K_5-free Subgraphs of Random Graphs

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Abstract

We prove a special case of a conjecture by Kohayakawa, Łuczak and Rödl which, if true, would allow the application of Szemerédi's regularity lemma for sparse random graphs with a certain edge density. We consider 5-partite graphs such that each partition contains n vertices, and every pair of partitions forms an ε-regular pair with m edges. Here, two sets of vertices U and W of size n form an ε-regular pair with m edges if all subsets $U' \subseteq U$ and $W' \subseteq W$ with $|U'|, |W'| \geq \varepsilon n$ satisfy $||E(U', W')| - |U'||W'|m/n^2| \leq \varepsilon|U'||W'|m/n^2$. We show that for each $\beta > 0$, there exist at most $\beta^m \binom{n^2}{m}^{10}$ such graphs which do not contain a K_5 as a subgraph if $m \geq Cn^{5/3}$ for a constant $C = C(\beta)$, ε is sufficiently small, and n is sufficiently large.

Mathematics Subject Classification: 05C80, 05C35

1 Introduction

Over the last decade Szemerédi's regularity lemma [19] has proved to be one of the most powerful tools in modern graph theory. (See also for example [1, 2, 16, 7] for the statement and the use of the lemma.) Unfortunately it can be only applied for dense graphs, that is graphs with $\Theta(n^2)$ edges, because in this case the neighbourhood of most vertices is large enough such that the lemma can be applied recursively. Independently Kohayakawa and Rödl introduced a variant which holds for graphs with additional structure, and in particular for sparse random graphs [8]. However, using this variant to show extremal and Ramsey type results as in the dense case, requires an additional key step since Łuczak showed that one cannot directly generalise the methods used for dense graphs to the sparse case, see [13] where Łuczak is quoted. The missing step has been formulated as a conjecture by Kohayakawa, Łuczak and Rödl [12], see also [7] where the conjecture is called *one of the most important open questions in the theory of random graphs.* Stating the conjecture in its full generality is rather technical, and we refer the reader to the cited literature. However, in the next section we state our main theorem, which is a special case of the conjecture, and we informally say what the conjecture is about.

Heretofore the conjecture was proved only for the special cases of cycles [3, 5, 6, 15, 11, 9, 10, 17] and the complete graph on four vertices [12, 4]. In this paper we prove the conjecture for the complete graph on five vertices thereby improving on a result which showed that almost all slightly denser graphs than demanded in the conjecture contain a K_5 [14].

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2 Main Result

To state our main theorem we need the following two definitions.

Definition 2.1 (ε-regularity) *A bipartite graph $B = (U \dot\cup W, E)$ with $|U| = |W| = n$ and $|E| = m$ is called (ε, n, m)-regular if for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon n$ and $|W'| \geq \varepsilon n$,*

$$\left| |E(U', W')| - m \cdot \frac{|U'| \cdot |W'|}{n^2} \right| \leq \varepsilon m \cdot \frac{|U'| \cdot |W'|}{n^2}.$$

Definition 2.2 *The set $\mathcal{S}_5(n, m; \varepsilon)$ consists of all 5-partite graph $G = (V_1 \dot\cup \ldots \dot\cup V_5, E)$ such that $G[V_i, V_j]$ is (ε, n, m)-regular for all $\{i, j\} \subset \{1, 2, 3, 4, 5\}$. The set $\mathcal{F}_5(n, m; \varepsilon)$ is the set of all graphs in $\mathcal{S}(n, m; \varepsilon)$ which do not contain the K_5 as a subgraph, that is*

$$\mathcal{F}_5(n, m; \varepsilon) := \{G \in \mathcal{S}_5(n, m; \varepsilon) \mid K_5 \not\subseteq G\}.$$

We are now able to state the main result of this paper.

Theorem 2.3 *For any $\beta > 0$ there exist constants $\varepsilon_0 > 0, C > 0, n_0 > 0$ such that*

$$|\mathcal{F}_5(n, m; \varepsilon)| \leq \beta^m \binom{n^2}{m}^{10}$$

for all $m \geq C n^{5/3}$, $n \geq n_0$ and $0 < \varepsilon \leq \varepsilon_0$.

The conjecture of Kohayakawa, Łuczak and Rödl mentioned in the introduction generalises Theorem 2.3 in the sense that a (general) graph H is considered (instead of the K_5) which should not appear as a subgraph, and (instead of $S_5(n, m; \varepsilon)$) a set of graphs which consist of $|V(H)|$-partitions such that any pair of partitions V_i, V_j is (ε, m, n)-regular if the corresponding edge is present in H. Here, as in our theorem, the graphs have to be sufficiently dense and the critical value of m depends on the (2-)density of the graph H.

The remainder of the paper is devoted to the proof of Theorem 2.3. In Section 3 we restate some technical tools from [4]. In Section 4 we introduce the basic concepts we use in our proof called homogeneous sets, covers and multicovers. In Section 5 we investigate the structure of neighbourhoods of vertices in most graphs in $\mathcal{S}_5(n, m; \varepsilon)$. We will see that the neighbourhoods of most vertices are homogeneous and form multicovers. We will also consider the common neighbourhood of two vertices, and of vertices with sets. This leads to cocovers and square-candidate covers. Having showed the existence of square-candidate covers we can proceed in Section 6 as in the proof of the result for the K_4 in [4] and complete the proof of Theorem 2.3.

3 Preliminaries

3.1 Conventions and notation

In this paper we will only consider graphs G in $\mathcal{S}_5(n, m; \varepsilon)$ with partitions $V_i, V_j, V_k, V_l, V_{l'}$. For neighbourhoods inside specific partitions we will use the abbreviations $\Gamma_i(v) := \Gamma(v) \cap V_i$ and $d_i(v) = |\Gamma_i(v)|$. For $F \subseteq E$ we let $\Gamma_i^F(v) = \{u \in V_i \mid \{v, u\} \in F\}$ and define $d_i^F(v) = |\Gamma_i^F(v)|$.

To gain control over the size of certain sets, we will often need to deterministically fix a subset of a given cardinality in a larger set of vertices. In order to do so we assume that the vertices in V_i are ordered in an arbitrary but unique way, say $V_i = \{1, \dots, n\}$. By $[A]_x$ we denote the set $B \subseteq A$ of size $|B| = x$ that contains the x smallest elements in A. If $|A| < x$, we define $[A]_x := A$.

We will not introduce floors and ceilings when we are talking about integral terms, e.g., cardinalities of sets. Since we are only interested in the asymptotic behaviour of those quantities, this would merely introduce lower order error terms which complicate the exposition. However, it would be a standard but laborious task to modify the proofs such that the integrality of all terms is respected. Similarly we will ignore whether certain quantities are integrally divisible, e.g., we will assume that we can partition a set of q vertices into exactly q/p sets of size p.

By a *property* Π for a vertex v we mean a condition which may be true or false for v. For a property Π let $V_i[\Pi]$ denote the vertices in V_i which satisfy Π. The contraposition of a property is given by $\neg\Pi$. The trivial property which is always satisfied is denoted by Ω. If the definition of a property Π incorporates another property Π', we will write $(\Pi \mid \Pi')$. For simplicity $(\Pi \mid \Omega)$ will be abbreviated by (Π). For a family of sets $\mathcal{X} \subseteq \binom{X}{t}$ (satisfying a certain property) we let $\bar{\mathcal{X}} := \binom{X}{t} \setminus \mathcal{X}$.

3.2 Auxiliary technical results

In this subsection we collect a few bounds on binomial coefficients, which will be used repeatedly throughout the proof of Theorem 2.3, and a few counting arguments which were introduced in [4].

Lemma 3.1 (Technical inequalities for binomial coefficients)

(i) If $0 \le b + x \le 0.9a$, *then*
$$\binom{a}{b} \le \binom{a-x}{b} e^{10b}.$$

(ii) If $0 \le x \le 1$, *then*
$$\binom{xa}{b} \le \binom{a}{b} x^b.$$

(iii) If $b := \sum_{i=1}^{k} b_i \le a$, *then*
$$\prod_{i=1}^{k} \binom{a}{b_i} \le \binom{a}{b} (2^k)^b.$$

The following lemma is inspired by a very simple observation. If in a bipartite graph $G = (U \mathbin{\dot\cup} W, E)$ all vertices in U have large degree, then there must also be many vertices in W with large degree. This is proved by an easy counting argument.

Lemma 3.2 (Overlap lemma) *[4] Let* $\varepsilon > 0$. *If a bipartite graph* $G = (U \mathbin{\dot\cup} W, E)$ *satisfies* $d(u) \ge (1-\varepsilon)|W|$ *for all* $u \in U$, *then*
$$|\{w \in W \mid d(w) \ge (1-\sqrt{\varepsilon})|U|\}| \ge (1-\sqrt{\varepsilon})|W|.$$

We want to investigate some properties of a typical regular graph and then show that a graph satisfying these properties contains a K_5. More precisely, we shall see that a graph is untypical or ‘bad’ if it contains *linearly* many vertices which have a somewhat ‘irregular’ neighbourhood.

Definition 3.3 *(*Neighbourhood function*) A* neighbourhood function $\mathcal{N}$ *is given a 5-partite graph $G = (V_1 \cup \ldots \cup V_5, E)$, a set $B \subseteq V_i$, and a value d_v for each $v \in B$ and it computes sets $\mathcal{N}(v) \subseteq \binom{V_j}{d_v}$ for each $v \in B$.*

Definition 3.4 *(*Bad neighbourhood function*) Let $\delta, \pi > 0$ and let $\mathcal{G} \subseteq \mathcal{S}_5(n, m; \varepsilon)$. A neighborhood function $\mathcal{N}$ is called a* bad neighbourhood function *for the graph family $\mathcal{G}$ and the parameters δ, π if the following condition holds:*

For each $G = (V, E) \in \mathcal{G}$ there exist $1 \leq i, j \leq 5$ and a set $B \subseteq V_i$ with $|B| \geq \delta n$ and $d_j(v) \geq m/(2n)$ for all $v \in B$ such that for each $v \in B$

$$\Gamma_j(v) \in \mathcal{N}(v) \quad \textit{and} \quad |\mathcal{N}(v)| \leq \pi^{d_v} \binom{n}{d_v},$$

where $\mathcal{N}(v)$ is applied to the graph $G' = (V, E')$ with edge set $E' := E \setminus E(B, V_j)$, the set B, and $d_v := d_j(v)$.

The following lemma shows how bad neighbourhood functions are used to prove that the cardinality of a certain set of graphs is small.

Lemma 3.5 *[4]* (Counting bad graphs) *Let $\pi, \delta > 0$ with $2\pi^{\delta/2} \leq \delta$. If $\mathcal{N}$ is a bad neighbourhood function for $\mathcal{G} \subseteq \mathcal{S}_5(n, m; \varepsilon)$ and parameters δ, π, then*

$$|\mathcal{G}| \leq \delta^m \binom{n^2}{m}^{10}$$

for $m = \omega(n \log n)$ and n sufficiently large.

3.3 Relations between numbers

In Lemma 3.5 we have already encountered two quantities π and δ which had to satisfy the condition $2\pi^{\delta/2} \leq \delta$. We shall need such conditions frequently and that is why we introduce the following relation.

Definition 3.6 (Relation $\lll$) *Let $\mu, \nu > 0$. We say that $\mu \lll \nu$ if*

$$\nu \geq 1000 \mu^{\nu^2/5}.$$

In order to obtain a constant μ from a constant ν with $\mu \lll \nu$ we define following procedure.

Definition 3.7 ($(\mu)^{\oplus}$ and $(\mu)^{\ominus}$) *Let $f(x) := x^{5/x^2}/1000$. For $0 < \mu < 1$, we define $(\mu)^{\ominus} := f(\mu)$. For $0 < \nu < 1/1000$, we define $(\nu)^{\oplus} := f^{-1}(\nu)$. Furthermore $(\mu)^{\ominus}$ and $(\nu)^{\oplus}$ may be iterated. We set $(\mu)^{\ominus 1} := (\mu)^{\ominus}$ and $(\mu)^{\ominus i+1} := ((\mu)^{\ominus i})^{\ominus}$. If ν is small enough then $(\nu)^{\oplus i}$ is defined analogously.*

Note that $\mu \lll \nu$ if and only if $\mu \leq (\nu)^{\ominus}$.

3.4 Cardinalities of sets

In this subsection we define some quantities which will be restated later but we collect them here for better reference. The following cardinalities of sets of vertices will be used in the proof, where λ will be replaced by one of various constants:

$$\begin{aligned}
q &:= \frac{m}{n}, \\
q_\lambda &:= (1-\lambda)\lambda q, \\
p_\lambda &:= 2\lambda^{-1}\tfrac{n^2}{m}, \\
r_\lambda &:= (1-\lambda)\lambda^2\tfrac{m^2}{n^3}, \\
t_\lambda &:= \lambda^{-1}\tfrac{n^2}{m}, \\
h_\lambda &:= \lambda^{-1}\tfrac{m^3}{n^5}, \\
r_\lambda(x) &:= \tfrac{x}{2p_\lambda} = \tfrac{1}{4}\lambda\tfrac{m}{n^2}x.
\end{aligned}$$

The following quantities depend on specific constants where we set $\sigma := (\mu)^{\ominus 5}$

$$\begin{aligned}
o_\mu &:= \sigma^{-5/2}\tfrac{n^2}{m}, \\
s_\mu &:= r_\sigma(r_\mu) = \tfrac{1}{4}(1-\mu)\mu^2\sigma\tfrac{m^3}{n^5}, \\
d_\mu &:= 2s_\mu \cdot \tfrac{q_\mu}{2o_\mu} = \tfrac{1}{4}(1-\mu)^2\mu^3\sigma^{7/2}\tfrac{m^5}{n^8}, \\
\tilde{p} &:= p_\mu/2 = \mu^{-1}\tfrac{n^2}{m}, \\
\tilde{r} &:= r_\mu(\sigma^2 q_\mu)/2 = \tfrac{1}{8}(1-\mu)\mu^2\sigma^2 q, \\
\hat{q} &:= (1-(\mu)^{\oplus 2})n/\tilde{p},
\end{aligned}$$

Obviously some of the above quantities are very similar or even the same, e.g., $\tilde{p} = t_\mu$. However, these quantities will have a different meaning in the proof and we hope that this redundancy will increase the readability of the exposition.

