu)] \leq \Pr[|P^*| < p_\nu/2].$$

It remains to show that the latter probability is very small. Indeed, if there are less than $p_\nu/2$ good vertices in $v_1, \dots, v_k$, then

$$|N_k| < p_\nu/2 \cdot q_\nu = (1-\nu)n.$$

For $\varepsilon \leq \nu$, we can apply Lemma 4.4 with $W' := V_j \setminus N_k$ to obtain that

$$B_k := \{v \in V_i \mid |\Gamma_j(v) \cap (V_j \setminus N_k)| < (1-\varepsilon)\nu q\}$$

satisfies $|B_k| \leq \varepsilon n$. Also $(1-\varepsilon)\nu q > (1-\nu)\nu q = q_\nu$, i.e., any bad vertex lies in B_k. Note that $p_\nu = 2\nu^{-1}\frac{n^2}{m} \leq \varepsilon n/2 \leq \sqrt{\varepsilon} n/2$ by choice of m. Since

$$|V' \setminus \{v_1, \dots, v_k\}| \geq \sqrt{\varepsilon} n - p_\nu \geq \sqrt{\varepsilon} n/2$$

for every $1 \leq k \leq p_\nu$, the probability for a particular choice of (at least $p_\nu/2$) bad vertices is at most

$$\left(\frac{\varepsilon n}{\sqrt{\varepsilon} n/2}\right)^{p_\nu/2} \leq (2\varepsilon^{1/4})^{p_\nu}$$

and we can conclude that

$$\Pr[P \in \bar{\mathcal{P}}_{i,j}(G;\nu)] \leq \binom{p_\nu}{p_\nu/2} \cdot (2\varepsilon^{1/4})^{p_\nu} \leq 2^{p_\nu} \cdot (2\varepsilon^{1/4})^{p_\nu} \leq (4\varepsilon^{1/4})^{p_\nu}.$$

By choosing ε^P suitably small, the claim follows. □

5.3 Multicovers

In the previous section we have introduced simple covers of size $p_\nu = \Theta(n^2/m)$, whereas in our proof we will often be concerned with sets of size $\Theta(q) = \Theta(m/n)$, i.e., neighbourhoods of vertices. The following definition therefore transfers the notion of a cover to such (larger) sets.

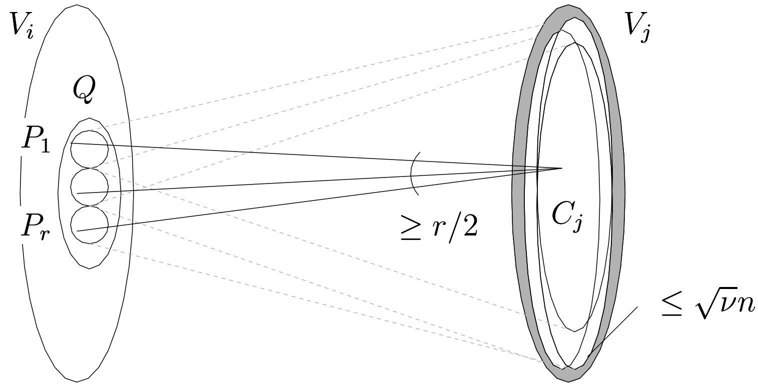

Figure 3: ν-multicover

Definition 5.4 (Multicovers) *Consider a graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$, $\{i, j\} \subseteq \{1, \ldots, \ell\}$ and $\nu > 0$. We call a set $Q \subseteq V_i$ a* ν-multicover *of V_j if there exist pairwise disjoint subsets $P_1, \ldots, P_r \subseteq Q$ such that*

(i) $r = r_\nu(|Q|) := \frac{|Q|}{2p_\nu}$,

(ii) $|P_i| = p_\nu$ *for all* $i = 1, \ldots, r$, *and*

(iii) P_i *is a* ν*-cover of* V_j *for all* $i = 1, \ldots, r$.

Furthermore, we let $Q^*(\nu) := \bigcup_{k=1}^{r} P_k^*$, where the sets P_k^* are defined as in Definition 5.1 and the parameter ν indicates that Q shall be decomposed into ν-covers P_k in an arbitrary but fixed way.

Figure 3 visualises the structure of a ν-multicover.

Consider the case when Q is a multicover of size $\Theta(m/n)$. Observe that for a vertex $w \in V_j$ the average number of neighbours within Q is $\Theta((m/n) \cdot (m/n^2)) = \Theta(m^2/n^3) = \Theta(r_\nu(|Q|))$. The following definition denotes by $C_j(Q; \nu)$ the set of vertices in V_j which (up to an appropriately chosen multiplicative constant) have at least as many neighbours in Q as expected.

Definition 5.5 (Covered neighbourhood of Q-sets) *Consider a graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$, $\{i, j\} \subseteq \{1, \ldots, \ell\}$ and $\nu > 0$. For a ν-multicover $Q \subseteq V_i$ of V_j, we define*

$$C_j(Q; \nu) := \{w \in V_j \mid |\Gamma_i(w) \cap Q| \geq r_\nu(|Q|)/2\}.$$

Lemma 5.6 (Large covered neighbourhoods) *Given a graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$ let $Q \subseteq V_i$ be a ν-multicover of V_j for some $\nu > 0$. Then*

$$|C_j(Q; \nu)| \geq (1 - \sqrt{\nu})n.$$

Proof Let $P_1, \ldots, P_r$ with $r := r_\nu(|Q|)$ denote pairwise disjoint sets of size p_ν in Q which are ν-covers of V_j. The following represents a typical application of Corollary 4.3. Consider the auxiliary bipartite graph $B = (\{P_1, \ldots, P_r\} \dot\cup V_j, E_B)$, where $\{P_k, w\} \in E_B$ if and only if $w \in \Gamma_j(P_k)$ in G. $P_1, \ldots, P_r$ can be interpreted as 'super-vertices' in G, which are obtained by merging the vertices in the sets P_i into a single vertex. By (8) we have

$|\Gamma_j(P_k)| \geq (1-\nu)n$, hence all vertices P_k have degree at least $(1-\nu)n$ in B. Thus, we can apply Corollary 4.3 to obtain that

$$Z := \{z \in V_j \mid d_B(w) \geq (1-\sqrt{\nu})r\}$$

satisfies $|Z| \geq (1-\sqrt{\nu})n$. If $z \in Z$, then

$$\Big|\{k = 1, \ldots, r \mid w \in \Gamma_j(P_k)\}\Big| \geq (1-\sqrt{\nu})r \geq r/2 = r_\nu(|Q|)/2,$$

and thus $z \in C_j(Q;\nu)$. Hence $Z \subseteq C_j(Q;\nu)$, which concludes the proof of the lemma. □

Let $p_\nu \leq s \leq q$. Note that $p_\nu \leq q$ for $m \geq \nu^{-1}n^{3/2}$. In the following we show that, analogously to Lemma 5.3, almost all sets of size s in V_i form a ν-multicover of V_j.

Definition 5.7 (Bad multicovers) *Let $s \in \mathbb{N}$ and $\nu > 0$, and consider $G \in \mathcal{S}_\ell(n,m;\varepsilon)$ and $\{i,j\} \subseteq \{1,\ldots,\ell\}$. Let $\mathcal{Q}_{i,j}(G;s,\nu)$ denote the family of sets in $\binom{V_i}{s}$ which form a ν-multicover of V_j. Furthermore, we define $\bar{\mathcal{Q}}_{i,j}(G;s,\nu) := \binom{V_i}{s} \setminus \mathcal{Q}_{i,j}(G;s,\nu)$.*

Lemma 5.8 ($\bar{\mathcal{Q}}$ is small) *For $\pi, \nu > 0$, there exists a constant $\varepsilon^Q = \varepsilon^Q(\pi,\nu)$ such that for any pair $\{i,j\} \subseteq \{1,\ldots,\ell\}$, and any graph $G \in \mathcal{S}_\ell(n,m;\varepsilon)$ with $\varepsilon \leq \varepsilon^Q$, $\nu^{-1}n^{3/2} \leq m \leq n^2/4$, and any s with $p_\nu \leq s \leq q$,*

$$|\bar{\mathcal{Q}}_{i,j}(G;s,\nu)| \leq \pi^s \binom{n}{s},$$

for n sufficiently large.

Proof We proceed analogously to the proof of Lemma 5.3 and show that

$$\Pr[Q \in \bar{\mathcal{Q}}_{i,j}(G;s,\nu)] \leq \pi^s$$

for a randomly chosen set $Q \subseteq V_i$ with $|Q| = s$. As far as the underlying probability space is concerned, we let $r' := s/p_\nu$ and assume that disjoint sets $P_1, \ldots, P_{r'}$ of size p_ν are chosen one after another uniformly at random without replacement. Note that as in the proof of Lemma 5.3 we do not consider the probability space of unordered sets of size s. However, the event '$Q \in \bar{\mathcal{Q}}_{i,j}(G;s,\nu)$' does not depend on the ordering which we introduce by choosing the sets $P_1, \ldots, P_{r'}$ in turn and thus, we get the same probability.