We will assume that all constants which are denoted by greek letters are smaller than $(1/1000)^{\ominus 5}$. Furthermore the constant ε will always refer to the regularity constant, i.e., to the constant used to define the graph class $\mathcal{S}_5(n,m;\varepsilon)$. We assume that ε is arbitrarily small, i.e., as small as we need it in all subsequent inequalities. In particular, ε is much smaller than all other constants.

4 Basic Concepts

4.1 Homogeneous sets

Unfortunately, the ε-regularity of a graph does not help much when considering subgraphs with $o(n)$ vertices. However, for our proofs we will mostly be content with a weaker property, which we call *homogeneity.* This property can be shown to be hereditary, i.e., it also holds for most subgraphs of a homogeneous graph.

The basic idea behind homogeneity consists in a characterisation of the neighbourhood of sets (not only of single vertices!).

Definition 4.1 (Vertices with a bad degree) *Consider a graph $G \in \mathcal{S}_5(n, m; \varepsilon)$ and let $X \subseteq V_j$. We define*

$$\bar{D}_i(X) := \{v \in V_i \mid \ |d_X(v) - |X| \cdot \tfrac{m}{n^2}| \geq \varepsilon \cdot |X| \tfrac{m}{n^2}\}.$$

It is easy to see that in an ε-regular graph almost all vertices have the 'right' degree into any other partition.

Lemma 4.2 ($\bar{D}$ is small) *Let $X \subseteq V_j$. If $|X| \geq \varepsilon n$ then $|\bar{D}_i(X)| \leq 2\varepsilon n$.*

Proof Follows directly from the definition of ε-regularity. □

Since almost all vertices in an ε-regular graph have $\Theta(m/n)$ neighbours, one can deduce that the neighbourhood of $\Theta(n^2/m)$ vertices typically has size $\Theta(m/n \cdot n^2/m) = \Theta(n)$. Furthermore, the neighbourhoods of the single vertices have disjoint parts of size $\Theta(m/n)$, i.e., every vertex contributes an equal part to the combined neighbourhood of all vertices. In the sequel we give a formal definition of this structure.

Definition 4.3 (Covering vertices) *Let $P \subseteq V_i$. For a property Π, the set $I_j(P; \nu \mid \Pi) \subseteq P \cap V_i[\Pi]$ is defined as a subset of $P \cap V_i[\Pi]$ of maximum cardinality such that for each $p \in I_j(P; \nu \mid \Pi)$, there exists a set $W_p \subseteq \Gamma_j(v_1)$ with $|W_p| = q_\nu := (1 - \nu)\nu q$, and such that the sets $(W_p)_{p \in I_j(P;\nu|\Pi)}$ are pairwise disjoint. If there are several such sets, then $I_j(P; \nu \mid \Pi)$ is assigned one of these sets in an arbitrary but unique way. For $I_j(P; \nu \mid \Omega)$ we write $I_j(P; \nu)$.*

Definition 4.4 (Covered neighbourhood) *Let $P \subseteq V_i$, we define the set $W_j(P; \nu)$ as*

$$W_j(P; \nu) := \bigcup_{i \in I_j(P;\nu)} W_i.$$

In [11] (cf. Lemma 11) an argument has been presented which implies that in an ε-regular graph almost all sets P with $|P| = p_\nu = \nu^{-1} \frac{n^2}{m}$ have a large neighbourhood, that is they satisfy $|W_j(P; \nu)| \geq (1 - \nu)n$. Intuitively, a set is homogeneous, if exactly this property holds. A weaker property than ε-regularity suffices to show the desired bound on $|W_j(P; \nu)|$ and this property forms the basis of the definition of homogeneity. In Section 4.2 we will then see how this property is used to prove the existence of sets P with large neighbourhoods.

Definition 4.5 (Homogeneous sets) *A set $X \subseteq V_i$ is called* ρ-homogeneous *for V_j if for all $\nu \geq \varepsilon$ and all $P \subseteq X$ with $|P| \leq p_\varepsilon := 2\varepsilon^{-1} n^2/m$ one of the following two conditions is satisfied:*

(i) $|W_j(P; \nu)| \geq (1 - \nu)n$,

or

(ii) $|(X \setminus P) \cap \bar{D}_i(V_j \setminus W_j(P; \nu))| \leq \rho |X|$.

The set $\mathcal{H}_j(Z, t; \rho) \subseteq \binom{Z}{t}$ contains all ρ-homogeneous sets in $\binom{Z}{t}$.

When we gain more and more knowledge about typical properties of most vertices in the graph we have to strengthen the definition of homogeneity. For Π-enhanced ρ-homogeneous sets we additionally require that the vertices satisfy a given property Π.

Definition 4.6 (Enhanced homogeneous sets) *A ρ-homogeneous set $X \subseteq V_i$ is called* Π-enhanced *for a property* Π, *if* $|X[\neg\Pi]| \leq \rho|X|$. *The set of* Π*-enhanced ρ-homogeneous sets in* $\binom{Z}{t}$ *is denoted by* $\mathcal{H}_j(Z, t; \rho \mid \Pi) \subseteq \binom{Z}{t}$.

Before we show the crucial property of homogeneity, namely that this property is hereditary, i.e., it typically also holds for subsets, we collect a few simple properties of homogeneity.

Proposition 4.7 (Simple properties of homogeneity)

(i) ε-regularity implies 2ε-homogeneity of V_i for V_j.

(ii) If $\rho \leq \rho'$, then any Π-enhanced ρ-homogeneous set is Π-enhanced ρ'-homogeneous.

(iii) If a set X is Π-enhanced ρ-homogeneous, then any subset $X' \subseteq X$ with $|X'| \geq \alpha|X|$ is Π-enhanced ρ/α-homogeneous.

Proof (i) is a simple consequence of Lemma 4.2. The other claims follow directly from the Definition 4.5 and Definition 4.6. □

Lemma 4.8 (Hereditary nature of homogeneity) *Let $\rho_1 > 0$, and let ρ_2 satisfy $\rho_2 \geq 32\rho_1^{\rho_2}$. Then for all $2\log n \leq t_2 \leq t_1 \leq n$ and any set $X \in \mathcal{H}_j(Z, t_1; \rho_1 \mid \Pi)$,*

$$|\overline{\mathcal{H}}_j(X; t_2, \rho_2 \mid \Pi)| \leq 2\rho_2^{t_2}\binom{t_1}{t_2}.$$

Proof First observe that since $|X[\neg\Pi]| \leq \rho_1 t_1$, the number of sets $Y \in \binom{X}{t_2}$ with $|Y[\neg\Pi]| \geq \rho_2 t_2$ can be bounded from above by

$$\binom{\rho_1 t_1}{\rho_2 t_2}\binom{t_1}{t_2 - \rho_2 t_2} \leq \rho_1^{\rho_2 t_2}\binom{t_1}{\rho_2 t_2}\binom{t_1}{t_2 - \rho_2 t_2} \leq \rho_1^{\rho_2 t_2} 4^{t_2}\binom{t_1}{t_2} \leq \rho_2^{t_2}\binom{t_1}{t_2}.$$

Hence it remains to prove that the conditions of Definition 4.5 are satisfied. For any set S in X of size t_2 which is not ρ_2-homogenous, there exist a $\nu \geq \varepsilon$ and a set S' of size $\bar{p} \leq \min\{p_\varepsilon, t_2 - \rho_2 t_2\}$ with $|W_j(S', \nu)| < (1-\nu)n$ such that for at least $\rho_2 t_2$ vertices in $S \setminus S'$ the degree is not as demanded. Observe that ν is only used to determine the cardinalities of the sets $W_j(P, \nu)$ and q_ν. Hence there are at most n^2 choices for ν. Also, since X is ρ_1-homogenous, there are at most $\rho_1|X|$ vertices in X with wrong degree. Thus

$$\begin{aligned}
|\overline{\mathcal{H}}_j(X; t_2, \rho_2 \mid \Pi)| &\leq n^2 \sum_{\substack{\bar{p} \leq p_\varepsilon \\ \bar{p} \leq t_2 - \rho_2 t_2}} \binom{t_1}{\bar{p}}\binom{t_1 - \bar{p}}{t_2 - \bar{p} - \rho_2 t_2}\binom{\rho_1 t_1}{\rho_2 t_2} \\
&\leq n^2 \sum_{\substack{\bar{p} \leq p_\varepsilon \\ \bar{p} \leq t_2 - \rho_2 t_2}} \binom{t_1}{\bar{p}}\binom{t_1}{t_2 - \bar{p} - \rho_2 t_2}\rho_1^{\rho_2 t_2}\binom{t_1}{\rho_2 t_2} \\
&\leq n^2 \sum_{\substack{\bar{p} \leq p_\varepsilon \\ \bar{p} \leq t_2 - \rho_2 t_2}} 8^{t_2}\rho_1^{\rho_2 t_2}\binom{t_1}{t_2} \leq n^2 t_2 8^{t_2}\rho_1^{\rho_2 t_2}\binom{t_1}{t_2} \\
&\leq 2^{t_2} 16^{t_2}\rho_1^{\rho_2 t_2}\binom{t_1}{t_2} \leq \rho_2^{t_2}\binom{t_1}{t_2}
\end{aligned}$$

□

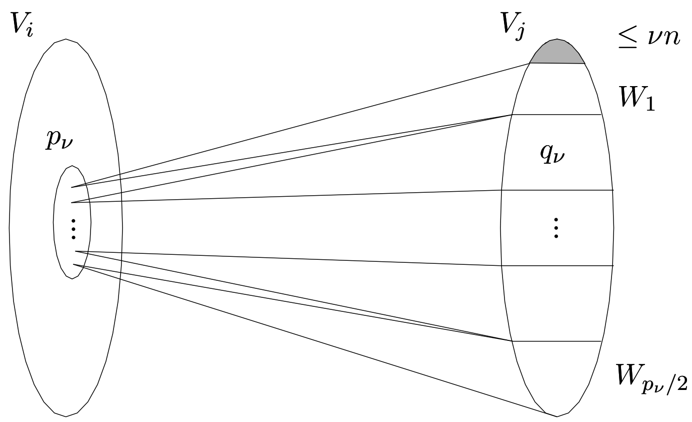

Figure 1: A ν-cover

4.2 Simple covers

In Section 4.1 we have already encountered the following intuitive argument: In a random graph with edge probability m/n^2 a set of size n^2/m has $(n^2/m) \cdot n \cdot (m/n^2) = n$ incident edges on the average. Hence, we may expect that a set P of size $\Theta(n^2/m)$ in partition V_i covers a class V_j for $j \neq i$, i.e., $\Gamma_j(P) \approx V_j$. Such covering sets or *covers* will play a crucial rôle in our proof. Figure 1 illustrates the structure of a cover.

Definition 4.9 (Covers) *A set $P \subseteq V_i$ with $|P| = p_\nu := 2\nu^{-1}\frac{n^2}{m}$ is called a* ν-cover *of V_j if $|I_j(P;\nu)| \geq p_\nu/2$. For a ν-cover P we let $P^* := [I_j(P;\nu)]_{p_\nu/2}$.*

Note that for a ν-cover P we have

$$|\Gamma_j(P)| \geq |W_j(P;\nu)| \geq (p_\nu/2) \cdot q_\nu = (1-\nu)n. \tag{1}$$

The following definition parallels the definition of Π-enhanced ρ-homogeneous sets. A cover is called Π-qualified if the covering vertices can be chosen to satisfy Π.

Definition 4.10 (Qualified covers) *A ν-cover P of V_j is* Π-qualified *for a given property Π if $|I_j(P;\nu \mid \Pi)| \geq p_\nu/2$. Then we set $P^* := [I_j(P;\nu \mid \Pi)]_{p_\nu/2}$. For the set of Π-qualified covers in X we introduce the notation $\mathcal{P}_j(X;\nu \mid \Pi) \subseteq \binom{X}{p_\nu}$. Instead of $\mathcal{P}_j(X;\nu \mid \Omega)$ we simply write $\mathcal{P}_j(X;\nu)$.*

The following lemma establishes the connection between homogeneous sets and covers. It shows that almost all sets of size p_ν (up to a superexponentially small fraction) in ρ-homogeneous sets are ν-covers.

Lemma 4.11 ($\bar{\mathcal{P}}$ is small) *Let $\nu \geq \varepsilon$. If X with $|X| > p_\nu$ is Π-enhanced ρ-homogeneous for V_j then*

$$|\bar{\mathcal{P}}_j(X; \nu \mid \Pi)| \leq (33\rho)^{p_\nu/2} \binom{|X|}{p_\nu},$$

for n sufficiently large.

Proof Consider a set P which is not a Π-qualified ν-cover. Then $|I_j(P, \nu \mid \Pi)| < p_\nu/2 \leq p_\varepsilon$ and hence $W_j(P, \nu) < (p_\nu/2) \cdot q_\nu = (1-\nu)n$. Also all vertices in $P \setminus I_j(P, \nu \mid \Pi)$ belong to $\bar{D}_i(V_j \setminus W_j(P, \nu))$ or do not satisfy Π. By Definition 4.5 and Definition 4.6 there are at most $2\rho|X|$ such vertices in X. Hence we can bound the number of sets which are not Π-qualified ν-covers from above by

$$\begin{aligned}
\sum_{p_\nu - 2\rho|X| \leq \bar{p} \leq p_\nu/2} \binom{|X|}{\bar{p}} \binom{2\rho|X|}{p_v - \bar{p}} &\leq \sum_{\bar{p} \leq p_\nu/2} \binom{|X|}{\bar{p}} \binom{|X|}{p_v - \bar{p}} (2\rho)^{p_\nu - \bar{p}} \\
&\leq \frac{p_\nu}{2} 4^{p_\nu} \binom{|X|}{p_\nu} (2\rho)^{p_\nu/2} \leq (33\rho)^{p_\nu/2} \binom{|X|}{p_\nu},
\end{aligned}$$

when n and thus p_ν is sufficiently large. $\square$

4.3 Multicovers

In the previous section we have introduced covers of size $\Theta(n^2/m)$, whereas in our proof we will often be concerned with sets of size $\Theta(m/n) = \Theta(q)$, i.e., neighbourhoods of vertices. The following definition therefore transfers the notion of a cover to such (larger) sets.