If $Q = P_1 \cup \ldots \cup P_{r'}$ belongs to $\bar{\mathcal{Q}}_{i,j}(G;s,\nu)$ then at least $r'/2$ sets P_k must belong to $\bar{\mathcal{P}}_{i,j}(G;\nu)$, since otherwise we have found $r'/2 = s/(2p_\nu) = r_\nu(|Q|)$ P-sets in Q which satisfy Definition 5.4 and so Q is a ν-multicover. Applying Lemma 5.3 repeatedly we obtain for $\bar{\pi} > 0$ and $\varepsilon < \varepsilon^P(\bar{\pi},\nu)$ that

$$\Pr[Q \in \bar{\mathcal{Q}}_{i,j}(G;s,\nu)] \leq 2^{r'} \left(\frac{\bar{\pi}^{p_\nu}\binom{n}{p_\nu}}{\binom{n-s}{p_\nu}} \right)^{r'/2} \overset{4.1(ii)}{\leq} 2^{r'} \cdot (\bar{\pi}^{p_\nu} e^{10p_\nu})^{r'/2} \leq (2e^5 \cdot \bar{\pi}^{1/2})^s.$$

By choosing $\bar{\pi}$ sufficiently small, the claim follows. □

Before discussing further properties and applications of multicovers we need the following simple definition which is used to control the overlap between a family of sets.

Definition 5.9 (Quasidisjoint sets) *A family of sets $A_1, \ldots, A_b \subseteq V$ is called* s-quasidisjoint *if*

$$|\{i \in \{1,\ldots,b\} \mid x \in A_i\}| \leq s \quad \textit{for all } x \in V.$$

Consider a ν-multicover Q with $Q^* = \bigcup_{k=1}^r P_k^*$ and $r = r_\nu(|Q|)$. For a fixed P_k^*, all neighbourhoods W_u for $u \in P_k^*$ are disjoint (cf. Definition 5.1). Since there are only r sets P_k^*, we conclude that the sets W_v, $v \in Q^*$ are r-quasidisjoint.

In our subsequent proofs we will often exclude a subset X of vertices in a partition. The following definition and lemma show that the structure of a multicover remains intact (up to constants) if this set X is not too big.

Definition 5.10 (Resistant multicovers) *A ν-multicover $Q \subseteq V_i$ of V_j is called* X-resistant *for $X \subseteq V_j$ if there exists a subset $Q^{**} \subseteq Q$ with $|Q^{**}| = \nu^2 q$ such that for the vertices $v \in Q^{**}$ there exist $r_\nu(|Q|)$-quasidisjoint* covering neighbourhoods *$W_v' \subseteq \Gamma_j(v) \setminus X$ with $|W_v'| = \nu^2 q$.*

Lemma 5.11 (All multicovers are resistant) *Consider a graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$, $\{i, j\} \subseteq \{1, \ldots, \ell\}$, $X \subseteq V_j$ with $|X| \leq (1 - 2\sqrt{\nu})n$, and a sufficiently small constant $\nu > 0$. Any ν-multicover $Q \subseteq V_i$ of V_j with $|Q| = q_\nu$ is also X-resistant.*

Proof We count the number of occurrences of vertices $x \in V_j$ in covering neighbourhoods W_u of the ν-multicover Q. More precisely, if a vertex x belongs to a covering neighbourhood W_u, such that u belongs to one of the ν-covers P which make up the ν-multicover, then this corresponds to one such occurrence.

Let us go back to the proof of Lemma 5.6. We showed there that $C_j(Q; \nu)$ is large by examining the set $Z \subseteq C_j(Q; \nu)$. Now we consider this set Z again. Every vertex $z \in Z$ corresponds to at least $(1 - \sqrt{\nu}) r_\nu(|Q|) \geq r_\nu(|Q|)/2$ occurrences of z in covering neighbourhoods.

Since $|Z| \geq (1 - \sqrt{\nu})n$, it follows that $|Z \setminus X| \geq \sqrt{\nu} n$. We conclude that at least

$$\sum_{u \in Q^*} |W_u \setminus X| \geq \sqrt{\nu} n \cdot r_\nu(|Q|)/2 \geq \frac{1}{8} \nu^{3/2} \frac{m}{n} |Q| \tag{9}$$

occurrences remain if we restrict the sets W_u to vertices in $V_j \setminus X$.

Let $Q' := \{u \in Q^* \mid |W_u \setminus X| \geq \nu^2 q\}$ and assume that $|Q'| \leq \nu^2 q$. Then we can bound the number of these occurrences by

$$\sum_{u \in Q^*} |W_u \setminus X| \quad \leq \quad |Q \setminus Q'| \cdot \nu^2 q + |Q'| \cdot q_\nu = |Q| \nu^2 q + \nu^2 q |Q| \leq 2\nu^2 \frac{m}{n} |Q|.$$

This obviously contradicts (9) for ν sufficiently small.

Finally observe that the he sets W_u' are $r_\nu(|Q|)$-quasidisjoint since they are subsets of the covering neighbourhoods W_u which are already $r_\nu(|Q|)$-quasidisjoint. □

The concepts introduced so far – covers and (resistant) multicovers – are only concerned with edges between *two* classes V_i and V_j. In order to get closer to finding clique candidates we have to consider structures between *three* classes, too. The following definition introduces multicovers where the covering neighbourhoods are themselves multicovers.

Definition 5.12 (Qualified multicovers) *X-resistant μ-multicover $Q \subseteq V_i$ of V_j is called* ν-qualified *for V_k if there is a choice of $r_\mu(|Q|)$-quasidisjoint covering neighbourhoods $W_1', \ldots, W_{\mu^2 q}'$ of cardinality $\mu^2 q$ such that all these sets are themselves ν-multicovers of V_k.*

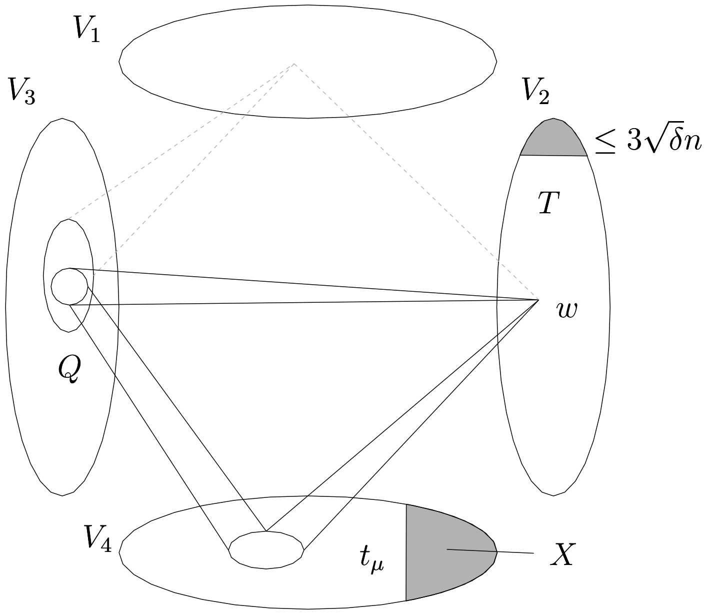

Figure 4: (μ, δ)-triangle cover

5.4 Triangle covers

Informally speaking, a *triangle cover* is a set $Q \subseteq V_3$ which has many vertices $w \in V_2$ such that each w spans many triangles between itself, Q and $V_4 \setminus X$ for some forbidden set X (cf. Figure 4).

Definition 5.13 (Triangle covers) *Let $\delta > 0$. Consider $G \in \mathcal{S}_4(n, m; \varepsilon)$, and $X \subseteq V_4$. We call a set $Q \subseteq V_3$ an* X-resistant (μ, δ)-triangle cover *of V_2 via V_4 if there exists a set $T = T(Q) \subseteq V_2$ with $|T(Q)| \geq (1 - 3\sqrt{\delta})n$ such that*

$$\forall w \in T: \ |TRI_4(w, Q) \setminus X| \geq t_\mu,$$

where $t_\mu := \mu^{-1}\frac{n^2}{m}$ and

$$TRI_4(w, Q) := \Gamma_4(w) \cap \Gamma_4(\Gamma_3(w) \cap Q).$$

Figure 4 illustrates the structure of triangle covers. Later in the proof we will consider triangle covers Q with $Q \subseteq \Gamma_3(v)$ for a vertex $v \in V_1$, and we will be interested in finding vertices $w \in T \cap \Gamma_2(v)$. Note that such vertices w complete t_μ K_4-candidates with v.

Unfortunately, we are not able to prove directly that there exist many triangle covers. Instead we are taking a little detour via so-called *triangle candidate covers*, defined below. For a set $X \subseteq V_4$, we fix a set $\tilde{Q} \subseteq Q$ with the following properties: Almost all vertices $u \in \tilde{Q}$ have a large common neighbourhood in $V_4 \setminus X$ with almost all vertices in V_2. Observe that a priori nothing is required about the edges between Q and w.

Definition 5.14 (Triangle candidate covers) *Consider a graph $G \in \mathcal{S}_4(n, m; \varepsilon)$, and $X \subseteq V_4$. A set $Q \subseteq V_3$ with $|Q| = q_\mu$ is called an* X-resistant (ν, μ)-triangle candidate cover *of V_2 via V_4 if there exist sets $\tilde{Q} \subseteq Q$ with $|\tilde{Q}| = \mu^2 q$ and $\tilde{T} = \tilde{T}(Q) \subseteq V_2$ with $|\tilde{T}| \geq$*

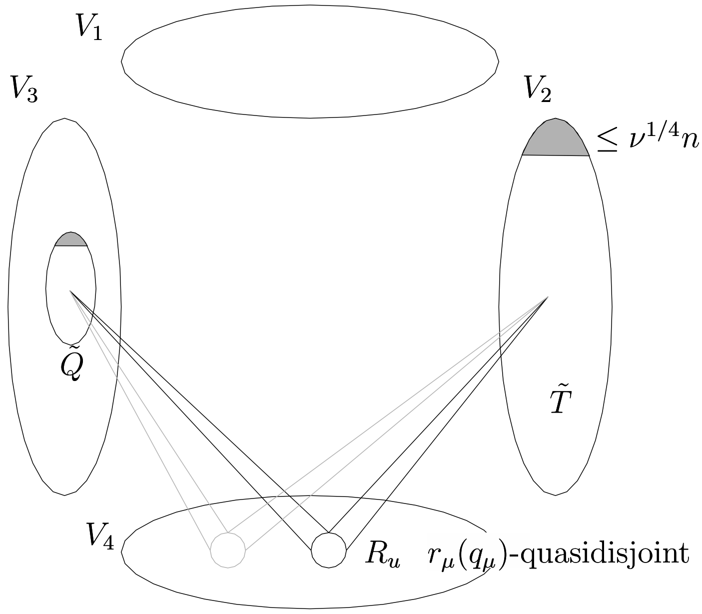

Figure 5: (ν, μ)-triangle candidate cover

$(1-\nu^{1/4})n$ *such that the following condition is satisfied:*

$$\forall w \in \tilde{T}\ \exists \tilde{Q}^* \subseteq \tilde{Q}, |\tilde{Q}^*| \geq (1-\nu^{1/4})\mu^2 q$$
$$\forall u \in \tilde{Q}^*\ \exists R_u \subseteq \Big(\Gamma_4(u) \cap \Gamma_4(w)\Big) \setminus X \text{ such that } |R_u| \geq \tilde{r}_1/2 \text{ and}$$
$$\text{the sets } R_u \text{ are } r_\mu(q_\mu)\text{-quasidisjoint for } u \in \tilde{Q}^*,$$

where $\tilde{r}_1 := r_\nu(\mu^2 q) = \frac{1}{4}\mu^2\nu\frac{m^2}{n^3}$.

Figure 5 visualises the structure of a triangle candidate cover.

While the above definition might lead the reader to believe that a triangle candidate cover is a very sophisticated structure, the following lemma shows that we have already encountered them.

Lemma 5.15 (Qualified multicovers are triangle candidate covers) *Consider a graph* $G \in \mathcal{S}_4(n, m; \varepsilon)$, *and* $X \subseteq V_4$ *with* $|X| \leq (1-2\sqrt{\mu})n$. *Let* $Q \subseteq V_3$ *with* $|Q| = q_\mu$ *be an* X*-resistant* μ*-multicover of* V_4 *which is* ν*-qualified for* V_2. *Then* Q *is also an* X*-resistant* (ν, μ)*-triangle candidate cover of* V_2 *via* V_4.

Proof We will show that Q^{**}, as given by Definition 5.10, satisfies the properties of $\tilde{Q}$ in Definition 5.14. There exist $r_\mu(q_\mu)$-quasidisjoint covering neighbourhoods $W'_1, \ldots, W'_{\mu^2 q} \subseteq V_4 \setminus X$ with cardinality $\mu^2 q$. As Q is ν-qualified for V_2 these sets are ν-multicovers of V_2 and by Lemma 5.6 we have $|C_2(W_v; \nu)| \geq (1-\sqrt{\nu})n$ for all $v = 1, \ldots, \mu^2 q$. By Corollary 4.3 we conclude that there is a set $\tilde{T} \subseteq V_2$ with $|\tilde{T}| \geq (1-\nu^{1/4})n$ so that every $w \in \tilde{T}$ belongs to at least $(1-\nu^{1/4})\mu^2 q$ sets $C_2(W_v; \nu)$.

In other words, for every $w \in \tilde{T}$ the set $\tilde{Q}^* := \{v \in Q^{**} \mid w \in C_2(W_v; \nu)\} \subseteq \tilde{Q} = Q^{**}$ satisfies $|\tilde{Q}^*| \geq (1-\nu^{1/4})\mu^2 q$.

Now for every $u \in \tilde{Q}^*$ define $R_u := \Gamma_4(w) \cap W_u$. Observe that by definition of $\tilde{Q}^*$, we have $w \in C_2(W_u; \nu)$, hence, by definition of the latter, $|R_u| = |\Gamma_4(w) \cap W_u| \geq r_\nu(\mu^2 q)/2 =$

$\tilde{r}_1/2$. On the other hand, the definition of W_u implies that $W_u \subseteq \Gamma_4(u) \setminus X$, and therefore $R_u \subseteq (\Gamma_4(u) \cap \Gamma_4(w)) \setminus X$, which proves the claim. □

Consider a triangle candidate cover Q and a vertex $w \in \tilde{T}(Q)$. Assume that w has $\Theta(m^2/n^3)$ neighbours R in $\tilde{Q}^*$. Since every vertex $u \in \tilde{Q}^*$ has $\tilde{r}_1/2 = \Theta(m^2/n^3)$ common neighbours with w in V_4, we should expect that R and w complete $\Theta(m^2/n^3 \cdot m^2/n^3) = \Theta(m^4/n^6)$ triangles if all neighbourhoods would be disjoint. Furthermore, note that $m^4/n^6 = \Omega(n^2/m)$ for $m = \Omega(n^{8/5})$. This leads to the following definition, where we introduce bad sets R which close significantly less than this expected number of triangles. Note that the quasidisjointness of the sets R_u in Definition 5.14 gives us the necessary control on the overlap of the sets R_u for $u \in R'$.

Definition 5.16 (Bad R-sets) *Consider a graph $G \in \mathcal{S}_4(n, m; \varepsilon)$, $X \subseteq V_4$, $w \in V_2$ and a set $Q \subseteq V_3$ with $|Q| = q_\mu$ which is an X-resistant (ν, μ)-triangle candidate cover of V_2 via V_4. We define*

$$\bar{\mathcal{R}}(w, Q; X, \lambda) := \{R \in \binom{\tilde{Q}}{\tilde{r}_2/2} \mid |(\Gamma_4(R) \cap \Gamma_4(w)) \setminus X| < t_\lambda\},$$

where $t_\lambda := \frac{1}{\lambda} \cdot \frac{n^2}{m}$ and $\tilde{r}_2 := r_\mu(\mu^2 q) = \frac{\mu^3}{4}\frac{m^2}{n^3}$.

The following lemma shows that such bad R-sets indeed occur very rarely.

Lemma 5.17 (Few bad R-sets) *Assume that $m \geq 2^8(\lambda\mu^5\nu)^{-1}n^{8/5} =: m^R(\nu, \mu, \lambda)$ and that n is sufficiently large. Now consider $w \in \tilde{T}(Q)$. Under the conditions given in Definition 5.16 we have*

$$|\bar{\mathcal{R}}(w, Q; X, \lambda)| \leq \left(8\nu^{1/4}\right)^{\tilde{r}_2/4} \binom{\mu^2 q}{\tilde{r}_2/2} \leq \kappa^{\tilde{r}_2} \binom{\mu^2 q}{\tilde{r}_2/2},$$

where $\kappa := 2\nu^{1/16}$.

Proof We choose $R = \{u_1, \ldots, u_{\tilde{r}_2/2}\} \subseteq \tilde{Q}$ uniformly at random and estimate $\Pr[R \in \bar{\mathcal{R}}(w, Q; X, \lambda)]$. We may assume that the vertices $u_1, \ldots, u_{\tilde{r}_2/2}$ are chosen one by one, since this event does not depend on the ordering of the vertices.

For $k = 1, \ldots, \tilde{r}_2/2$, we define $TC_k \subseteq V_4 \setminus X$ iteratively by

$$\begin{aligned} TC_0 &:= \emptyset \\ TC_k &:= TC_{k-1} \cup \begin{cases} [R_{u_k} \setminus TC_{k-1}]_{\tilde{r}_1/4}, & \text{if } |R_{u_k} \setminus TC_{k-1}| \geq \tilde{r}_1/4 \\ \emptyset, & \text{if } |R_{u_k} \setminus TC_{k-1}| < \tilde{r}_1/4, \end{cases} \end{aligned}$$

where R_u is defined as in Definition 5.14. Furthermore, we define the set of *good* vertices at step k by

$$GQ_k := \{u \in \tilde{Q} \setminus \{u_1, \ldots, u_{k-1}\} \mid |R_u \setminus TC_{k-1}| \geq \tilde{r}_1/4\},$$

which are those vertices $u \in \tilde{Q}$ that, if chosen at step k, would contribute to TC_{k-1}. Observe that $\tilde{Q}^* \subseteq GQ_1$, and thus $|GQ_1| \geq |\tilde{Q}^*| \geq (1 - \nu^{1/4})\mu q$. Assume that during steps $1, \ldots, \tilde{r}_2/2$ at least

$$k_0 := 16(\lambda\mu^2\nu)^{-1}\frac{n^5}{m^3}$$

good vertices ($u_k \in GQ_k$) have been chosen. Then

$$|(\Gamma_4(R) \cap \Gamma_4(w)) \setminus X| \geq |TC_{\tilde{r}_2/2}| \geq k_0 \cdot \frac{\tilde{r}_1}{4} = \frac{16}{\lambda\mu^2\nu}\frac{n^5}{m^3} \cdot \frac{1}{16}\mu^2\nu\frac{m^2}{n^3} = \lambda^{-1} \cdot \frac{n^2}{m} = t_\lambda.$$

Therefore

$$\Pr[R \in \bar{\mathcal{R}}(w, Q; X, \lambda)] \leq \Pr[\text{'}\leq k_0 \text{ vertices } u_k \in GQ_k\text{'}]. \tag{10}$$