Definition 4.12 (Multicovers) *Let $\nu > 0$. We call a set $Q \subseteq V_i$ a* ν-multicover *of V_j if there exist pairwise disjoint subsets $P_1, \ldots, P_r \subseteq Q$ with $r = r_\nu(|Q|) := \frac{|Q|}{2p_\nu}$ and $|P_1| = \ldots = |P_r| = p_\nu$ such that P_i is a ν-cover of V_j for all $i = 1, \ldots, r$.*

Furthermore, we let $Q^*(\nu) := \bigcup_{k=1}^r P_k^*$, where the parameter ν indicates that Q shall be decomposed into ν-covers P_k in an arbitrary but unique way.

Figure 2 shows the structure of a ν-multicover. Note that the P-sets actually occupy only one half of Q.

Observe that a multicover Q with $|Q| = \Theta(m/n)$ can be decomposed into $\Theta(m/n \cdot m/n^2) = \Theta(m^2/n^3)$ covers $P_1, \ldots, P_r$. Since $\Gamma_j(P_l) \approx V_j$ for all $l = 1, \ldots, r$, we expect that a typical vertex has degree r into Q (up to some constant factor). Observe that the average number of neighbours is $\Theta((m/n) \cdot (m/n^2)) = \Theta(m^2/n^3) = \Theta(r)$. This intuition is confirmed by the following definition and lemma.

Definition 4.13 (Covered neighbourhood of multicovers) *Let $\nu > 0$. For a ν-multicover $Q \subseteq V_i$, and a set $F \subseteq E$ of edges, we define*

$$C_j^F(Q; \nu) := \{w \in V_j \mid |\Gamma_i^F(w) \cap Q| \geq r_\nu(|Q|)/2\}.$$

For $C_j^E(Q, \nu)$ we write $C_j(Q, \nu)$.

Lemma 4.14 (Large covered neighbourhoods) *Let $Q^*(\nu) = \bigcup_k P_k^* \subseteq V_i$ be a ν-multicover of V_j for some $\nu > 0$ with $\sqrt{\nu} \leq \frac{1}{2}$. Then*

$$|C_j(Q; \nu)| \geq (1 - \sqrt{\nu})n.$$

For $F \subseteq E$ with $d_j(v) - d_j^F(v) \leq \nu^2 q$ we have

$$|C_j^F(Q; \nu)| \geq (1 - 3\sqrt{\nu})n.$$

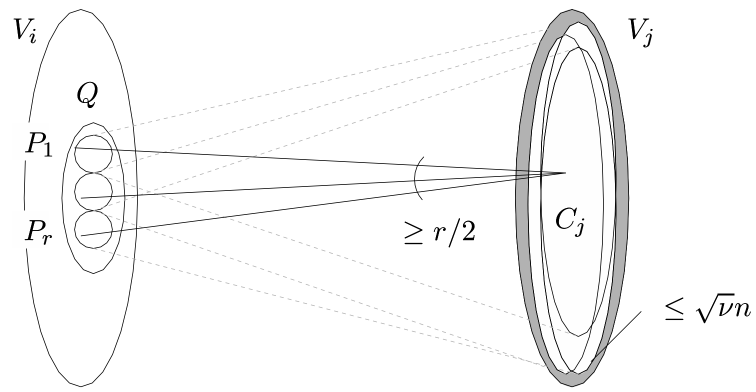

Figure 2: ν-multicover

Proof The first claim was already proved in [4]. The second claim follows from the observation that

$$|\Gamma_j^F(P_i)| \geq (1-\nu)n - p_\nu \cdot \nu^2 q = (1-\nu)n - 2\nu \frac{n^2}{m} \cdot \frac{m}{n} = (1-3\nu)n,$$

and the rest of the proof is completely analogous to the case $F = \emptyset$. □

For a ν-multicover Q the constant ν can be interpreted as the *accuracy* of the cover. The smaller ν is the more vertices in V_j belong to C_j. However, we have to pay a price for better accuracy. As ν decreases the size $p_\nu = 2\nu^{-1}\frac{n^2}{m}$ of the covers inside the multicover increases. This leads to a decrease of $r_\nu(|Q|)$. Thus vertices in C_j see less vertices inside Q as ν gets smaller. Due to this tradeoff we will need multicovers of different accuracy in our proof. Supercovers are ν-multicovers for a variety of constants ν.

Definition 4.15 (Supercovers) *Let $\mathcal{S}$ be a finite set of constants, and let $\tau > 0$. A set Q is called a $(\tau, \mathcal{S})$*-supercover *if every subset $Q' \subseteq Q$ with $|Q'| \geq \tau|Q|$ is a ν-multicover for all $\nu \in \mathcal{S}$.*

At later stages of the proof we will not be content with finding multicovers or supercovers. Additionally, we will need that the vertices which make up the structure of the multicover, i.e., which possess the necessary covering neighbourhoods, satisfy certain properties. To this aim we introduce the following definition.

Definition 4.16 (Qualified multi- and supercovers) *A ν-multicover Q of V_j is* Π-qualified *if the ν-covers that make up Q may be chosen such that they are Π-qualified. A $(\tau, \mathcal{S})$-supercover Q is Π-qualified if every subset Q' with $|Q'| \geq \tau|Q|$ is a Π-qualified ν-multicover* *for all $\nu \in \mathcal{S}$. Let $X \subseteq V_i$. Then $\mathcal{MQ}_j(X; t, \nu \mid \Pi)$ denotes the set of all subsets of X which are Π-qualified ν-multicovers of V_j with t vertices. For the set of Π-qualified $(\tau, \mathcal{S})$-supercovers we write*

$$\mathcal{Q}_j(X; t, \tau, \mathcal{S} \mid \Pi) := \{Z \in \binom{X}{t} \quad \mid \quad \forall Z' \subseteq Z, |Z'| \geq \tau t : \\ Z' \in \bigcap_{\nu \in \mathcal{S}} \mathcal{MQ}_j(X; |Z'|, \nu \mid \Pi)\}.$$

Analogously to Lemma 4.11 for covers, we now show that almost all subsets of a ρ-homogeneous set are ν-multicovers.

Lemma 4.17 ($\bar{\mathcal{Q}}$ is small) *Let $0 < \alpha < 1$, and let $\mathcal{S}$ be a finite set consisting of elements ν with $\varepsilon \leq \nu \leq 1$. Also let $\rho_1 > 0$ and ρ_2 satisfy $\rho_2 \geq (34\rho_1/\alpha)^{\rho_2/4} 8|\mathcal{S}|$. Then for any Π-enhanced ρ_1-homogeneous set X, and any t with $p_\varepsilon \leq t \leq (1-\alpha)|X|$,*

$$|\bar{\mathcal{Q}}_j(X; t, \rho_2, \mathcal{S} \mid \Pi)| \leq \rho_2^t \binom{|X|}{t}$$

for n and thus t and $|X|$ sufficiently large.

Proof Let $\rho_2 t \leq t' \leq t$. Consider a randomly chosen subset $Q \subset X$ of size t'. If $Q \in \overline{\mathcal{MQ}}_j(X; t', \nu \mid \Pi)$ for a constant $\nu \in \mathcal{S}$, then any partition of Q into $r' = t'/p_\nu$ sets of size p_ν yields at least $r'/2$ sets which are not Π-enhanced ν-covers. Hence the probability that $Q \in \overline{\mathcal{MQ}}_j(X; t', \nu \mid \Pi)$ is less than the probability that more than half of r' randomly chosen disjoint sets of size p_ν are not Π-enhanced ν-covers. Observe that if $l \leq r'$ sets $P_1, \ldots, P_l$ are chosen then $X \setminus (P_1, \ldots, P_l)$ is still Π-enhanced (ρ_1/α)- homogeneous. Hence by Lemma 4.11 the probability that P_{l+1} is not a Π-enhanced ν-cover is less than $(33\rho_1/\alpha)^{p_\nu/2}$. Therefore we can bound the probability that $Q \in \overline{\mathcal{MQ}}_j(X; t', \nu \mid \Pi)$ from above by

$$2^{r'} \left((33\rho_1/\alpha)^{p_\nu/2} \right)^{r'/2} \leq (34\rho_1/\alpha)^{t'/4},$$

for n and thus t sufficiently large. Hence

$$\begin{aligned} |\bar{\mathcal{Q}}_j(X; t, \rho_2, \mathcal{S} \mid \Pi)| &\leq \sum_{\nu \in \mathcal{S}} \sum_{\rho_2 t \leq t' \leq t} |\overline{\mathcal{MQ}}_j(X; t', \nu \mid \Pi)| \binom{|X|}{t - t'} \\ &\leq |\mathcal{S}| \sum_{\rho_2 t \leq t' \leq t} (34\rho_1/\alpha)^{t'/4} \binom{|X|}{t'} \binom{|X|}{t - t'} \\ &\leq |\mathcal{S}| \sum_{\rho_2 t \leq t' \leq t} (34\rho_1/\alpha)^{t'/4} 4^t \binom{|X|}{t} \\ &\leq |\mathcal{S}| t (34\rho_1/\alpha)^{\rho_2 t/4} 4^t \binom{|X|}{t} \leq \rho_2^t \binom{|X|}{t}. \end{aligned}$$

□

Remark 4.18 *We will only consider a fixed set $\mathcal{S}$ of finite size (less than 6 in fact). Observe that in Lemma 4.17 only the size of the set $\mathcal{S}$ is important not the actual elements (as long as they are bigger than ε). So often we will just say that a set is a τ-supercover and leave the set $\mathcal{S}$ implicit.*

5 Neighbourhoods

5.1 Structure of simple neighbourhoods

Now it is time to introduce the first few of several properties by which we intend to characterise 'good' graphs, i.e., graphs for which the occurrence of complete subgraphs can be guaranteed. These properties will refer to the size and the structure of the neighbourhood of a single vertex.

We say that a vertex $v \in V_i$ satisfies the *degree property* if the following condition (D) is met:

$$(D) \qquad (1-\varepsilon)q \le d_j(v) \le (1+\varepsilon)q \text{ for all } j \neq i. \tag{2}$$

The next lemma shows that most vertices in ε-regular graphs satisfy (D).

Lemma 5.1 (Most vertices satisfy (D)) *For a graph $G \in \mathcal{S}_5(n, m; \varepsilon)$, we have*

$$|V_i[(\neg D)]| \le 8\varepsilon n.$$

Proof Follows directly from Lemma 4.2. $\square$

A vertex $v \in V_i$ satisfies the *homogeneity property* $(H_\rho^{j,k})$ if the following condition is met:

$$(H_\rho^{j,k} \mid \Pi) \quad \Gamma_j(v) \text{ is } \Pi\text{-enhanced } \rho\text{-homogeneous for } V_k. \tag{3}$$

For the conjunction of two properties $(A) \wedge (B)$, we introduce the shortcut (A, B). For example if a vertex satisfies $(H_\rho^{j,k})$ and (D), then it satisfies $(H_\rho^{j,k}, D)$. The following simple fact is stated here for future reference.

Proposition 5.2 *Let $v \in V_i$ satisfy $(H_\rho^{j,k}, D)$. Then any set $Q' \subseteq \Gamma_j(v)$ with $|Q'| \ge \tau q$ is $2\rho/\tau$-homogeneous.*

Proof Note that $|\Gamma_j(v)| \le (1+\varepsilon)q \le 2q$ since v satisfies property (D). Now Proposition 5.2 follows directly from Proposition 4.7. $\square$

We say that a vertex $v \in V_i$ satisfies the *cover property* $(C_{\rho,\mathcal{S}}^{j,k})$ if the following condition is met:

$$(C_{\rho,\mathcal{S}}^{j,k} \mid \Pi) \quad \Gamma_j(v) \text{ is a } (\Pi\text{-qualified}) \ (\rho, \mathcal{S})\text{-supercover of } V_k. \tag{4}$$

Observe that $(H_\rho^{j,k} \mid \Pi)$ and $(C_{\rho,\mathcal{S}}^{j,k} \mid \Pi)$ are monotonous, i.e., for $\rho' \ge \rho$, $(H_\rho^{j,k} \mid \Pi)$ implies $(H_{\rho'}^{j,k} \mid \Pi)$ and $(C_{\rho,\mathcal{S}}^{j,k} \mid \Pi)$ implies $(C_{\rho',\mathcal{S}}^{j,k} \mid \Pi)$. This is due to Proposition 4.7 (ii) and the fact that a $(\rho, \mathcal{S})$-supercover is also a $(\rho', \mathcal{S})$-supercover for $\rho' \ge \rho$. Hence the constants cannot be too small.

Since the homogeneity property and the cover property bear some similarity (namely both deal with the existence of covers and multicovers), we combine them to the property

$$(Q_{\rho,\mathcal{S}}^{j,k} \mid \Pi) := (H_\rho^{j,k} \mid \Pi) \wedge (C_{\rho,\mathcal{S}}^{j,k} \mid \Pi).$$

As for all subsequently defined properties we introduce a set of bad graphs with respect to $(Q_{\rho,\mathcal{S}}^{j,k} \mid \Pi)$, namely

$$\mathcal{B}_{i,j,k}^{Q|\Pi}(n, m; \mathcal{S}, \varepsilon, \rho) := \{G \in \mathcal{S}_5(n, m; \varepsilon) \mid |V_i[(\neg Q_{\rho,\mathcal{S}}^{j,k} \mid \Pi, D)]| \ge (\rho)^{\oplus} n\}, \tag{5}$$

and a set of good graphs, in this case

$$\mathcal{S}^Q(n, m; \mathcal{S}, \varepsilon, \rho) := \mathcal{S}_5(n, m; \varepsilon) \setminus \bigcup_{i \neq j \neq k} \mathcal{B}_{i,j,k}^{Q|\Omega}(n, m; \mathcal{S}, \varepsilon, \rho).$$

As mentioned in Remark 4.18 we will often leave the set $\mathcal{S}$ implicit. If we do so, we just write

$$\mathcal{B}^{Q|\Pi}_{i,j,k}(n,m;\varepsilon,\rho) \text{ and } \mathcal{S}^{Q}(n,m;\varepsilon,\rho).$$

We say that a vertex $v \in V_i$ has property (Q_ρ) if it has property $(Q^{j,k}_\rho \mid \Omega)$ for all $j \neq i$, $k \neq j, i$. Observe that all graphs $G \in \mathcal{S}^Q(n,m,\varepsilon,\rho)$ satisfy

$$|V_i[\neg(Q_\rho, D)]| \leq |V_i[(\neg D)]| + |V_i[(\neg Q_\rho, D)]| \leq 8\varepsilon n + 12(\rho)^{\oplus} n \leq (\rho)^{\oplus 2} n \tag{6}$$

for all $i = 1, \ldots, 5$, and hence V_i is (Q_ρ, D)-enhanced $(\rho)^{\oplus 2}$-homogeneous.