Due to Definition 5.14 we conclude that for every vertex $u \in \tilde{Q}^*$ there exists a set $R_u \subseteq (\Gamma_4(u) \cap \Gamma_4(w)) \setminus X$ with $|R_u| = \tilde{r}_1/2$. If $u \notin GQ_k$, at least $\tilde{r}_1/4$ vertices in R_u must belong to TC_{k-1}. As long as at most k_0 good vertices have been chosen in the first $k-1$ steps, we have $|TC_{k-1}| \leq k_0 \cdot \tilde{r}_1/4$. Since the sets R_u are $r_\mu(q_\mu)$-quasidisjoint we conclude that in this case

$$\begin{aligned} |\tilde{Q}^* \setminus GQ_k| &\leq \frac{|TC_{k-1}| \cdot r_\mu(q_\mu)}{\tilde{r}_1/4} + (k-1) \leq k_0 \cdot r_\mu(q_\mu) + \tilde{r}_2 = \Theta\left(\frac{n^5}{m^3} \cdot \frac{m^2}{n^3}\right) \\ &= \Theta(n^2/m). \end{aligned} \tag{11}$$

Observe that $n^2/m = o(q)$ for $m = \Omega(n^{8/5})$. Hence we obtain $|\tilde{Q}^* \setminus GQ_k| \leq \nu\mu^2 q$ for n sufficiently large. From (11) we deduce that

$$|GQ_k| \geq |\tilde{Q}^*| - |\tilde{Q}^* \setminus GQ_k| \geq (1-\nu^{1/4})\mu^2 q - \nu\mu^2 q \geq (1-2\nu^{1/4})\mu^2 q = (1-2\nu^{1/4})|\tilde{Q}|. \tag{12}$$

Note that

$$\tilde{r}_2/4 = \frac{1}{16}\mu^3\frac{m^2}{n^3} \geq k_0 \cdot 2^{-8}\mu^5\nu\lambda\frac{m^5}{n^8} \geq k_0,$$

by choice of m, and hence $\tilde{r}_2/2 - k_0 \geq \tilde{r}_2/4$. From (10) and (12) it therefore follows that

$$\Pr[R \in \bar{\mathcal{R}}(w, Q; X, \lambda)] \leq 2^{\tilde{r}_2/2} \cdot \left(2\nu^{1/4}\right)^{\tilde{r}_2/4} = \left(8\nu^{1/4}\right)^{\tilde{r}_2/4},$$

which completes the proof. □

Although we have already defined triangle covers we have not yet proved their occurrence in typical graphs. Instead we have shown that there exist many triangle candidate covers $Q \subseteq V_3$ and that most sets of size $\Theta(m^2/n^3)$ inside Q close $\Theta(n^2/m)$ triangle candidates. Observe that a typical vertex $w \in V_2$ has $\Theta(m^2/n^3)$ neighbours in Q. If these neighbours indeed close $\Theta(n^2/m)$ triangle candidates and this is true for most vertices in V_2, this suffices to show that Q is a triangle cover.

Unfortunately, by considering one triangle candidate cover alone we do not obtain a sufficiently small probability for bad graphs. This is due to the fact that by Lemma 5.17 we only gain a factor $\Theta(m^2/n^3)$ in the exponent for a single bad neighbourhood of size $\Theta(m^2/n^3)$ inside a specific triangle candidate cover Q. But, as in the case of multicovers of size $\Theta(m/n)$ (cf. Lemma 5.8), we need $\Theta(m/n)$ in the exponent to obtain sufficiently small probabilities. This technical difficulty is overcome by considering a partition of V_3 into $\Theta(n^2/m)$ triangle candidate covers Q. By applying the above arguments to all these sets at once we gain a factor of $\Theta(n^2/m \cdot m^2/n^3) = \Theta(m/n)$ in the exponent, which will suffice to complete the proof. The following definition introduces such partitions of V_3.

Definition 5.18 (Cover families) *Let $\nu, \mu > 0$ and $\tilde{p} := p_\mu/4$. A (ν, μ)*-cover family *which is resistant to $(X_i)_{i=1,\ldots,\tilde{p}}$ consists of pairwise disjoint sets $Q_1, \ldots, Q_{\tilde{p}} \subseteq V_3$ and forbidden vertex sets $X_1, \ldots, X_{\tilde{p}} \subseteq V_4$ such that the following conditions are satisfied for all $i = 1, \ldots, \tilde{p}$:*

(i) $|Q_i| = q_\mu$,

(ii) Q_i *is an* X_i*-resistant* (ν, μ)*-triangle candidate cover of* V_2 *via* V_4,

(iii) $\tilde{Q}_i$, *which is given by Definition 5.14, is a* μ*-multicover of* V_2.

A set $P^* = \{v_1, \ldots, v_{\tilde{p}}\} \subseteq V_1$ induces *a* (ν, μ)*-cover family if* $Q_i \subseteq \Gamma_3(v_i)$ *for* $i = 1, \ldots, \tilde{p}$.

Our next aim is to show that cover families contain many triangle covers. A cover family consists of triangle candidate covers which we would like to turn into triangle covers by showing that the necessary edges between V_3 and V_2 do exist, i.e., that the sets $\tilde{T}(Q_i)$ from Definition 5.14 can be turned into the sets $T(Q_i)$ from Definition 5.13. The following lemma introduces certain 'bad' vertices in V_2. If we can show that there are only few such vertices, then the cover family behaves as desired and we can prove that there exist many triangle covers.

Lemma 5.19 (Cover families contain many triangle covers) *Let* $\sqrt{\nu} \le \mu$ *and* $2\mu^{1/4} \le \delta$. *Consider a* (ν, μ)*-cover family* $Q_1, \ldots, Q_{\tilde{p}} \subseteq V_3$ *with forbidden vertex sets* $X_1, \ldots, X_{\tilde{p}} \subseteq V_2$. *A vertex* $w \in V_2$ *is called* non-spreading *if there exist at least* $\delta\tilde{p}$ *sets* Q_i *such that*

$$w \in \tilde{T}(Q_i) \ \wedge \ \exists R \subseteq \Gamma_3(w) \cap \tilde{Q}_i, |R| = \tilde{r}_2/2 : \ R \in \bar{\mathcal{R}}(w, Q_i; X_i, \mu^2).$$

If there are at most δn *non-spreading vertices, then*

$$|\{i \in \{1, \ldots, \tilde{p}\} \mid Q_i \text{ is an } X_i\text{-resistant } (\mu^2, \delta)\text{-triangle cover}\}| \ge (1 - 2\sqrt{\delta})\tilde{p}.$$

Proof For $i = 1, \ldots, \tilde{p}$ consider the sets

$$\tilde{C}_i := \tilde{T}(Q_i) \cap C_2(\tilde{Q}_i; \mu) \subseteq V_2,$$

where $\tilde{T}(Q_i)$ is defined according to Definition 5.14. The idea is that the vertices in $\tilde{T}(Q_i)$ are guaranteed to share many common neighbours (inside V_4) with the set $\tilde{Q}_i$ while the set $C_2(\tilde{Q}_i; \mu)$ sees many vertices in $\tilde{Q}_i$, hence their intersection is a good way of getting closer to the set $T(Q_i)$ from Definition 5.13. By Lemma 5.6 and Definition 5.14 we conclude that

$$|\tilde{C}_i| \ge |\tilde{T}(Q_i)| - |V_2 \setminus C_2(\tilde{Q}_i; \mu)| \ge (1 - \nu^{1/4} - \sqrt{\mu})n \ge (1 - 2\sqrt{\mu})n.$$

Hence by Corollary 4.3 there are at least

$$(1 - \sqrt{2\sqrt{\mu}})n \ge (1 - 2\mu^{1/4})n \ge (1 - \delta)n$$

vertices $w \in V_2$ such that w belongs to at least $(1 - 2\mu^{1/4})\tilde{p} \le (1 - \delta)\tilde{p}$ sets $\tilde{C}_i$. Let S denote the set of these vertices. If $w \in S$ satisfies $w \in \tilde{C}_i$, we say that Q_i is *apt* for w. Recall that in this case we know that $w \in C_2(\tilde{Q}_i; \mu)$, hence by Definition 5.5 we have that

$$|\tilde{Q}_i \cap \Gamma_3(w)| \ge r_\mu(\mu^2 q)/2 = \tilde{r}_2/2.$$

So we fix a set $R_i(w) \subseteq \tilde{Q}_i \cap \Gamma_3(w)$ with $|R_i(w)| = \tilde{r}_2/2$ for every set Q_i which is apt for w. Now we remove all non-spreading vertices from S, and call the resulting set S'. Note that $|S'| \ge |S| - \delta n \ge (1 - 2\delta)n$.

For any vertex $w \in S'$ there are at least $(1 - \delta)\tilde{p}$ apt sets Q_i. Recall that since $w \in S'$ we have that $w \in \tilde{T}(Q_i)$ but also that w is spreading. These two conditions together imply that there exist at least $(1 - \delta - \delta)\tilde{p} = (1 - 2\delta)\tilde{p}$ sets Q_i for which

$$\forall R \subseteq \Gamma_3(w) \cap \tilde{Q}_i, |R| = \tilde{r}_2/2 : \ R \notin \bar{\mathcal{R}}(w, Q_i; X_i, \mu^2).$$

Letting $R := R_i(w)$ this shows that by Definition 5.16 of $\bar{\mathcal{R}}$ that we have

$$|(\Gamma_4(R_i(w)) \cap \Gamma_4(w)) \setminus X_i| \geq t_{\mu^2}.$$

Now recall that

$$TRI_4(w, Q_i) := \Gamma_4(\Gamma_3(w) \cap Q_i) \cap \Gamma_4(w) \supseteq \Gamma_4(R_i(w)) \cap \Gamma_4(w)$$

to obtain that $|TRI_4(w, Q_i) \setminus X_i| \geq t_{\mu^2}$. Hence for any vertex $w \in S'$ we have $w \in T(Q_i)$ for at least $(1-2\delta)\tilde{p}$ sets Q_i. We apply Corollary 4.3 one more time and obtain that there are at least $(1-\sqrt{2\delta})\tilde{p}$ sets Q_i such that

$$|T(Q_i)| \geq (1-\sqrt{2\delta})|S'| \geq (1-\sqrt{2\delta})(1-2\delta)n \geq (1-3\sqrt{\delta})n.$$

Hence these sets Q_i are (μ^2, δ)-triangle covers and the proof is complete. □

6 General counting lemma

We want to investigate some properties of a typical regular graph and then show that a graph satisfying these properties contains a K_4. More precisely, we shall see that a graph is untypical or 'bad' if it contains *linearly* many vertices which have a somewhat 'irregular' neighbourhood.

The aim of this section is to provide a rather general counting lemma which can be applied to various definitions of what we mean by an 'irregular' neighbourhood.

We will construct and thus count the number of untypical graphs by first fixing their edges up to $E[B, V_j]$, where $B \subseteq V_i$ for $i \neq j$ denotes a set of 'bad' vertices with $|B| = \Theta(n)$. The key idea then is that an appropriate definition of 'irregular' implies that, once all edges except those between B and V_j have been chosen, the number of 'irregular' neighbourhoods is very small compared to the total number of possible neighbourhoods (more precisely, $\pi^{d_v}\binom{n}{d_v}$ instead of $\binom{n}{d_v}$ for some suitably small constant $\pi > 0$, where $d_v = |\Gamma_j(v)|$ for $v \in B$.)

Definition 6.1 *(*Neighbourhood function*) Consider a 4-partite graph $G = (V_1 \cup \ldots \cup V_4, E)$ and a set $B \subseteq V_i$. A* neighbourhood function *$\mathcal{N}$ assigns sets $\mathcal{N}(v) \subseteq \binom{V_j}{d_v}$ to every $v \in B$, given v and a value d_v. The dependency on d_v will be dropped in the notation.*

Definition 6.2 *(*Bad neighbourhood function*) Let $\delta, \pi > 0$ and let $\mathcal{G} \subseteq \mathcal{S}_4(n, m; \varepsilon)$. A neighbourhood function $\mathcal{N}$ is called a* bad neighbourhood function *for the set $\mathcal{G}$ and the parameters δ, π if the following condition holds:*

For each $G = (V, E) \in \mathcal{G}$ there exist $1 \leq i, j \leq 4$, $i \neq j$ and a set $B \subseteq V_i$ with $|B| \geq \delta n$ and $d_j(v) \geq q/2$ for all $v \in B$ such that for each $v \in B$

$$\Gamma_j(v) \in \mathcal{N}(v) \quad \textit{and} \quad |\mathcal{N}(v)| \leq \pi^{d_v}\binom{n}{d_v},$$

where $\mathcal{N}(v)$ is applied to the graph $G' = (V, E')$ with edge set $E' := E \setminus E(B, V_j)$, the set B, and $d_v := d_j(v)$.

Lemma 6.3 (Counting bad graphs) *Let $\beta, \delta > 0$, and let $\pi := \pi(\beta, \delta) > 0$ be sufficiently small such that $2 \cdot \pi^{\delta/2} \leq \beta$. If $\mathcal{N}$ is a bad neighbourhood function for $\mathcal{G} \subseteq \mathcal{S}_4(n, m; \varepsilon)$ and parameters δ, π, then*

$$|\mathcal{G}| \leq \beta^m \binom{n^2}{m}^6$$

for n sufficiently large and $m = \omega(n \log n)$.

Proof We construct all graphs in $\mathcal{G}$ as follows: We start by choosing i, j with $i \neq j$, the set $B \subseteq V_i$ and the degree values $d_v := |\Gamma_j(v)|$ for $v \in V_i$. Observe that there are at most $4 \cdot 3 \cdot n! \cdot n^n \leq 2^m$ possibilities for that, with lots of room to spare. Then we fix the edges in $E \setminus E(V_i, V_j)$ (at most $\binom{n^2}{m}^5$ possibilities). For the vertices $v \in V_i \setminus B$ we have at most $\binom{n}{d_v}$ choices to fix their remaining neighbourhood $\Gamma_j(v)$. For each vertex $v \in B$ we choose a set from $\mathcal{N}(v)$ as its neighbourhood $\Gamma_j(v)$. From the assumption that $\mathcal{N}$ is a bad neighbourhood function it follows that we may assume that there are at most $\pi^{d_v}\binom{n}{d_v}$ possibilities for that. Hence, we obtain that the total number of possibilities for choosing the edges between V_i and V_j is bounded by

$$\begin{aligned}\left(\prod_{v\in V_i\setminus B}\binom{n}{d_v}\right)\cdot\left(\prod_{v\in B}\pi^{d_v}\binom{n}{d_v}\right) &\leq \pi^{|B|\cdot\min_{v\in B}\{d_v\}}\prod_{v\in V_i}\binom{n}{d_v}\\ &\leq \pi^{\delta n\cdot q/2}\binom{n^2}{m} = \pi^{\delta m/2}\binom{n^2}{m}.\end{aligned}$$

Thus it follows that

$$|\mathcal{G}| \leq 2^m \cdot \binom{n^2}{m}^5 \cdot \pi^{\delta m/2}\binom{n^2}{m} \leq \beta^m \cdot \binom{n^2}{m}^6,$$

which proves the claim. □

7 Proof of the Main Theorem

7.1 Constants

For the proof of Theorem 2.4 we will need the following constants which depend on β:

$$\delta, \delta_r, \delta_c, \nu_q, \mu_q, \varepsilon_q > 0.$$

We characterise these constants by means of several properties (The function $\pi(\beta,\delta)$ is given in Lemma 6.3 and ν^R is defined in Lemma 7.7.):

$$2\mu_q^{1/2} \leq \pi(\beta,\delta_c), \tag{13}$$

$$\varepsilon_q \leq \mu_q^2, \tag{14}$$

$$\nu_q^{1/2} \leq \mu_q, \tag{15}$$

$$2\mu_q^{1/4} \leq \delta_r, \tag{16}$$

$$200\delta \leq \mu_q, \tag{17}$$

$$\nu_q < \nu^R(\beta,\delta_r,\mu_q), \tag{18}$$

$$4\delta_r^{1/2} \leq \delta_c, \tag{19}$$

$$30\delta_r^{1/4} \leq \pi(\beta,\delta_c). \tag{20}$$

$$\left(2^{20/(\delta_r\mu_q^2)}\nu_q^{1/16}\right)^{\delta_r\mu_q^2/10} \leq \pi(\beta,\delta_r) \tag{21}$$

It can be seen that suitable values for the constants exist by fixing them in the following order:

$$\delta_c, \delta_r, \mu_q, \varepsilon_q, \delta, \nu_q.$$

Furthermore, we fix a sufficiently large constant $C \geq 1$ such that the conditions of Lemma 5.17 are satisfied, i.e., $Cn^{8/5} \geq m^R(\nu,\mu,\lambda)$. In the sequel we will then assume that $Cn^{8/5} \leq m \leq n^2/4$.

7.2 Counting bad sets

As mentioned before we want to investigate properties of a typical ε-regular graph. In this subsection we show that for nearly all ε-regular graphs the neighbourhoods of most vertices don't deviate from their expected cardinalities, and are multicovers. The next step will then be to prove that in these graphs only very few vertices are non-spreading, which by Lemma 5.19 gives rise to many triangle covers. Finally, this implies that a typical ε-regular graphs contains a linear number of vertices in V_1 which have many clique candidates.

7.2.1 Vertices with untypical degree

We say that a vertex $v \in V_i$ satisfies the *degree property* if the following condition (D) is met:

$$(D) \qquad \forall j \neq i: \quad (1-\varepsilon)q \leq d_j(v) \leq (1+\varepsilon)q.$$

Definition 7.1 (Graphs with vertices of wrong degree)

$$\begin{aligned}\mathcal{B}^D(n,m;\varepsilon) \quad := \quad & \{G \in \mathcal{S}_4(n,m;\varepsilon) \mid \\ & \exists i \textit{ s.t. at least } 6\varepsilon n \textit{ vertices in } V_i \textit{ do not satisfy } (D)\}.\end{aligned}$$

Lemma 7.2 ($\mathcal{B}^D$ is very small indeed)

$$\mathcal{B}^D(n,m;\varepsilon) = \emptyset.$$

Proof It follows from Lemma 4.4 that $|\{v \in V_i \mid |d_j(v) - q| > \varepsilon q\}| < 2\varepsilon n$ for all $i \neq j$. $\square$

7.2.2 Vertices without multicovers in the neighbourhood

By Lemma 5.8 we know that almost all sets of size αq for some $0 < \alpha < 1$ are ν-multicovers. Using Lemma 6.3 we conclude that almost all vertices have multicovers in their neighbourhood. The following definition and lemma formalise this intuitive argument.

We say that a vertex $v \in V_i$ satisfies the *multicover property* if the following condition (MC_ν) is met:

$$(MC_\nu) \quad \forall j \neq i \; \forall Q \subseteq \Gamma_j(v) \text{ s.t. } \varepsilon_q q \leq |Q| \leq q \; \forall k \neq j: \; Q \in \mathcal{Q}_{j,k}(G; |Q|, \nu),$$

where $\mathcal{Q}_{j,k}$ was defined in Definitions 5.7 and 5.4.