We will now show that $\mathcal{B}^{Q}_{i,j,k}(n,m;\mathcal{S},\varepsilon,\rho)$ is small.

Lemma 5.3 ($\mathcal{B}^Q$ is small) *Let Π be a property which can be verified for a vertex $v \in V_i$ without knowing the edges between V_i and V_j, let V_j be Π-enhanced α-homogeneous with $(\alpha)^{\oplus} \leq \rho$, and let $\mathcal{S}$ consist of less than, say 100 constants greater than ε. Then*

$$|\mathcal{B}^{Q|\Pi}_{i,j,k}(n,m;\mathcal{S},\varepsilon,\rho)| \leq ((\rho)^{\oplus})^m \binom{n^2}{m}^{10}.$$

Proof Consider the set of bad vertices $B \subseteq V_i[(\neg Q^{j,k}_\rho, D)]$ with $|B| \geq (\rho)^{\oplus} n$ implied by (5).

Case 1: There are at least $(\rho)^{\oplus} n/2$ bad vertices in V_i for which $\Gamma_j(v)$ is not Π-enhanced ρ-homogeneous for V_k, i.e., $(H^{j,k}_\rho \mid \Pi)$ is not satisfied. We define the neighbourhood function

$$\mathcal{N}(v) := \{X \in \binom{V_j}{d_v} \quad \mid \quad X \in \overline{\mathcal{H}}_k(V_j; d_v, \rho \mid \Pi)\}$$

Observe that $d_v \geq q/2$ since $v \in V_i[D]$. Since V_j is Π-enhanced α-homogeneous, we deduce by Lemma 4.8 that $|\mathcal{H}_k(V_j; d_v, \rho \mid \Pi)| \leq 2\rho^{d_v}\binom{n}{d_v}$. We obtain $|\mathcal{N}(v)| \leq 2\rho^{d_v}\binom{n}{d_v}$. Now the claim follows for this case by Lemma 3.5.

Case 2: There are at least $(\rho)^{\oplus} n/2$ bad vertices for which $\Gamma_j(v)$ is no Π-qualified ρ-supercover for V_k, i.e. $(C^{j,k}_{\rho,\mathcal{S}})$ is not satisfied. The corresponding neighbourhood function is given by

$$\mathcal{N}(v) := \{X \in \binom{V_j}{d_v} \quad \mid \quad X \in \bar{\mathcal{Q}}_k(V_j; d_v, \rho, \mathcal{S} \mid \Pi)\}$$

Recall that V_j is Π-enhanced α-homogeneous. By Lemma 4.17 we have $|\bar{\mathcal{Q}}_k(V_j; d_v, \rho, \mathcal{S} \mid \Pi)| \leq (\rho)^{d_v}\binom{n}{d_v}$ and hence $|\mathcal{N}(v)| \leq (\rho)^{d_v}\binom{n}{d_v}$ similar to the previous case. This completes the proof. □

The next property we define for specific values of i, j, k only since we shall only need it for this set of values. Assume that vertex $v \in V_4$ has a ν-multicover Q of V_2 in its neighbourhood $\Gamma_5(v)$. Since $C_2(Q;\nu)$ contains $(1-\nu)n$ vertices in V_2 and ν is assumed to be small, we expect that the neighbourhood $\Gamma_2(v)$ and $C_2(Q;\nu)$ overlap significantly. We say that a vertex $v \in V_4$ satisfies the *triangle property* if the following condition (T_ρ) is met:

$$(T_\rho) \quad \forall Q \subseteq \Gamma_5(v),\ Q \text{ is a } \rho\text{-multicover of } V_2 :\ |\Gamma_2(v) \setminus C_2(Q;\rho)| \leq (\rho)^{\oplus} q. \tag{7}$$

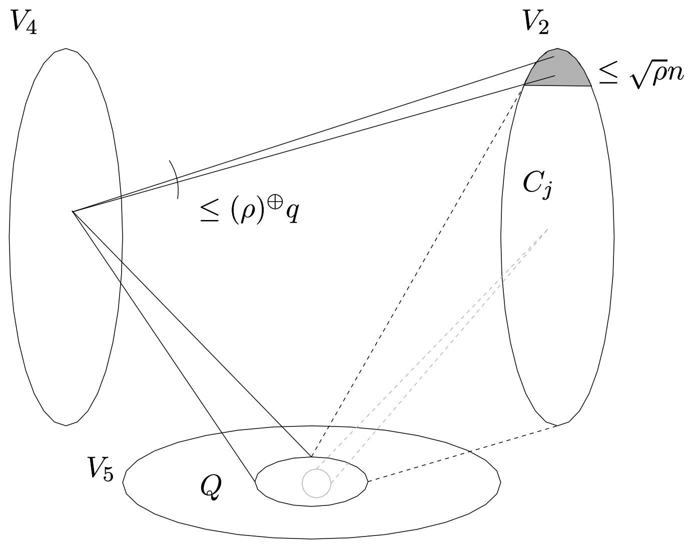

Figure 3: Property (T_ρ)

Figure 3 illustrates property (T_ρ). This leads to the bad set

$$\mathcal{B}^T(n, m; \varepsilon, \rho) \ := \ \{G \in \mathcal{S}_5(n, m; \varepsilon) \mid \ |V_4[(\neg T_\rho, D)]| \geq (\rho)^{\oplus 2} n\}, \tag{8}$$

and the good set

$$\mathcal{S}^{QT}(n, m; \varepsilon, \rho) := \mathcal{S}^Q(n, m; \varepsilon, \rho) \setminus \mathcal{B}^T(n, m; \varepsilon, \rho)).$$

Observe that all graphs $G \in \mathcal{S}^{QT}$ satisfy

$$|V_4[\neg(Q_\rho, T_\rho, D)]| \leq |V_4[(\neg D)]| + |V_4[(\neg Q_\rho, D)]| + |V_4[(\neg T_\rho, D)]| \leq 3(\rho)^{\oplus 2} n \leq (\rho)^{\oplus 3} n \tag{9}$$

by Lemma 5.1, (5) and (8). Hence for all graphs $G \in \mathcal{S}^{QT}$, V_4 is (Q_ρ, T_ρ, D)-enhanced $(\rho)^{\oplus 3}$-homogeneous.

Lemma 5.4 ($\mathcal{B}^T$ is small)

$$|\mathcal{B}^T(n, m; \varepsilon, \rho)| \leq ((\rho)^{\oplus 2})^m \binom{n^2}{m}^{10}.$$

Proof We again use Lemma 3.5, i.e., we have to show that there exists an appropriate definition of a bad neighbourhood function $\mathcal{N}$ for $(\rho)^{\oplus 2} n$ bad vertices $B \subseteq V_4[(\neg T_\rho, D)]$. We let

$$\mathcal{N}(v) := \{X \in \binom{V_2}{d_v} \quad | \quad \exists Q \subseteq \Gamma_5(v),\ Q \text{ is a } \rho\text{-multicover of } V_2 :$$
$$|X \setminus C_2(Q_k; \rho)| \geq (\rho)^{\oplus} q\}.$$

When counting $|\mathcal{N}(v)|$ we first choose the set Q. Since $Q \subseteq \Gamma_5(v)$, and $d_5(v) \le (1+\varepsilon)q$, we can bound the number of choices for Q by $2^{(1+\varepsilon)q}$. As $|C_2(Q;\rho)| \ge (1-\sqrt{\rho})n$ by Lemma 4.14, we deduce that

$$|\mathcal{N}(v)| \le 2^{(1+\varepsilon)q}\binom{\sqrt{\rho}n}{(\rho)^{\oplus}q}\binom{n}{d_v-(\rho)^{\oplus}q} \overset{\text{L3.1(ii,iii)}}{\le} 4^{d_v}\rho^{(\rho)^{\oplus}q/2}\cdot 4^{d_v}\binom{n}{d_v} \le ((\rho)^{\oplus})^{d_v}\binom{n}{d_v},$$

proving the claim. □

5.2 Overlapping neighbourhoods

After we have collected several properties in the preceding section concerning the neighbourhood of single vertices we will now examine the common neighbourhood of two vertices.

Definition 5.5 (Overlapping neighbourhoods) *Let $\rho,\sigma > 0$, let $X \subseteq V_j$ and let $P \subseteq V_k$ be an $(H_\rho^{j,l}, H_\rho^{j,l'})$-qualified σ-cover of V_j. A vertex $v \in V_i$ has an X-*resistant (ρ,σ)-overlapping neighbourhood with P in V_j for V_l and $V_{l'}$ *if*

$$\begin{aligned}
&\exists \tilde{P} \subseteq P^*, |\tilde{P}| = 2\sqrt{\sigma}p_\sigma\\
&\forall u \in \tilde{P}\ \exists R_u \subseteq (\Gamma_j(v)\cap\Gamma_j(u))\setminus X, |R_u| = r_\sigma := (1-\sigma)\sigma^2\tfrac{m^2}{n^3}:\\
&\qquad R_u \textit{ is a } (\rho)^{\oplus}\textit{-supercover of } V_l \textit{ and } V_{l'}, \textit{ and}\\
&\qquad \textit{the sets } R_u \textit{ are pairwise disjoint for all } u.
\end{aligned}$$

The overlapping neighbourhood is called Π-qualified *if the sets R_u are Π-qualified $(\rho)^{\oplus}$-supercovers.*

Following the Remark 4.18 we left the set $\mathcal{S}$ implicit when talking about a $(\rho)^{\oplus}$-supercover in the previous definition as we will often do in the future. Observe, that the definition of an X-resistant (ρ,σ)-overlapping neighbourhood involves all five partitions of a graph in $\mathcal{S}_5(n,m;\varepsilon)$, and the sets R_u have to be supercovers for all partitions V_l with $l \ne i,j,k$. To simplify notation we will only specify the partitions in which the vertex v, the set P and the set X lie, and leave the remaining two partitions which have to be covered by R_u implicit.

A vertex $v \in V_i$ satisfies the *overlapping neighbourhood property* if the following condition $(N_{\rho,\sigma}^{k,j})$ is met:

$$\begin{aligned}
(N_{\rho,\sigma}^{k,j}[X])\qquad &\forall P \subseteq V_k,\ P \text{ is a } (H_\rho^{j,l}, H_\rho^{j,l'})\text{-qualified } \sigma\text{-cover of } V_j:\\
&\quad v \text{ has an } X\text{-resistant } (\rho,\sigma)\text{-overlapping neighbourhood}\\
&\quad \text{with } P \text{ in } V_j \text{ for } V_l \text{ and } V_{l'}
\end{aligned}\tag{10}$$

Figure 4 illustrates the structure of an overlapping neighbourhood. Let

$$\begin{aligned}
\mathcal{B}_{i,k,j}^{N}(n,m;\varepsilon,\rho,\sigma) \quad:=\quad &\{G \in \mathcal{S}_5(n,m;\varepsilon) \mid \exists X \subseteq V_j, |X| \le (1-(\sigma)^{\oplus})n:\\
&|V_i[(\neg N_{\rho,\sigma}^{k,j}[X], D)]| \ge (\sigma)^{\oplus 2}n\}
\end{aligned}\tag{11}$$

denote the set of graphs which are bad with respect to property $(N_{\rho,\sigma}^{k,j}[X])$.

Lemma 5.6 *Let $(\rho)^{\oplus} \lll \sigma$. Then*

$$|\mathcal{B}_{i,k,j}^{N}(n,m;\varepsilon,\rho,\sigma)| \le ((\sigma)^{\oplus 2})^m\binom{n^2}{m}^{10}.$$

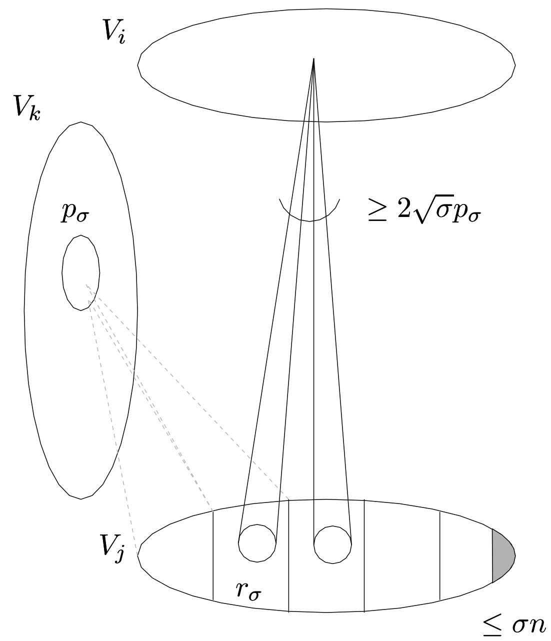

Figure 4: Property $(N^{k,j}_{\rho,\sigma}[X])$

Proof First assume that X is fixed and satisfies

$$|X| =: (1-\lambda)n \leq (1-(\sigma)^{\oplus})n. \tag{12}$$

Since $(\sigma)^{\oplus} \geq \varepsilon$, it follows from Lemma 4.2 that $|\bar{D}_i(V_j \setminus X)| \leq 2\varepsilon n$. Hence we can find $(1-10\varepsilon)n \geq (1-(\sigma)^{\oplus 2}/2)n$ vertices $v \in V_i[(D)]$ with

$$|\Gamma_j(v) \setminus X| \geq (1-\varepsilon)\frac{m}{n^2}|V \setminus X| \geq (1-\varepsilon)\lambda q.$$

There are two possibilities why a vertex $v \in V_i$ with $|\Gamma_j(v) \setminus X| \geq (1-\varepsilon)\lambda q$ might fail to satisfy $(N^{k,j}_{\rho,\sigma}[X])$: Either there are less than $2\sqrt{\sigma}p_\sigma$ vertices $u \in P$ such that v and u have a common neighbourhood R_u of size at least r_σ, or the sets R_u are not supercovers. We will now show that both scenarios happen very rarely.

More precisely we now want to show that there are less than

$$t := \left(\frac{(\sigma)^{\oplus 2}}{4}\right)^m \binom{n^2}{m}^{10}$$

graphs such that for more than $(\sigma)^{\oplus 2}/4$ vertices $v \in V_i$ with $|\Gamma_j(v) \setminus X| \geq (1-\varepsilon)\lambda q$ there exists an $(H^{j,l}_\rho, H^{j,l'}_\rho)$-qualified σ-cover $P \subseteq V_k$ of V_j such that

$$|\{u \in P^* \mid |(W_u \cap \Gamma_j(v)) \setminus X| \geq r_\sigma\}| \leq 3\sqrt{\sigma}p_\sigma. \tag{13}$$

Consider the following neighbourhood function in order to apply Lemma 3.5.

$$\mathcal{N}(v) := \{Y \in \binom{V_j}{d_v} \quad | \quad |Y \setminus X| \geq (1-\varepsilon)\lambda q \ \wedge$$

$$
\begin{aligned}
&\exists P \subseteq V_k,\ P \text{ is an } (H_\rho^{j,l}, H_\rho^{j,l'})\text{-qualified } \sigma\text{-cover of } V_j\\
&\exists \bar{P} \subseteq P^*, |\bar{P}| = |P^*| - 3\sqrt{\sigma}p_\sigma = (1-6\sqrt{\sigma})p_\sigma/2\\
&\forall u \in \bar{P}:\ |(W_u \cap Y) \setminus X| \le r_\sigma\}.
\end{aligned}
$$

In order to count $|\mathcal{N}(v)|$ we first choose P and $\bar{P}$. There are at most $n^{p_\sigma} \cdot 2^{p_\sigma}$ possibilities to choose these sets. Observe that $n^{p_\sigma} \cdot 2^{p_\sigma} \le 2^q$ for sufficiently large n. Let $F := \bigcup_{u \in \bar{P}} W_u$ and observe that

$$
|F| = (1-6\sqrt{\sigma})p_\sigma/2 \cdot q_\sigma = (1-6\sqrt{\sigma})(1-\sigma)n \ge (1-7\sqrt{\sigma})n.
$$

Now we fix

$$
y_v := |Y \setminus X| \ge (1-\varepsilon)\lambda q \ge (1-\varepsilon)(\sigma)^{\oplus} q \quad \text{and} \quad f_v := |(Y \setminus X) \cap F|
$$

(at most $n \cdot n \le 2^q$ possibilities; recall that d_v is part of the input to $\mathcal{N}(v)$ and thus does not have to be counted.). We choose $d_v - y_v$ neighbours in X and y_v neighbours in $V_j \setminus X$. These y_v vertices are further subdivided. f_v neighbours are selected from $F \setminus X$ and $y_v - f_v$ neighbours come from $V_j \setminus (F \cup X)$. Since

$$
f_v = |(Y \setminus X) \cap F| \le |\bar{P}| \cdot r_\sigma \le |P^*| \cdot r_\sigma = p_\sigma/2 \cdot r_\sigma \le \sigma q
$$

and

$$
y_v - f_v \ge ((1-\varepsilon)(\sigma)^{\oplus} - \sigma)q \ge (\sigma)^{\oplus} d_v/2,
$$

we obtain

$$
\begin{aligned}
|\mathcal{N}(v)| &\le 2^q \sum_{f_v \le \sigma q} \sum_{\substack{(\sigma)^{\oplus} d_v/2 + f_v\\ \le y_v \le d_v}} \binom{n}{d_v - y_v}\binom{n}{f_v}\binom{7\sqrt{\sigma}n}{y_v - f_v}\\
&\le 2^{2d_v}\, \sigma q\, d_v\, 8^{d_v} \left((7\sqrt{\sigma})^{(\sigma)^{\oplus}/2}\right)^{d_v} \cdot \binom{n}{d_v} \le ((\sigma)^{\oplus})^{d_v}\binom{n}{d_v}.
\end{aligned}
$$

Let $\rho' = (\rho)^{\oplus}$. We next prove that there are less than

$$
t = \left(\frac{(\sigma)^{\oplus 2}}{4}\right)^m \binom{n^2}{m}^{10}
$$

graphs for which there are $((\sigma)^{\oplus 2}/4)n$ vertices such that for all $(H_\rho^{j,l}, H_\rho^{j,l'})$-qualified σ-covers $P \subseteq V_k$

$$
|\{u \in P^* \ |\ |(W_u \cap \Gamma_j(v)) \setminus X| \ge r_\sigma\}| \ge 3\sqrt{\sigma}p_\sigma
$$

but there exists a $(H_\rho^{j,l}, H_\rho^{j,l'})$-qualified σ-cover $P \subseteq V_k$ such that

$$
\begin{aligned}
&\exists \bar{P} \subseteq P^*, |\bar{P}| = \sqrt{\sigma}p_\sigma\\
&\forall u \in \bar{P}\ \exists R_u \subseteq (W_u \cap \Gamma_j(v)) \setminus X, |R_u| = r_\sigma:\ R_u \text{ is } \underline{\text{no}}\ \rho'\text{-supercover of } V_l \text{ or } V_{l'}\}
\end{aligned}
$$

Again we will apply Lemma 3.5 and define the neighbourhood function

$$
\begin{aligned}
\mathcal{N}(v) := \{Y \in \binom{V_j}{d_v} \quad | \quad &\exists P \subseteq V_k,\ P \text{ is an } (H_\rho^{j,l}, H_\rho^{j,l'})\text{-qualified } \sigma\text{-cover}\\
&\exists \bar{P} \subseteq P^*, |\bar{P}| = \sqrt{\sigma}p_\sigma\\
&\forall u \in \bar{P}\ \exists R_u \subseteq (W_u \cap Y) \setminus X, |R_u| = r_\sigma:\\
&\quad R_u \text{ is } \underline{\text{no}}\ \rho'\text{-supercover of } V_l \text{ or } V_{l'}\ \}.
\end{aligned}
$$

As in the previous case the number of possible choices for P and $\bar{P}$ can be bounded by 2^q for n sufficiently large. Since P is assumed to be $(H_\rho^{j,l}, H_\rho^{j,l'})$-qualified, $\Gamma_j(u)$ is ρ-homogeneous for V_l and $V_{l'}$ for all $u \in P^*$. By Proposition 4.7, W_u with

$$|W_u| = q_\sigma \geq \tfrac{3}{4}\sigma q \geq \tfrac{\sigma}{2} d_j(u)$$

is thus $2\rho/\sigma \leq \rho^{1/2}$-homogeneous and the number of sets $R \in \binom{W_u}{r_\sigma}$ which are not ρ'-supercovers is bounded by $2\rho'^{r_\sigma}\binom{q_\sigma}{r_\sigma}$ due to Lemma 4.17.

Hence

$$\begin{aligned} |\mathcal{N}(v)| &\leq 2^q \left((2\rho')^{r_\sigma}\binom{q_\sigma}{r_\sigma}\right)^{\sqrt{\sigma}p_\sigma} \binom{n}{d_v - \sqrt{\sigma}p_\sigma r_\sigma} \leq 2^q (2\rho')^{r_\sigma\sqrt{\sigma}p_\sigma}\binom{\sqrt{\sigma}p_\sigma q_\sigma}{\sqrt{\sigma}p_\sigma r_\sigma}\binom{n}{d_v - \sqrt{\sigma}p_\sigma r_\sigma} \\ &\leq (32\rho'^{\sigma^2/2})^{d_v}\binom{n}{d_v} \leq \sigma^{d_v}\binom{n}{d_v}, \end{aligned}$$

since

$$p_\sigma \cdot r_\sigma = 2\sigma^{-1}\frac{n^2}{m} \cdot (1-\sigma)\sigma^2\frac{m^2}{n^3} = 2(1-\sigma)\sigma q \geq \sigma d_v.$$

Now we can easily prove the theorem. We first showed that there are $(1-(\sigma)^{\oplus 2}/2)n$ vertices $v \in V_i[(D)]$ with

$$|\Gamma_j(v) \setminus X| \geq (1-\varepsilon)\frac{m}{n^2}|V \setminus X| \geq (1-\varepsilon)\lambda q.$$

We then showed that there are at most t graphs for which there are more than $((\sigma)^{\oplus 2}/4)n$ of these vertices v with $(H_\rho^{j,l}, H_\rho^{j,l'})$-qualified σ-covers $P \subseteq V_k$ of V_j such that

$$|\{u \in P^* \mid \ |(W_u \cap \Gamma_j(v)) \setminus X| \geq r_\sigma\}| \leq 3\sqrt{\sigma}p_\sigma.$$

We then showed that there are at most t graphs for which there are more than $((\sigma)^{\oplus 2}/4)n$ of the remaining vertices which do not satisfy the required conditions. Hence we have for every fixed set X that for all but

$$2t \leq \left(\frac{(\sigma)^{\oplus 2}}{2}\right)^m \binom{n^2}{m}^{10}$$

graphs there are more than $(1-(\sigma)^{\oplus 2})n$ vertices which have an X resistant (ρ,σ)-overlapping neighbourhood with every $(H_\rho^{j,l}, H_\rho^{j,l'})$-qualified σ-cover P in V_k. The claim now follows since there are at most $2^n \leq 2^m$ different sets X in V_j. □

In the following section we will deal with graphs in

$$\mathcal{S}^{QTN}(n,m;\varepsilon,\rho,(\mu)^{\ominus 5}) := \mathcal{S}^{QT}(n,m;\varepsilon,\rho) \setminus \mathcal{B}^N_{3,4,5}(n,m,\varepsilon,\rho,(\mu)^{\ominus 5}),$$

and vertices $v \in V_1$ which satisfy $(N^{3,4}_{\rho,\mu}[V_4[\neg(Q_\rho,T_\rho)]])$. As we have observed before (6), $|V_4[\neg(Q_\rho,T_\rho)]| \leq (\rho)^{\oplus 2}$ for all graphs in $\mathcal{S}^{QTN}(n,m;\varepsilon,\rho,\mu)$. Hence if $(\rho)^{\oplus 2} \leq \mu$ and $(\rho)^{\oplus 2} \leq 1-(\mu)^{\oplus}$ then by Lemma 5.6 less than

$$((\mu)^{\oplus 2})^m \binom{n^2}{m}^{10} \tag{14}$$

graphs in $\mathcal{S}^{QTN}(n,m;\varepsilon,\rho,\mu)$ have more than $(\mu)^{\oplus 2}n$ vertices in V_1 which do not satisfy $(N^{3,4}_{\rho,\mu}[V_4[\neg(Q_\rho,T_\rho)]])$ but (D). In addition, if $(\rho)^{\oplus} \lll (\mu)^{\ominus 5}$ then

$$|\mathcal{S}^{QTN}(n,m;\varepsilon,\rho,(\mu)^{\ominus 5})| \geq (1-3((\mu)^{\ominus 3})^m)\binom{n^2}{m}^{10}. \tag{15}$$

5.3 Cocovers

In Section 5.2 we have seen that a constant fraction of a typical set $P \subseteq V_4$ with $\Theta(n^2/m)$ vertices has a common neighbourhood of size $\Theta(m^2/n^3)$ with a given vertex $u \in V_3$, yielding $\Theta(n^2/m \cdot m^2/n^3) = \Theta(m/n)$ common neighbours. If these common neighbourhoods were (rather) disjoint for $\Theta(n^2/m)$ vertices $u_1, u_2, \ldots$, we could expect that all common neighbourhoods together 'tile' the n vertices in V_5, since $\Theta(m/n \cdot n^2/m) = \Theta(n)$. The aim of this section is to show that this is indeed the case.

More specifically, we will fix a vertex $v \in V_1$ and a vertex $u \in V_3$ with common neighbourhood $R \in V_4$ of size $\Theta(m^2/n^3)$. Note that $m^2/n^3 \geq n^2/m$ for $m \geq n^{5/3}$. Hence, we can divide R into several subsets $P_1, P_2, \ldots$ of size $\Theta(n^2/m)$ for which the above arguments hold, i.e., the common neighbourhoods with u contain $\Theta(n^2/m \cdot m^2/n^3) = \Theta(m/n)$ vertices. If we take $\Theta(n^2/m)$ vertices $u_1, u_2, \ldots \in V_3$, we expect their neighbourhoods to tile V_5 (cf. Figure 5).

Sets $\{u_1, u_2, \ldots\} \subseteq V_3$ for which such a structure as described above can be found will be called cocovers. Before turning this into a formal definition, we will introduce a new auxiliary notion, which will allow us to control the overlap between sets.

Definition 5.7 (Quasidisjoint sets) *A family of sets $A_1, \ldots, A_b \subseteq V$ is called* s-quasidisjoint *if*

$$|\{i \in \{1, \ldots, b\} \mid x \in A_i\}| \leq s \quad \textit{for all } x \in V.$$

It is called (s, μ)-balanced in V *if it is $2s$-quasidisjoint and*

$$|\{x \in V \mid \ |\{i \in \{1, \ldots, b\} \mid x \in A_i\}| \geq s\}| \geq (1 - \mu)|V|.$$

The k-fold overlap *of the sets $A_1, \ldots, A_b$ is defined as*

$$OV_k(A_1, \ldots, A_b) := \{x \in V \mid |\{i \in \{1, \ldots, b\} \mid x \in A_i\}| \geq k\}.$$

Observe that if the s-fold overlap of a $2s$-quasidisjoint family of sets $A_1, \ldots, A_b \subseteq V$ is at least $(1 - \mu)|V|$ then the set is (s, μ)-balanced for V.