Definition 7.3 (Vertices with a bad Q-set)

$$\begin{aligned}\mathcal{B}^{MC}(n,m;\varepsilon,\nu) \quad := \quad & \{G \in \mathcal{S}_4(n,m;\varepsilon) \mid \\ & \exists i \textit{ s.t. at least } 3\delta n \textit{ vertices in } V_i \textit{ do satisfy } (D) \textit{ but not } (MC_\nu)\}.\end{aligned}$$

Lemma 7.4 ($\mathcal{B}^{MC}$ is small) *There is a constant* $\varepsilon^{MC} = \varepsilon^{MC}(\varepsilon_q, \beta, \delta, \nu) > 0$ *such that*

$$|\mathcal{B}^{MC}(n,m;\varepsilon,\nu)| \leq \beta^m \binom{n^2}{m}^6$$

for n *sufficiently large and* $\varepsilon \leq \varepsilon^{MC}$.

Proof By Lemma 6.3 it suffices to show that there exists an appropriate bad neighbourhood function $\mathcal{N}$. In order to define $\mathcal{N}$, observe first that for each graph $G \in \mathcal{B}^{MC}(n, m; \varepsilon, \nu)$ there exist i and j such that the set $B \subseteq V_i$ of those vertices that satisfy (D) but fail (MC_ν) for the given value of j and at least one set $Q \subseteq \Gamma_j(v)$ has cardinality $|B| \geq \delta n$.

We hence let

$$\mathcal{N}(v) := \{U \in \binom{V_j}{d_v} \mid \exists Q \subseteq U \text{ s.t. } Q \in \bar{\mathcal{Q}}_{j,k}(G; t, \nu) \text{ for some } k \neq i, j \text{ and } \varepsilon_q q \leq t \leq q\}$$

and apply it to graphs $G \in \mathcal{B}^{MC}(n, m; \varepsilon, \nu)$ with i, j, B as indicated above and $d_v := |\Gamma_j(v)|$ for $v \in B$.

Let $\pi = \pi(\beta, \delta)$ be as in Lemma 6.3, choose $\gamma > 0$ sufficiently small such that $3\gamma^{\varepsilon_q/2} \leq \pi$ and assume that $\varepsilon < \varepsilon^Q(\gamma, \nu)$, where the latter is as in Lemma 5.8. By Lemma 5.8 and Lemma 4.1 (i) we have

$$\begin{aligned} |\mathcal{N}(v)| &\overset{5.8}{\leq} 2 \sum_{\varepsilon_q q \leq t \leq q} \left[\gamma^t \binom{n}{t} \cdot \binom{n-t}{d_v - t}\right] \overset{4.1}{\leq} q \cdot \gamma^{\varepsilon_q q} 2^{d_v + 1} \binom{n}{d_v} \\ &\overset{(D)}{\leq} \frac{d_v}{1-\varepsilon} \gamma^{\varepsilon_q \frac{d_v}{1+\varepsilon}} 2^{d_v+1} \binom{n}{d_v} \leq (3\gamma^{\varepsilon_q/(1+\varepsilon)})^{d_v} \binom{n}{d_v} \leq \pi^{d_v} \binom{n}{d_v}. \end{aligned}$$

This concludes the proof of the lemma. □

7.2.3 Vertices without 'good' neighbourhoods

In the preceding lemmas we have shown that most vertices in a typical graph $G \in \mathcal{S}_4(n, m; \varepsilon)$ satisfy properties (D) and (MC). For the remainder of this paper we concentrate our attention on *good vertices*, which are denoted by

$$GV_i := \{v \in V_i \mid v \text{ satisfies } (D), (MC_{\nu_q}), \text{ and } (MC_{\mu_q})\}.$$

Let

$$\mathcal{S}_4^{DM}(n, m; \varepsilon) := \mathcal{S}_4(n, m; \varepsilon) \setminus [\mathcal{B}^D(n, m; \varepsilon) \cup \bigcup_{\nu \in \{\nu_q, \mu_q\}} \mathcal{B}^{MC}(n, m; \varepsilon, \nu)].$$

The notation for the 'bad' sets $\mathcal{B}$ and the 'good' sets $\mathcal{S}$ follows the convention that the superscript indicates the property resp. properties which characterise this set of graphs. For bad sets the given property is violated by a significant number of vertices, whereas for good sets the properties indicated by the superscript are satisfied by almost all vertices.

Consider a graph $G \in \mathcal{S}_4^{DM}(n, m; \varepsilon)$. From Lemma 7.2 and Lemma 7.4 we can immediately conclude that, say,

$$|GV_i| \geq (1 - 99\delta)n \qquad \text{for all } i = 1, \ldots, 4$$

with lots of room to spare. We say that a vertex $v \in V_i$ satisfies the *good degree property* if the following condition (D') is met:

$$(D') \qquad \forall j: \ |\Gamma_j(v) \setminus GV_j| \leq 100\delta q.$$

The subset of GV_i which consists of all vertices satisfying (D') is denoted by GV_i', i.e.,

$$GV_i' := \{v \in GV_i \mid v \text{ satisfies } (D')\}.$$

Lemma 7.5 (GV_i' is big) *For $G \in \mathcal{S}_4^{DM}(n,m;\varepsilon)$ we have for all $i = 1, \dots, 4$,*

$$|GV_i'| \leq (1 - 100\delta)n.$$

Proof Let $j \in \{1, \dots, 4\}$, $j \neq i$, and consider any set $Z \subseteq V_j$ with $Z \supseteq V_j \setminus GV_j$ and $|Z| = 99\delta n$. By Lemma 4.4 we obtain that $|\Gamma_j(v) \cap Z| \leq (1+\varepsilon)99\delta q \leq 100\delta q$ for all but at most εn vertices in v_i. The lemma immediately follows. $\square$

7.2.4 Vertices without triangle covers

Due to Lemma 5.19 we find the desired triangle covers if we are able to prove that there are few non-spreading vertices. The following definition and lemma show that this is indeed the case for most graphs.

Definition 7.6 (Too many non-spreading vertices)

$$\begin{aligned}\mathcal{B}^R(n,m;\varepsilon) \quad := \quad & \{G \in \mathcal{S}_4(n,m;\varepsilon) \mid \exists X_1, \dots, X_{\tilde{p}} \subseteq V_4, \\ & \exists\ (\nu_q, \mu_q)\text{-cover family } Q_1, \dots, Q_{\tilde{p}} \subseteq V_2 \text{ resistant to } (X_l)_{l=1,\dots,\tilde{p}}\text{:} \\ & \text{there exist at least } \delta_r n \text{ non-spreading vertices } w \in V_2\}\end{aligned}$$

Analogously to Lemma 5.19 we say that a vertex $w \in V_2$ is non-spreading if there exist at least $\delta_r \tilde{p}$ sets Q_i such that

$$w \in \tilde{T}(Q_i) \ \wedge \ \exists R \subseteq \Gamma_3(w) \cap \tilde{Q}_i, |R| = \tilde{r}_2/2 : \ R \in \bar{\mathcal{R}}(w, Q_i; X_i, \mu_q^2).$$

Lemma 7.7 ($\mathcal{B}^R$ is small)

$$|\mathcal{B}^R(n,m;\varepsilon)| \leq \beta^m \binom{n^2}{m}^6,$$

whenever $m \geq m^R(\nu_q, \mu_q, \mu_q^2)$ and n sufficiently large.

Proof There are at most $2^{n \cdot \tilde{p}} \leq 2^m$ choices for the sets X_l and at most $(\tilde{p}+1)^n$ choices for the cover family $Q_1, \dots Q_{\tilde{p}}$. Hence we have at most, say, 3^m possibilities for fixing the (ν_q, μ_q)-cover family with the forbidden sets, and, consequently, it suffices to count the number of graphs which are bad with respect to a specific cover family.

To this aim we plan to apply Lemma 6.3, where the set of bad vertices B corresponds to $\delta_r n/2$ non-spreading vertices which satisfy (D). Observe that there are at least $\delta_r n - 6\varepsilon n \geq \delta_r n/2$ such vertices. We define the neighbourhood function

$$\begin{aligned}\mathcal{N}(w) \quad := \quad & \{X \in \binom{V_3}{d_w} \mid \exists I \subseteq \{1, \dots, \tilde{p}\}, |I| = \delta_r \tilde{p} \\ & \forall i \in I\ \exists R \subseteq X \cap \tilde{Q}_i, |R| = \tilde{r}_2/2 : \ R \in \bar{\mathcal{R}}(w, Q_i; X_i, \mu_q^2)\}.\end{aligned}$$

Now let us estimate $\mathcal{N}(w)$. First we choose the set I (at most $2^{\tilde{p}}$ possibilities). Then we fix the $\delta_r \tilde{p}$ bad R-sets. Recall that $w \in \tilde{T}(Q_i)$. Thus, by Lemma 5.17 there are at most $\kappa^{\tilde{r}_2}\binom{\mu_q^2 q}{\tilde{r}_2/2}$ possibilities for each bad R-set. Observe that $\delta_r \tilde{p} \cdot \mu_q^2 q = \delta_r \mu_q n/2$ and that