Definition 5.8 (Cocovers) *Let $v \in V_1$, let $\tilde{V}_3 \subseteq V_3$ and let $\sigma := (\mu)^{\ominus 5}$. A set $O \subseteq V_3$ with*

$$|O| = o_\mu := \sigma^{-5/2} \frac{n^2}{m}$$

is called a (ρ, μ)-cocover of V_5 via v and V_4 *if*

$$\begin{aligned}
&\exists O^+ \subseteq O, |O^+| \geq \sigma o_\mu / 5 \\
&\forall u \in O^+ \ \exists R_u \subseteq \Gamma_4(u) \cap \Gamma_4(v), |R_u| = \sqrt{\sigma} r_\mu \\
&\qquad R_u \textit{ is a } (T_\rho)\textit{-qualified } \sqrt{(\rho)^{\oplus}}\textit{-supercover of } V_2 \\
&\forall y \in R_u \ \exists W_{u,y} \subseteq \Gamma_5(y) \cap \Gamma_5(u), |W_{u,y}| = r_\sigma \\
&\qquad W_{u,y} \textit{ is a } (\rho)^{\oplus}\textit{-supercover of } V_2, \\
&\qquad \textit{the sets } W_{u,y} \textit{ are } (s_\mu, \mu)\textit{-balanced in } V_5.
\end{aligned}$$

where $s_\mu := r_\sigma(r_\mu) = r_\mu / 2p_\sigma = \frac{1}{4}(1 - \mu)\mu^2 \sigma \frac{m^3}{n^5}$.

Let $\mathcal{CO}_{3,5}(v, \tilde{V}_3; \rho, \mu) := \{O \in \binom{\tilde{V}_3}{o_\mu} \mid O$ *is a (ρ, μ)-cocover via v*$\}$.

Figure 5 illustrates the structure of a (ρ, μ)-cocover.

The following lemma shows that almost all sets (of suitable size) are cocovers.

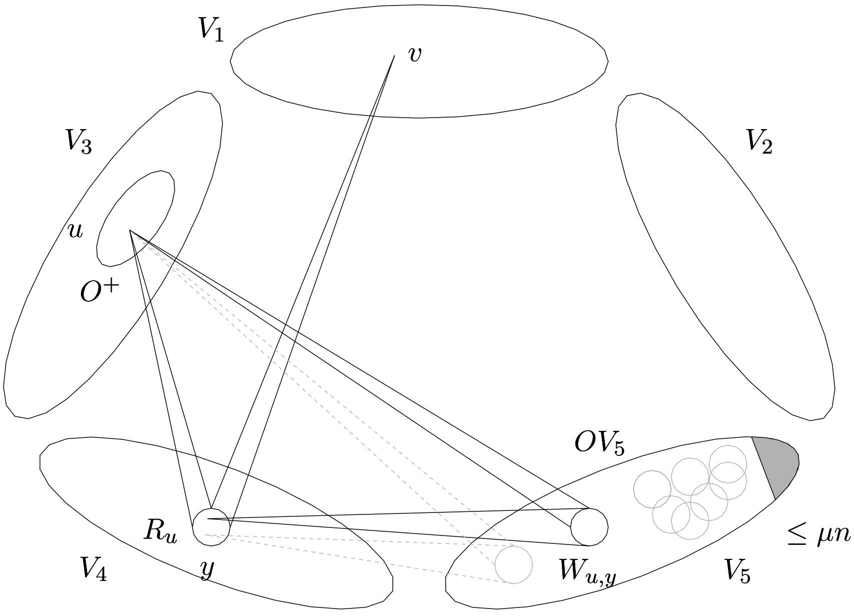

Figure 5: (ρ,μ)-cocover

Lemma 5.9 ($\overline{\mathcal{CO}}$ is small) *Let $\rho \lll \sigma = (\mu)^{\ominus 5}$, and let $\tilde V_3 \subseteq V_3$ satisfy $|\tilde V_3| \ge (1/2)|V_3|$. Consider a graph $G \in \mathcal{S}^{QTN}(n,m;\varepsilon,\rho,(\mu)^{\ominus 5})$, and a vertex $v \in V_1$ with property $(N^{3,4}_{\rho,\mu}[V_4[\neg(Q_\rho,T_\rho)]])$. Then*

$$|\overline{\mathcal{CO}}_{3,5}(v,\tilde V_3;\rho,\mu)| \le ((\mu)^{\ominus})^{o_\mu}\binom{|\tilde V_3|}{o_\mu}.$$

Proof Consider a vertex $u \in V_3$. If there exists a set $R_u \subseteq \Gamma_4(u)\cap\Gamma_4(v)$ with $|R_u| = \sqrt{\sigma} r_\mu$ such that R_u is a (T_ρ)-qualified $\sqrt{(\rho)^{\oplus}}$-supercover of V_2 and such that for all $y \in R_u$ there exists a $(\rho)^{\oplus}$-supercover $W_{u,y} \subseteq \Gamma_5(y)\cap\Gamma_5(u)$ with $|W_{u,y}| = r_\sigma$ of V_2 (i.e. the conditions for the sets R_u in the definition of a cocover are satisfied), then we set $C(u) = \{W_{u,y} \mid y \in R_u\}$ otherwise we set $C(u) = \emptyset$. For a set $U \subseteq V_3$, we define $C(U) \subseteq U$ as a set for which the sets $\{W \mid W \in C(u), u \in U\}$ are $2s_\mu$-quasidisjoint and the cardinality of the s_μ-fold overlap $OC(U)$ of $\{W \mid W \in C(u), u \in U\}$ is as large as possible. If there is more than one set with this property, then we choose one in an arbitrary but fixed way.

We call a set $O \in \binom{\tilde V_3}{o_\nu}$ *useless* if for every subset $U \in O$ with $U \le \mu o_\mu$, $|OC(U)| \le (1-\mu)n$ and more than $(\sigma)^{\oplus 3} o_\mu$ vertices in $O \setminus U$ do not satisfy $N^{4,5}_{\rho,\sigma}[OC(U)]$. Observe that the number of vertices in V_3 which do not satisfy $N^{4,5}_{\rho,\sigma}[OC(U)]$ is bounded from above by $(\sigma)^{\oplus 2} n \le 2(\sigma)^{\oplus 2}|\tilde V_3|$ since $|OC(U)| \le (1-\mu)n \le (1-(\sigma)^{\oplus})n$ and we consider graphs in $\mathcal{S}^{QTN}(n,m;\varepsilon,\rho,(\mu)^{\ominus 5})$. Hence we can bound the number of useless sets in $\tilde V_3$ from above by

$$\sum_{t \le \mu o_\mu} \binom{|\tilde V_3|}{t}\binom{2(\sigma)^{\oplus 2}|\tilde V_3|}{(\sigma)^{\oplus 3} o_\mu}\binom{|\tilde V_3|}{o_\mu - t - (\sigma)^{\oplus 3} o_\mu} \quad \le \quad 16^{o_\mu}(2(\sigma)^{\oplus 2})^{(\sigma)^{\oplus 3} o_\mu}\binom{|\tilde V_3|}{o_\mu}$$

$$\leq \frac{1}{2}((\mu)^{\ominus})^{o_\mu}\binom{|\tilde{V}_3|}{o_\mu}.$$

We now want to bound the number of sets $O \in \binom{\tilde{V}_3}{o_\mu}$ which are not $(H_\rho^{4,5}, H_\rho^{4,2})$-qualified μ-multicovers of V_4. Since we only consider graphs in $\mathcal{S}^{QTN}(n, m; \varepsilon, \rho, (\mu)^{\ominus 5})$, the set V_3 is $(H_\rho^{4,5}, H_\rho^{4,2})$- enhanced $(\rho)^{\oplus 3}$-homogeneous. Hence $\tilde{V}_3$ is $(H_\rho^{4,5}, H_\rho^{4,2})$- enhanced $2(\rho)^{\oplus 3}$-homogenous, and by Lemma 4.17 we can bound the number of sets of size o_μ which are not $(H_\rho^{4,5}, H_\rho^{4,2})$-qualified μ-multicovers of V_4 from above by

$$\frac{1}{2}(\mu)^{\ominus o_\mu}\binom{|\tilde{V}_3|}{o_\mu}.$$

We claim that every set $O \in \binom{\tilde{V}_3}{o_\mu}$ which is an $(H_\rho^{4,5}, H_\rho^{4,2})$-qualified μ-multicover and which is not useless is in $\mathcal{CO}_{3,5}(v, \tilde{V}_3; \rho, \mu)$. If $|OC(O)| \geq (1-\mu)n$ then

$$|C(O)| \geq \frac{(1-\mu)n \cdot s_\mu}{\sqrt{\sigma}r_\mu r_\sigma} \geq \frac{(1-\mu)\sigma}{4(1-\sigma)}o_\mu \geq \frac{\sigma}{5}o_\mu$$

since the sets in $OC(O)$ are form a s_μ-fold overlap. Hence $O \in \mathcal{CO}_{3,5}(v, \tilde{V}_3; \rho, \mu)$ if $|OC(O)| \geq (1-\mu)n$. So assume $|OC(O)| < (1-\mu)n$. Since O is an $(H_\rho^{4,5}, H_\rho^{4,2})$-qualified μ-multicover, it contains $r' := r_\mu(o_\mu)$ many $(H_\rho^{4,5}, H_\rho^{4,2})$-qualified μ-covers $P_1, \ldots, P_{r'}$. The vertex v has an $(H_\rho^{5,2}, T_\rho)$-qualified (ρ, μ)-overlapping neighbourhood with each of the sets P_l, $l = 1, \ldots, r'$ in V_4 because v satisfies $(N_{\rho,\mu}^{3,5,4}[V_4[\neg(Q_\rho, T_\rho)]])$. Hence there are at least $r'2\sqrt{\mu}p_\mu$ vertices $u \in O$ for which a set $\bar{R}_u \subseteq (\Gamma_4(v) \cap \Gamma_4(u))$ with $|\bar{R}_u| = r_\mu$ exists such that $\bar{R}_u$ is an $(H_\rho^{5,2}, T_\rho)$-qualified $(\rho)^{\oplus}$-supercover of V_5 and V_2. Let O^* denote the set of these vertices. Observe that $|C(O)|\sqrt{\sigma}r_\mu r_\sigma \leq 2s_\mu(1-\mu)n$ since the sets $W_{u,y}$ are $2s_\mu$ quasidisjoint and $|OC(O)| \leq (1-\mu)$. Hence

$$|C(O)| \leq \frac{2s_\mu(1-\mu)n}{\sqrt{\sigma}r_\mu r_\sigma} \leq \frac{(1-\mu)\sigma o_\mu}{2(1-\sigma)} \leq \frac{\sigma}{2}o_\mu \leq \mu o_\mu,$$

and $|O^* \setminus C(O)| \geq r'2\sqrt{\mu}p_\mu - \sigma o_\mu/2 = (\sqrt{\mu} - \sigma/2)o_\mu$. Since O is not useless, all but $(\sigma)^{\oplus 3}o_\mu$ vertices $u \in O^* \setminus C(O)$ satisfy $N_{\rho,\sigma}^{4,5}[OC(O)]$. Recall that $\bar{R}_u$ is an $(H_\rho^{5,2})$-qualified $(\rho)^{\oplus}$-supercover of V_5 and hence $\bar{R}_u$ contains at least

$$\frac{r_\mu}{2p_\sigma} = \frac{(1-\mu)\mu^2\sigma}{4}\frac{m^3}{n^5} = s_\mu$$

disjoint $(H_\rho^{5,2})$-qualified σ-covers. Therefore there are at least $s_\mu 2\sqrt{\sigma}p_\sigma = \sqrt{\sigma}r_\mu$ vertices y in $\bar{R}_u$ such that there exists a $(\rho)^{\oplus}$-supercover $W_{u,y} \subseteq V_5$ of V_2 with $|W_{u,y}| = r_\sigma$ and $W_{u,y} \cap OC(U) = \emptyset$, and such that the sets $W_{u,y}$ are s_μ-quasidisjoint. Call the set of these vertices Y_u. We will show that Y_u satisfies all the conditions of a set R_u in the definition of cocovers. Since $\bar{R}_u$ is a $(\rho)^{\oplus}$-supercover of V_2, Y_u is a $(\rho)^{\oplus}/\sqrt{\sigma}$-supercover and thus a $\sqrt{(\rho)^{\oplus}}$-supercover. In addition, the vertices in Y_u satisfy (T_ρ) as $\bar{R}_u$ was (T_ρ)-qualified. This together with the fact that $W_{u,y} \cap OC(U) = \emptyset$ implies that any vertex $O^* \setminus C(O)$ which satisfies $N_{\rho,\sigma}^{4,5}[OC(O)]$ can be added to $C(O)$. Thus adding some of these vertices cannot increase the size of $OC(O)$, but since the number of vertices in $O^* \setminus C(O)$ which satisfy $N_{\rho,\sigma}^{4,5}[OC(O)]$ is at least $(\sqrt{\mu} - \sigma/2 - (\sigma)^{\oplus 3})o_\mu$, $|OC(O^* \setminus C(O))|$ is at least

$$\frac{(\sqrt{\mu} - \sigma/2 - (\sigma)^{\oplus 3})o_\mu\sqrt{\sigma}r_\mu r_\sigma}{ns_\mu} = \frac{4(\sqrt{\mu} - \sigma/2 - (\sigma)^{\oplus 3})(1-\sigma)}{\sigma} \geq 1,$$

a contradiction. $\square$

The strength of the cocovers relies on the fact that their structure is difficult to destroy, since they incorporate many (quasi-)disjoint sets. Indeed we will show that in every subset of V_5 which is not too small a resistant cocover can be found. A resistant cocover has the same basic structure as a cocover, only the sizes of the various sets are suitably scaled down.

Definition 5.10 (Resistant Cocovers) *Let $v \in V_1$ and $X \subseteq V_5$. A set $O \subseteq V_3$ with $|O| = o_\mu$ is called* X-resistant (ρ,μ)-cocover of V_5 via v *if*

$$\begin{array}{l} \exists O^* \subseteq O, |O^*| = 2\sigma^2 o_\mu \\ \forall u \in O^* \; \exists R_u \subseteq \Gamma_4(u) \cap \Gamma_4(v), |R_u| = \sigma r_\mu \\ \qquad R_u \text{ is a } (T_\rho)\text{-qualified } \nu\text{-multicover of } V_2, \text{ where } \nu := (\rho)^{\oplus} \\ \forall y \in R_u \; \exists W_{u,y} \subseteq \Gamma_5(y) \cap \Gamma_5(u) \setminus X, |W_{u,y}| = \sigma r_\sigma \\ \qquad W_{u,y} \text{ is a } \rho\text{-multicover of } V_2, \text{ and the sets } W_{u,y} \text{ are } 2s_\mu\text{-quasidisjoint.} \end{array}$$

The following lemma shows that every cocover is X-resistant provided that X is not too big.

Lemma 5.11 (All cocovers are resistant) *Let $(\rho)^{\oplus} \lll (\mu)^{\ominus 5}$, and let $O \subseteq V_3$ with $|O| = o_\mu$ be a (ρ,μ)-cocover. Then O is an X-resistant (ρ,μ)-cocover for any set $X \subseteq V_5$ with $|X| \leq (1-2\mu)$.*

Proof As before let $\sigma = (\mu)^{\ominus 5}$. We count the number of occurrences of vertices $x \in V_5$ in sets $W_{u,y}$ for the (ρ,μ)-cocover O. By counting occurrences we mean that a single vertex $x \in V_5$ is counted as many times as it occurs in a set $W_{u,y}$ for different vertices u, y.

Since the s_μ-fold overlap of the sets $W_{u,y}$, $u \in O^+, y \in R_u$ is at least $(1-\mu)|V_5|$ and every vertex in this overlap corresponds to at least s_μ such occurrences, we conclude that at least $s_\mu \cdot \mu n$ occurrences remain if we restrict the sets $W_{u,y}$ to vertices in $V_5 \setminus X$.

Assume that there are not enough vertices left from the original sets R_u and $W_{u,y}$ (i.e. in the (ρ,μ)-cocover) for the X-resistant (ρ,μ)-cocover. Let us denote these (original) sets by $\hat{R}_u$ and $\hat{W}_{u,y}$ in order to distinguish them from the sets to be found for the X-resistant cocover. Recall that $|\hat{R}_u| = \sqrt{\sigma} r_\mu$ and $|\hat{W}_{u,y}| = r_\sigma$.

If the number of suitable sets R_u and $W_{u,y}$ is too small, we obtain the following bound on the number of remaining occurrences in sets $W_{u,y} \subseteq \hat{W}_{u,y} \setminus X$. Note that for every vertex $u \in O^+$ and all $\sqrt{\sigma} r_\mu$ neighbours $y \in \hat{R}_u$ there may remain σr_σ neighbours in $\hat{W}_{u,y}$. Secondly there may be σr_μ neighbours $y \in \hat{R}_u$ such that all r_σ neighbours in $\hat{W}_{u,y}$ remain. Finally, for at most $2\sigma^2 o_\mu$ vertices in O^+ and all $\sqrt{\sigma} r_\mu$ neighbours $y \in \hat{R}_u$ all r_σ neighbours in $\hat{W}_{u,y}$ may remain. Combining these numbers of occurrences of vertices $x \in V_5$ in W-sets leads to

$$\begin{aligned} \#\text{occurrences} &\leq |O^+| \cdot \sqrt{\sigma} r_\mu \cdot \sigma r_\sigma + |O^+| \cdot \sigma r_\mu \cdot r_\sigma + 2\sigma^2 o_\mu \cdot \sqrt{\sigma} r_\mu \cdot r_\sigma \\ &\leq \sigma^2 o_\mu r_\mu r_\sigma. \end{aligned}$$

Combining this with the lower bound $s_\mu \cdot \mu n$ we obtain

$$\mu s_\mu n \leq \sigma^2 o_\mu r_\mu r_\sigma \;\Rightarrow\; \mu \leq \sigma^2 \cdot \frac{4(1-\sigma)}{\sigma^{3/2}} \leq 4\sqrt{\sigma}.$$

This is obviously a contradiction.

It remains to show that the sets R_u and $W_{u,y}$ are indeed multicovers. This easily follows from the fact that the corresponding sets in the original (ρ,μ)-cocover were $(\rho)^{\oplus}$- or at least $\sqrt{(\rho)^{\oplus}}$-supercovers. Observe that the remaining sets R_u and $W_{u,y}$ are at least a σ-fraction of the original sets. As $\sqrt{(\rho)^{\oplus}} \leq (\rho)^{\oplus 2} \leq \sigma$ this completes the proof. $\square$

5.4 Square (candidate) covers

The aim of this section is to find (many) squares K_4 in the graph. If many such subgraphs exist and their occurrences are scattered throughout the graph it seems plausible that at least some of them can be enlarged to a subgraph K_5. To this aim we introduce the set $SQU_5(w, Q, v)$:

$$\begin{aligned} SQU_5(w, Q, v) := \{x \in V_5 \quad | \quad & \exists u \in \Gamma_3(w) \cap Q\ \exists y \in \Gamma_4(v) \cap \Gamma_4(u) \cap \Gamma_4(w) : \\ & x \in \Gamma_5(u) \cap \Gamma_5(y) \cap \Gamma_5(w)\}. \end{aligned}$$

We will use this definition for $Q \subseteq \Gamma_3(v)$ and $w \in \Gamma_2(v)$. Observe that $SQU_5(w, Q, v)$ then directly corresponds to the K_5-candidates which we want to construct in the end (cf. Figure 6).

The following definition introduces a structure which immediately leads to many K_5-candidates.

Definition 5.12 (Square covers) *For a set $X \subseteq V_5$, let*

$$S_2(Q, v; X, \rho) := \{w \in V_2 \mid |SQU_5(w, Q, v) \setminus X| \geq t_\rho\},$$

*where $t_\rho := \rho^{-1}\frac{n^2}{m}$. We call a set $Q \subseteq V_3$ an X-resistant (ρ, μ)-*square cover *of V_2 via v if*

$$|S_2(Q, v; X, \rho)| \geq (1 - \mu)n.$$

See Figure 6 for an illustration of the structure of a square cover.

Intuitively, for a square cover Q almost all vertices in V_2 belong to squares. Consider a vertex $v \in V_1$ and a square cover $Q \subseteq \Gamma_3(v)$. It would be easy to show that with very high probability $\Gamma_2(v)$ and $S_2(Q, v; \rho)$ intersect, yielding t_ρ K_5-candidates.

Unfortunately, we are not able to give a direct proof that almost all sets Q are square covers. Instead we will show a somewhat weaker statement which is based on so-called square candidate covers. Such a square candidate cover can be interpreted as a square cover where the edges from V_2 to V_3 are ignored, i.e., their presence is not yet guaranteed.

Definition 5.13 (Square candidate covers) *Let $v \in V_1$ and $X \subseteq V_5$. A set $Q \subseteq V_3$ with $|Q| = q_\mu$ is called an X-*resistant (ρ, μ)-square candidate cover of V_2 via v *if there exist sets $\tilde{Q} \subseteq Q$ and $\tilde{T} = \tilde{T}(Q) \subseteq V_2$ with $|\tilde{Q}| = \sigma^2 q_\mu$, $|\tilde{T}| = (1 - 6\nu^{1/8})n$ where $\nu := \sqrt{(\rho)^\oplus}$ such that*

$$\begin{aligned} &\forall w \in \tilde{T}\ \exists \tilde{Q}^* \subseteq \tilde{Q}, |\tilde{Q}^*| \geq (1 - 6\nu^{1/8})\sigma^2 q_\mu \\ &\forall u \in \tilde{Q}^*\ \exists Y_{u,w} \subseteq \Gamma_4(v) \cap \Gamma_4(u) \cap \Gamma_4(w), |Y_{u,w}| = \nu^3 \tfrac{m^3}{n^5} =: h_\nu \\ &\forall y \in Y_{u,w}\ \exists S_{u,w,y} \subseteq \Gamma_5(u) \cap \Gamma_5(y) \cap \Gamma_5(w) \setminus X, |S_{u,w,y}| \geq h_\rho \\ &\qquad \text{the sets } S_{u,w,y} \text{ are } d_\mu\text{-quasidisjoint for fixed } w, \end{aligned}$$

where $d_\mu := 2s_\mu \cdot \frac{q_\mu}{2o_\mu}$.

Let $\mathcal{SC}_{3,2}(v, X; \rho, \mu)$ denote the set of X-resistant (ρ, μ)-square candidates covers of V_2 via v in V_3.

Note that

$$d_\mu = 2s_\mu \cdot \frac{q_\mu}{2o_\mu} = \frac{1}{4}(1 - \mu)\mu^2\sigma\frac{m^3}{n^5} \cdot (1 - \mu)\mu\frac{m}{n} \cdot \sigma^{5/2}\frac{m}{n^2} \geq \sigma^4\frac{m^5}{n^8}, \tag{16}$$

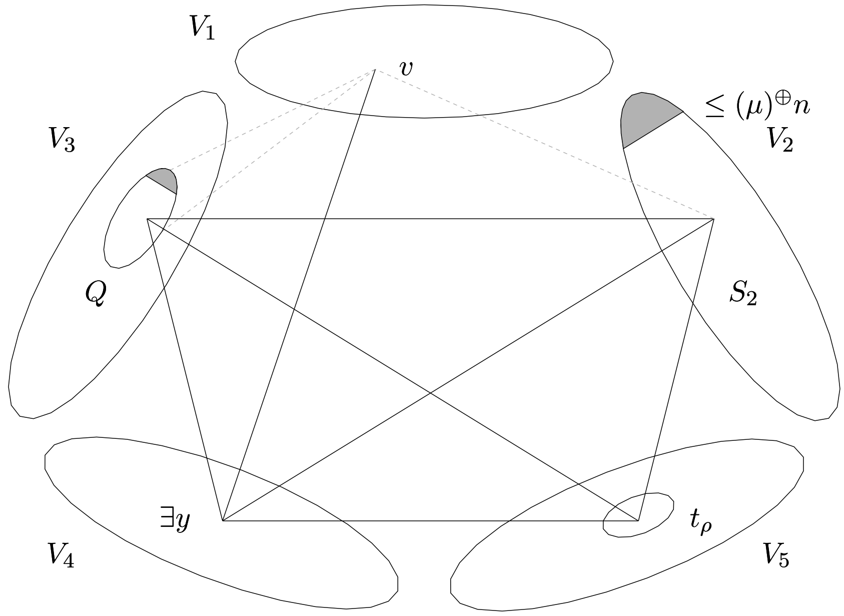

Figure 6: (ρ, μ)-square cover

where as before $\sigma = (\mu)^{\ominus 5}$ and,

$$d_\mu \leq \sigma \frac{m^5}{n^8}. \tag{17}$$

Even if Definition 5.13 looks somewhat scary, the structure of square candidate covers is actually not too difficult, as Figure 7 illustrates. In fact, it turns out that square candidate covers are not much more than cocovers. By comparing the structure of (X-resistant) cocovers (cf. Figure 5) and that of square candidate covers (cf. Figure 7) the astute reader will immediately notice that the differences are only moderate. Firstly, there are new edges between V_2 and V_4, respectively V_2 and V_5. However, these edges do not come as a surprise. Since the sets R_u and $W_{u,y}$ in a cocover are multicovers. Hence, we should expect the edges in the square candidate cover to be indeed present.

The second difference is that square candidate covers have a larger cardinality than cocovers. Thus we can construct the former by combining sufficiently many cocovers.

The following lemma shows that a sufficiently large set Q which contains many cocovers is indeed a square candidate cover. This implies that the number of sets which are no square candidate cover is small.

Definition 5.14 (Sets containing many cocovers) *Let $\mathcal{SC}_{3,2}(v; \rho, \mu) \subseteq \binom{V_3}{q_\mu}$ denote all sets which can be decomposed into sets $O_1, \ldots, O_{q_\mu/o_\mu}$ with $|O_1| = \ldots = |O_{q_\mu/o_\mu}| = o_\mu$ such that at least half of these sets are (ρ, μ)-cocovers.*

In the sequel we will see that almost all sets in $\binom{V_3}{q_\mu}$ belong to $\mathcal{SC}_{3,2}(v; \rho, \mu)$ and that the latter set are contained in the X-resistant square candidate covers we have been looking for.