$\tilde{\delta}_r\tilde{p}\cdot r_2/2 = \frac{1}{16}\mu_q^2\delta_r q$. This leads to the bound

$$\begin{aligned}
|\mathcal{N}(v)| &\leq 2^{\tilde{p}}\cdot\left(\kappa^{\tilde{r}_2}\binom{\mu_q^2 q}{\tilde{r}_2/2}\right)^{\delta_r\tilde{p}}\cdot\binom{n-\tilde{r}_2/2\cdot\delta_r\tilde{p}}{d_v-\tilde{r}_2/2\cdot\delta_r\tilde{p}} \\
&\leq 2^{\tilde{p}}\cdot\kappa^{\tilde{r}_2\cdot\delta_r\tilde{p}}\cdot\binom{\delta_r\mu_q n/2}{\tilde{r}_2\delta_r\tilde{p}/2}\cdot\binom{n-\tilde{r}_2\delta_r\tilde{p}/2}{d_v-\tilde{r}_2\delta_r\tilde{p}/2} \\
&\leq (2\kappa)^{\tilde{r}_2\cdot\delta_r\tilde{p}}\cdot\binom{n}{\tilde{r}_2\delta_r\tilde{p}/2}\cdot\binom{n-\tilde{r}_2\delta_r\tilde{p}/2}{d_v-\tilde{r}_2\delta_r\tilde{p}/2} \overset{4.1(i)}{\leq} (2\kappa)^{\delta_r\mu_q^2 q/8}\cdot 2^{d_v}\binom{n}{d_v} \\
&\overset{(D)}{\leq} (2\kappa)^{\delta_r\mu_q^2 d_v/10}\cdot 2^{d_v}\binom{n}{d_v} \overset{5.17}{=} \left(4\nu_q^{1/16}\right)^{\delta_r\mu_q^2 d_v/10}\cdot 2^{d_v}\binom{n}{d_v} \\
&\leq \left(2^{20/(\delta_r\mu_q^2)}\nu_q^{1/16}\right)^{\delta_r\mu_q^2 d_v/10}\binom{n}{d_v} \overset{(21)}{\leq} \pi(\beta,\delta_r)^{d_v}\binom{n}{d_v}
\end{aligned}$$

Hence we can apply Lemma 6.3. □

Let

$$\mathcal{S}_4^{DMR}(n,m;\varepsilon) := \mathcal{S}_4^{DM}(n,m;\varepsilon)\setminus\mathcal{B}^R(n,m;\varepsilon).$$

For graphs in $\mathcal{S}_4^{DMR}$ we deduce from Lemma 5.19 that cover families contain many triangle covers. In order to find many cover families we will consider $\tilde{p}$ vertices in V_1. The following lemma shows that such vertices induce a cover family.

Lemma 7.8 (Induced cover families) *Let $P\subseteq GV_1'$ be a $2\mu_q$-cover of V_3 with $P^* =: \{v_1,\dots,v_{\tilde{p}}\}$, and for $i=1,\dots\tilde{p}$, let $X_1,\dots X_{\tilde{p}}\subseteq V_4$ be sets with $|X_i|\leq(1-2\sqrt{\mu_q})n$. Then P^* induces a (ν_q,μ_q)-cover family $Q_1,\dots,Q_{\tilde{p}}\subseteq V_3$ with forbidden vertex sets $X_1,\dots,X_{\tilde{p}}$.*

Proof Denote the covering neighbourhoods of the vertices in P^* by $W_1,\dots,W_{\tilde{p}}$, where $|W_i| = q_{2\mu_q} = (1-2\mu_q)2\mu_q q$ for $i=1,\dots,\tilde{p}$. Let

$$Q_i' := [W_i\cap GV_3]_{q_{\mu_q}}.$$

Note that $|W_i\setminus GV_3|\leq 100\delta q$, since $v_i\in GV_1'$. Hence

$$|W_i\cap GV_3|\geq(1-2\mu_q)2\mu_q q - 100\delta q \overset{(17)}{\geq} q_{\mu_q}$$

and $|Q_i'| = q_{\mu_q} \overset{(14)}{\geq} \varepsilon_q q$.

It suffices to show that the sets $Q_1',\dots,Q_{\tilde{p}}'$ satisfy the requirements of Definition 5.18.

Condition (i) has already been shown.

Since $P^*\subseteq GV_1'$ it follows from (MC_{μ_q}) that Q_i' is a μ_q-multicover of V_4 for $i=1,\dots,\tilde{p}$. As $|X_i|\leq(1-2\sqrt{\mu_q})n$, Lemma 5.11 implies that Q_i' is also X_i-resistant. The X_i-resistant covering neighbourhoods of Q_i' in V_4 have size

$$\mu_q^2 q \overset{(14)}{\geq} \varepsilon_q q.$$

Hence $Q_i'\subseteq GV_3$ implies that the covering neighbourhoods are ν_q-multicovers of V_2. Consequently, Q_i' is ν_q-qualified for V_2. By Lemma 5.15 we conclude that Q_i' is an X_i-resistant (ν_q,μ_q)-triangle candidate cover, which shows (ii).

Finally, $|\tilde{Q}_i'|\geq\mu_q^2 q \overset{(14)}{\geq} \varepsilon_q q$ and $v_i\in GV_1'$. Hence $\tilde{Q}_i'$ is a μ_q-multicover of V_2 and condition (iii) is satisfied. □

Now we are finally in a position to construct many triangle covers. To this aim we first characterise the 'good' vertices $v \in V_1$ which we are looking for. A vertex $v \in V_1$ is said to satisfy the *triangle cover property* (R) if

$$\begin{array}{ll} (R) & \forall X \subseteq V_4, |X| \leq (1-2\sqrt{\mu_q})n\ \exists Q_v \subseteq \Gamma_3(v): \\ & \quad Q_v \text{ is an } X\text{-resistant } (\nu_q^2, \delta_r)\text{-triangle cover.} \end{array}$$

The following lemma shows that most vertices satisfy (R).

Lemma 7.9 (Construct triangle covers) *For a graph $G \in \mathcal{S}_4^{DMR}(n, m; \varepsilon)$ there exist $(1 - 2\delta_c)n$ vertices $v \in V_1$ which satisfy (R).*

Proof First we will show a similar property, namely,

$$\begin{array}{ll} (R') & \forall X_{v_1}, \ldots, X_{v_n} \subseteq V_4, |X_{v_1}|, \ldots, |X_{v_n}| \leq (1-2\sqrt{\mu_q})n \\ & \exists RV_1 \subseteq V_1, |RV_1| \geq (1-2\delta_c)n\ \forall v \in RV_1\ \exists Q_v \subseteq \Gamma_3(v): \\ & \quad Q_v \text{ is an } X_l\text{-resistant } (\mu_q^2, \delta_r)\text{-triangle cover.} \end{array}$$

Here we have assumed that V_1 is ordered, i.e., $V_1 = \{v_1, \ldots, v_n\}$.

In order to prove (R') we construct $2\mu_q$-covers $P_1, \ldots, P_{\tilde{q}} \subseteq GV_1'$, where $\tilde{q} := (1 - \delta_c)n/\tilde{p}$, such that the sets $P_1^*, \ldots, P_{\tilde{q}}^*$ are disjoint. This can be achieved by induction. Assume that $P_1, \ldots, P_l$ have already been chosen. Note that by Lemma 7.5

$$\begin{aligned} |GV_1' \setminus (P_1^* \cup \ldots \cup P_l^*)| &\geq (1-100\delta)n - l \cdot \tilde{p} = (1-100\delta)n - \tilde{q} \cdot \tilde{p} \\ &\geq (\delta_c - 100\delta)n \overset{(17)}{\geq} \delta_c n/2. \end{aligned}$$

Hence we can find a $2\mu_q$-cover $P_{l+1} \subseteq GV_1' \setminus (P_1^* \cup \ldots \cup P_l^*)$ due to Lemma 5.3.

Now consider arbitrary sets $X_1, \ldots, X_n \subseteq V_4$. Using Lemma 7.8 we conclude that the sets P_l^* induce (ν_q, μ_q)-cover families for $l = 1, \ldots, \tilde{q}$. By Definition 7.6 it follows that for these cover families at most $\delta_r n$ non-spreading vertices $w \in V_2$ exist. Hence, due to Lemma 5.19 there exist at least $(1 - 2\sqrt{\delta_r})\tilde{p}$ vertices $v \in P_l^*$ for such that $\Gamma_3(v)$ contains an X_i-resistant (μ_q^2, δ_r)-triangle cover. Observe that the constants ν_q, μ_q and δ_r satisfy the requirements of Lemma 5.19 due to (15) and (16).

If we combine these vertices for all sets $P_1, \ldots, P_{\tilde{q}}$, we obtain $\tilde{q} \cdot (1 - 2\sqrt{\delta_r})\tilde{p} \geq (1 - 2\delta_c)n$ vertices for which a triangle cover exists. This completes the proof of (R').

Now assume that (R') holds but the claim of the lemma is not true. Thus there exist more than $2\delta_c n$ vertices $v \in V_1$ for which (R) is not satisfied. For every such vertex there exists a set $X_v \subseteq V_4$ with $|X_v| \leq (1 - 2\sqrt{\mu_q})n$ such that $\Gamma_3(v)$ does not contain an X_v-resistant (ν_q^2, δ_r)-triangle cover. Taking these sets $X_{v_1}, \ldots, X_{v_n}$ we obtain a contradiction to (R'). Note that the existence of a triangle cover for a vertex $v \in V_1$ depends only on 'its own' X-set. □

7.2.5 Vertices without clique candidates

Now that we have found triangle covers we intend to finally find clique candidates. As described in Section 3 we want to show that for almost every vertex $v \in V_1$ there exist many clique candidates $x \in V_4$ such that adding the edge $\{v, x\}$ would result in a K_4.

Definition 7.10 (Clique candidates) *For $v \in GV_1'$ and a set $A \subseteq V_2$ we define the set $CC(v, A) \subseteq V_4$ by*

$$CC(v, A) := \{x \in V_4 \mid \exists w \in A\ \exists u \in \Gamma_3(v)\ s.t.\ \{u, w\}, \{x, u\}, \{x, w\} \in E\}.$$

Definition 7.10 allows us to construct the set of clique candidates iteratively. Note that $CC(v, \Gamma_2(v))$ contains all possible clique candidates. We will now show how we can construct a set $A \subseteq \Gamma_2(v)$, adding vertex by vertex until $CC(v, A)$ is large enough. The set A will be called the *processed neighbourhood* of v. For the remainder of the paper we restrict our attention to sets with $|A| \leq \mu_q^2 q$.

A vertex $v \in V_1$ is said to satisfy the *clique candidate property* (C) if

$$\begin{array}{ll}(C) & \forall A \subseteq \Gamma_2(v), |A| \leq \mu_q^2 q \\ & \forall Q \subseteq \Gamma_3(v) \text{ s.t. } Q \text{ is a } CC(v, A)\text{-resistant } (\mu_q^2, \delta_r)\text{-triangle cover :} \\ & \quad (\Gamma_2(v) \setminus A) \cap T(Q) \neq \emptyset\end{array}$$

Definition 7.11 (Vertices without clique candidates)

$$\begin{aligned}\mathcal{B}^{CC}(n, m; \varepsilon) \quad := \quad & \{G \in \mathcal{S}_4^{DMR}(n, m; \varepsilon) \mid \\ & \textit{at least } \delta_c n \textit{ vertices } v \in V_1 \textit{ satisfy } (D) \textit{ but not } (C)\}\end{aligned}$$

Lemma 7.12 ($\mathcal{B}^{CC}$ is small)

$$|\mathcal{B}^{CC}(n, m; \varepsilon)| \leq \beta^m \binom{n^2}{m}^6$$

for all sufficiently large n.

Proof Once again we will use Lemma 6.3 to prove that $\mathcal{B}^{CC}(n, m; \varepsilon)$ is small. For the $\delta_c n$ bad vertices indicated by Definition 7.11 we define the neighbourhood function

$$\begin{aligned}\mathcal{N}(v) \quad := \quad & \{X \in \binom{V_2}{d_v} \mid \exists A \subseteq X, |A| \leq \mu_q^2 q : \\ & \exists Q \subseteq \Gamma_3(v) \text{ s.t. } Q \text{ is a } CC(v, A)\text{-resistant } (\mu_q^2, \delta_r)\text{-triangle cover} \\ & (X \setminus A) \cap T(Q) = \emptyset.\}\end{aligned}$$

In order to estimate $|\mathcal{N}(v)|$ we first choose the set $Q \subseteq \Gamma_3(v)$ (at most 2^{2q} possibilities). Then we fix $a_v := |A|$ for $v \in B$. Clearly, there are at most $\mu_q^2 q < n$ choices for the value a_v (Note that $n 2^{2q} \leq 5^{d_v}$ for n sufficiently large). Recall that $|T(Q_v)| \geq (1 - 3\sqrt{\delta_r})n$, therefore we choose a_v vertices for the set A, and then $d_v - a_v$ in $V_2 \setminus (A \cup T(Q))$. Thus

$$\begin{aligned}|\mathcal{N}(v)| \quad & \leq \quad n 2^{2q} \binom{n}{a_v} \cdot \binom{3\sqrt{\delta_r} n}{d_v - a_v} \overset{4.1(iii)}{\leq} 5^{d_v} \binom{n}{a_v} \cdot (6\delta_r^{1/2})^{d_v - a_v} \binom{n/2}{d_v - a_v} \\ & \leq \quad 5^{d_v} \binom{n}{a_v} \cdot (6\delta_r^{1/2})^{d_v/2} \binom{n - a_v}{d_v - a_v} \overset{4.1(i)}{\leq} 10^{d_v} (3\delta_r^{1/4})^{d_v} \binom{n}{d_v} \overset{(20)}{\leq} \pi(\beta, \delta_c)^{d_v} \binom{n}{d_v},\end{aligned}$$

for n sufficiently large. □

Let $\mathcal{S}_4^{DMRC}(n, m; \varepsilon) := \mathcal{S}_4^{DMR}(n, m; \varepsilon) \setminus \mathcal{B}^{CC}(n, m; \varepsilon)$. Furthermore, we define

$$GV_1'' := \{v \in V_1 \mid v \text{ satisfies } (R) \text{ and } (C)\}.$$

For a graph $G \in \mathcal{S}_4^{DMRC}(n, m; \varepsilon)$ observe that $|GV_1''| \geq (1 - 2\delta_c - \delta_c - 6\varepsilon)n = (1 - 4\delta_c)n$ due to Lemma 7.2, Lemma 7.9 and Definition 7.11.

The following lemma is devoted to the proof that the successive construction of the clique candidates $CC(v, \Gamma_2(v))$ using a processed neighbourhood $A \subseteq \Gamma_2(v)$.

Lemma 7.13 (Many candidates for a K_4) *For every vertex $v \in GV_1''$ there exists a processed neighbourhood $A \subseteq \Gamma_2(v)$ such that*

$$|CC(v, A)| \geq (1 - 2\sqrt{\mu_q})n. \tag{22}$$

Proof We show that as long as $|A| \leq \mu_q^2 q$ and $|CC(v, A)| < (1 - 2\sqrt{\mu_q})n$ we can find a vertex $w \in \Gamma_2(v) \setminus A$ such that $|CC(v, A \cup \{w\}) \setminus CC(v, A)| \geq t_{\mu_q^2}$. This suffices to prove the claim, since $\mu_q^2 q \cdot t_{\mu_q^2} = n \geq (1 - 2\sqrt{\mu_q})n$ and we may thus iteratively add the vertices w to the set A until $CC(v, A)$ is large enough.

Let $X := CC(v, A)$ and recall that $|X| \leq (1 - 2\sqrt{\mu_q})n$. Thus we conclude by the definition of (R) that there exists an X-resistant (ν_q^2, δ_r)-triangle cover $Q \subseteq \Gamma_3(v)$. Due to property (C) it follows that there exists a vertex $w \in (\Gamma_2(v) \setminus A) \cap T(Q)$. Clearly, this vertex satisfies the necessary properties and may be added to A. $\square$

7.2.6 Vertices without a K_4

Let

$$CC_v := CC(v, \Gamma_2(v)).$$

Now that we know that CC_v comprises almost all vertices in V_4, we are basically done. A final (simple) argument shows that $\Gamma_4(v)$ and CC_v must overlap.

We say that a vertex $v \in V_i$ satisfies the *clique property* if the following condition (K) is met:

$$(K) \qquad \Gamma_4(v) \cap CC_v \neq \emptyset$$

Definition 7.14 (Vertices with no K_4)

$$\mathcal{B}^K(n, m; \varepsilon) := \{G \in \mathcal{S}_4^{DMRC}(n, m; \varepsilon) \mid \textit{at least } \delta_c n \textit{ vertices in } GV_1'' \textit{ do not satisfy } (K)\}$$

Lemma 7.15 ($\mathcal{B}^K$ is small)

$$|\mathcal{B}^K(n, m; \varepsilon)| \leq \beta^m \binom{n^2}{m}^6$$

Proof A simple (and final) application of Lemma 6.3 proves the claim. The bad set B with $|B| \geq \delta_c n$ is defined as indicated by Definition 7.14. Let

$$\mathcal{N}(v) := \{X \in \binom{V_4}{d_v} \mid X \cap CC_v = \emptyset\}.$$

From Lemma 7.13 it follows that

$$|\mathcal{N}(v)| \leq \binom{2\sqrt{\mu_q}n}{d_v} \leq (4\mu_q)^{d_v/2} \binom{n}{d_v} \overset{(13)}{\leq} \pi(\beta, \delta_c)^{d_v} \binom{n}{d_v},$$

which completes the proof. $\square$

This also completes the proof of Theorem 2.4: for any graph $G \in \mathcal{S}_4^{DMRC}(n, m; \varepsilon) \setminus \mathcal{B}^K(n, m; \varepsilon)$ all but $\delta_c n$ vertices $v \in GV_1''$ have a vertex $x \in \Gamma_4(v) \cap CC_v$ that builds a K_4 containing v. Hence

$$\begin{aligned} \mathcal{F}_4(n, m; \varepsilon) \quad \subseteq \quad & \mathcal{B}_4^D(n, m; \varepsilon) \cup \mathcal{B}_4^{MC}(n, m; \varepsilon) \cup \mathcal{B}_4^R(n, m; \varepsilon) \cup \\ & \mathcal{B}_4^{CC}(n, m; \varepsilon) \cup \mathcal{B}_4^K(n, m; \varepsilon) \end{aligned}$$

and the result follows from the bounds on the sets $\mathcal{B}_4^*(n, m; \varepsilon)$.

Note that our proof actually yields a somewhat stronger result than Theorem 2.4. The condition (K) could be strengthened to

$$(K') \qquad |\Gamma_4(v) \setminus CC_v| \leq \varepsilon_n q$$

for a suitable constant $\varepsilon_n > 0$ which is large in comparison to $2\sqrt{\mu_q}$. Then Lemma 7.15 is still satisfied. Using this we obtain that almost all neighbours $x \in \Gamma_4(v)$ of a vertex in $v \in GV_1''$ which satisfies conditions (K') are part of a subgraph K_4 together with v. Details are left to the interested reader.

8 Conclusion

In this paper we have proved the special case $G = K_4$ of Conjecture 2.3 and we believe that our techniques might also be useful for the general case $G = K_\ell$. Intuitively speaking, the K_4 is the smallest graph such that the intersection of neighbourhoods has to be dealt with in an intermediary state of the proof. The other cases for which Conjecture 2.3 has been verified so far, i.e., cycles, only involve the intersection of neighbourhoods in the final step of the proof, when the cycle is closed. Since the basic concepts of our proof for the K_4, namely covers and the counting lemma, can in principle be transferred to arbitrary complete graphs K_ℓ, we are currently working on a generalisation of our proof for the case $\ell = 5$ and possibly even larger values of ℓ.

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