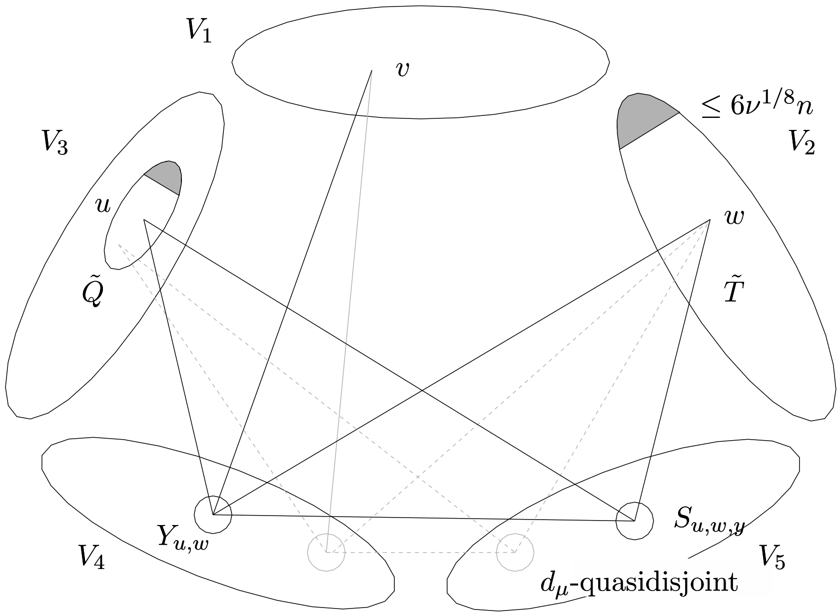

Figure 7: (ρ, μ)-square candidate cover

Lemma 5.15 (Many square candidate covers) *Let* $(\rho)^{\oplus} \lll \sigma = (\mu)^{\ominus 5}$*. Consider* $G \in \mathcal{S}^{QTN}(n, m; \varepsilon, \rho, (\mu)^{\ominus 5})$ *and a vertex* $v \in V_1$ *which satisfies* $(N^{3,4}_{\rho,\mu}[V_4[\neg(Q_\rho, T_\rho)]])$*. Then*

$$|\overline{\mathcal{SC}}_{3,2}(v; \rho, \mu)| \leq \mu^{2q} \binom{n}{q_\mu}$$

and

$$\mathcal{SC}_{3,2}(v; \rho, \mu) \subseteq \mathcal{SC}_{3,2}(v, X; \rho, \mu) \quad \textit{for all } X \subseteq V_5 \textit{ with } |X| \leq (1 - 2\mu)n,$$

i.e., every set in $\mathcal{SC}_{3,2}(v; \rho, \mu)$ *is an* X*-resistant* (ρ, μ)*-square candidate cover.*

Proof To prove the first claim we choose the q_μ/o_μ sets of size o_μ one after the other. Observe that $q_\mu \leq (1/2)n$. Hence by Lemma 5.9 the probability that we choose a set which is not a cocover is bounded by $((\mu)^{\ominus})^{o_\mu}$. We have to choose at least $q_\mu/(2o_\mu)$ sets which are not cocovers to obtain a set in $\overline{\mathcal{SC}}_{3,2}(v; \rho, \mu)$, and the probability of doing so is bounded by $2^{q_\mu/o_\mu}((\mu)^{\ominus})^{q_\mu/2} \leq \mu^{2q}$.

For the proof of the second claim let us consider arbitrary sets $Q \in \mathcal{SC}_{3,2}(v; \rho, \mu)$ and $X \subseteq V_5$ with $|X| \leq (1 - 2\mu)n$. By assumption, Q contains at least $q_\mu/(2o_\mu)$ (ρ, μ)-cocovers, which must be X-resistant due to Lemma 5.11. For the time being consider a single X-resistant (ρ, μ)-cocover O and a vertex $u \in O^*$. We let

$$F_u := \{\{y, w\} \in E(V_4, V_2) \mid y \in R_u, w \in \Gamma_2(y) \cap C_2(W_{u,y}, \rho)\}.$$

We intend to show the following claim:

(i) For at least $(1 - 3\sqrt{\nu})n$ vertices $w \in C_2^{F_u}(R_u; \nu)$ suitable sets $Y_{u,w}$ and $S_{u,w,y}$ for $y \in Y_{u,w}$ exist which satisfy the requirements of Definition 5.13.

Note that $d_2(y) - d_2^{F_u}(y) \le (\rho)^{\oplus} q \le \nu^2 q$ for $y \in R_u$ (recall that $\nu := \sqrt{(\rho)^{\oplus}}$, cf. Definition 5.13), since $R_u \subseteq \Gamma_4(u) \cap \Gamma_4(v)$ is (T_ρ)-qualified and $W_{u,y} \subseteq \Gamma_5(y) \cap \Gamma_5(u) \setminus X$ is a ρ-multicover. Hence by Lemma 4.14 we deduce that

$$|C_2^{F_u}(R_u;\nu)| \ge (1 - 3\sqrt{\nu})n.$$

For every vertex $w \in C_2^{F_u}(R_u;\nu)$ we have that

$$|\Gamma_4^{F_u}(w) \cap R_u| \ge r_\nu(\sigma r_\mu)/2 = \frac{1}{8}\nu\sigma r_\mu \cdot \frac{m}{n^2} \ge \frac{1}{9}\nu\sigma\mu^2\frac{m^3}{n^5} \ge \nu^3\frac{m^3}{n^5} = h_\nu,$$

as R_u is a ν-multicover (of V_2). Hence, we can find a set $Y_{u,w} \subseteq \Gamma_4(v) \cap \Gamma_4(u) \cap \Gamma_4(w)$ with $|Y_{u,w}| = h_\nu$ such that every vertex $y \in Y_{u,w}$ is incident to an edge $\{y, w\} \in F_u$. Due to the definition of F_u we deduce that $w \in C_2(W_{u,y}, \rho)$. This leads to

$$|\Gamma_5(w) \cap W_{u,y}| \ge r_\rho(\sigma r_\sigma)/2 = \frac{1}{8}\rho\sigma r_\sigma \cdot \frac{m}{n^2} \ge \frac{1}{9}\rho\sigma^3\frac{m^3}{n^5} \ge \rho^3\frac{m^3}{n^5} \ge h_\rho.$$

Thus we can find a set $S_{u,w,y} \subseteq \Gamma_5(u) \cap \Gamma_5(y) \cap \Gamma_5(w) \setminus X$ with $|S_{u,w,y}| = h_\rho$ as required by Definition 5.13 for at least $(1 - 3\sqrt{\nu})n$ vertices $w \in C_2^{F_u}(R_u;\nu)$. This shows claim (i).

Now we want to show the following claim.

(ii) There exists a set $GV_2(O) \subseteq V_2$ with $|GV_2(O)| = (1 - 3\nu^{1/4})n$ such that for every vertex $w \in GV_2(O)$ there exist $(1 - 3\nu^{1/4})2\sigma^2 o_\mu$ vertices $u \in O^*$ such that suitable sets $Y_{u,w}$ and $S_{u,w,y}$ satisfying the requirements of Definition 5.13 can be found. The sets $S_{u,w,y}$ are $2s_\mu$-quasidisjoint for fixed w.

By Lemma 3.2 it follows from (i) that at least $(1 - 3\nu^{1/4})n$ vertices in V_2 belong to $C_2^{F_u}(R_u;\nu)$ for at least $(1 - 3\nu^{1/4})2\sigma^2 o_\mu$ vertices $u \in O^*$. This holds for every X-resistant (ρ, μ)-cocover O. Also, the sets $(S_{u,w,y})_{u \in O^*, y \in Y_{u,w}}$ are $2s_\mu$-quasidisjoint because they are subsets of $2s_\mu$-quasidisjoint sets $W_{u,y}$ which shows Claim (ii).

Assume that Q contains at least $z := q_\mu/(2o_\mu)$ X-resistant (ρ, μ)-cocovers $O_1, \dots, O_z$. By Lemma 3.2 and (ii) we conclude that there are at least $(1 - 3\nu^{1/8})n$ vertices $w \in V_2$ such that $w \in GV_2(O_l)$ for at least $(1 - 3\nu^{1/8})z$ indices $l \in \{1, \dots, z\}$. Thus for such a vertex w there are at least

$$(1 - 3\nu^{1/8})z \cdot (1 - 3\nu^{1/4})2\sigma^2 o_\mu \ge (1 - 6\nu^{1/8})\sigma^2 q_\mu$$

vertices $u \in \bigcup_{l=1}^{z} O_l^*$ such that suitable sets $Y_{u,w}$ and $S_{u,w,y}$ exist. These sets $S_{u,w,y}$ are $2s_\mu z$-quasidisjoint by construction.

In order to show that such a set Q satisfies Definition 5.13 it suffices to identify $\tilde{Q}$ with $\bigcup_{l=1}^{z} O_l^*$, since $|\bigcup_{l=1}^{z} O_l^*| = z \cdot 2\sigma^2 o_\mu = \sigma^2 q_\mu$. □

The following definition introduces square candidates of a set $R \subseteq V_3$ and a vertex $w \in V_2$, i.e., vertices $x \in V_5$ which would complete a square with w and a vertex $u \in R$ (with an additional edge to a given vertex $v \in V_1$) if the edge $\{u, w\}$ were indeed present. However, note that these edges, i.e., the edges between V_2 and V_3 are not yet taken into account.

Definition 5.16 (Square candidates) *For $v \in V_1$, $w \in V_2$ and $R \subseteq V_3$, the set of square candidates is given by*

$$\begin{aligned} SQC(w, R, v) := \{x \in V_5 \quad | \quad & \exists u \in R\ \exists y \in \Gamma_4(v) \cap \Gamma_4(u) \cap \Gamma_4(w) : \\ & x \in \Gamma_5(u) \cap \Gamma_5(y) \cap \Gamma_5(w)\}. \end{aligned}$$

Square candidate covers Q were constructed in such a way that for a given vertex $w \in V_2$ almost all vertices in $\tilde{Q}$ complete $h_\nu \cdot h_\rho = \Theta(m^6/n^{10})$ square candidates with w.

Now assume that we choose several vertices $R \subseteq Q$ and require that the square candidates which belong to the vertices in R are 'rather' disjoint. If $|R|$ is sufficiently small, the quasidisjointness of the square candidates will suffice to show that this is indeed the case. Recall that the square candidates are $\Theta(m^5/n^8)$-quasidisjoint and at the threshold $m = \Theta(n^{8/5})$ it follows that they are $\Theta(1)$-quasidisjoint. Thus, the overlap should not cause real problems. Our aim is to choose $|R| = \Theta(m^2/n^3)$ vertices which complete $\Theta(n^2/m)$ square candidates. Note that this should be possible since $\Theta(m^2/n^3 \cdot m^6/n^{10}) = \Theta(m^8/n^{13})$ and $n^2/m = o(m^8/n^{13})$ for $m = \omega(n^{5/3})$.

The following definition introduces bad R-sets, i.e., sets $R \subseteq \tilde{Q}$ with few square candidates.

Definition 5.17 (Non-spreading R-sets) *Consider $v \in V_1$ and $X \subseteq V_5$. Let $Q \subseteq V_3$ be an X-resistant (ρ, μ)-square candidate cover and consider a vertex $w \in \tilde{T}(Q) \subseteq V_2$. We define*

$$\bar{\mathcal{R}}^5(w, Q, v; X, \rho) := \{R \in \binom{\tilde{Q}}{\tilde{r}} \mid |SQC(w, R, v) \setminus X| < t_\rho\},$$

where $t_\rho := \rho^{-1}\frac{n^2}{m}$ and $\tilde{r} := r_\mu(\sigma^2 q_\mu)/2 = \frac{1}{8}(1-\mu)\mu^2\sigma^2\frac{m^2}{n^3}$.

In order to prove that bad R-sets occur indeed very rarely, it will turn out to be essential that the sets $S_{u,w,y}$ in the square candidate cover have been shown to be quasidisjoint. Otherwise, neighbourhoods of the vertices in a set R could cluster and we would not obtain the desired cardinality. The following lemma shows how the quasidisjointness can be employed.

Lemma 5.18 (Sequential selection lemma) *Consider two separate sets V and B with $|B| = b$. The elements $x \in B$ are assigned corresponding sets $A_{x,1}, \ldots, A_{x,h} \subseteq V$. Assume that there is a subset $B^* \subseteq B$ with $|B^*| = (1-\nu)b$ such that $|A_{x,1}| = \ldots = |A_{x,h}| = a$ for $x \in B^*$ and the sets $(A_{x,l})_{x \in B^*, l \in \{1,\ldots,h\}}$ are s-quasidisjoint.*

A set $C \subseteq B$ with $|C| = c$ is called t-spreading *if $|\bigcup_{z \in C}(A_{z,1} \cup \ldots \cup A_{z,h})| \geq t$. If*

(i) $h \leq s$,

(ii) $2ts \leq \nu abh$,

(iii) $c \geq 4t/a$,

then

$$|\{C \in \binom{B}{c} \mid C \text{ is not } t\text{-spreading}\}| \leq (256\nu)^{c/2}\binom{|B|}{c}.$$

Proof For a set $C' \subseteq B$, let $T(C') \subseteq C'$ be the set of maximal cardinality such that for each $c \in T(C')$ there exists an index $i \in \{1, \ldots, l\}$ and a set $A_c \subset A_{c,i}$ with $|A_c| = a/2$ and the sets A_c, $c \in T(C')$ are pairwise disjoint. If there is more than one such set, we choose one in an arbitrary but fixed way. Let $C \in \binom{B}{c}$ be a non-t-spreading set. Then $|T(C)| \leq 2t/a$ because otherwise

$$|\bigcup_{z \in C}(A_{z,1} \cup \ldots \cup A_{z,h})| \geq |\bigcup_{c \in T(C)} A_c| \geq \frac{a}{2}\frac{2t}{a} = t.$$

Observe that every element in $X := \bigcup_{c\in T(C)} A_c$ is an element of at most s sets $A_{x,l}$, $x \in B^*$, $l \in \{1,\dots,h\}$. Consequently since $T(C) \le 2t/a$ then at most $2st/a$ sets $A_{x,l}$ contain less than $a/2$ elements which do not belong to X. We call these sets $A_{x,l}$ destructed. For every element $x \in C \setminus (T(C) \cap B^*$ all sets h sets must be destructed. Note that

$$|T(C) \cap B^*| \le \frac{2st}{ah} \le \frac{\nu abh}{ah} = \nu b,$$

and hence $|B \setminus T(C)| \ge (1-2\nu)b$. Thus we can bound the number of sets $C \in \binom{B}{c}$ which are not t-spreading from above by

$$\sum_{l\le 2t/a} \binom{|B|}{l}\binom{2\nu b}{c-l} \le \sum_{l\le 2t/a} \binom{|B|}{l}(2\nu)^{c-l}\binom{|B|}{c-l} \le \frac{c}{2}4^c(2\nu)^{c-2t/a}.$$

□

Now we are finally in a position to prove that bad R-sets occur indeed very rarely.

Lemma 5.19 (Few non-spreading R-sets) *Let $(\rho)^{\oplus} \lll (\mu)^{\ominus 5}$, and let Q be an X-resistant (ρ,μ)-square candidate cover and consider a vertex $w \in \tilde{T}(Q) \subseteq V_2$. Let $\nu := \sqrt{(\rho)^{\oplus}}$. Then*

$$|\bar{\mathcal{R}}^5(w,Q,v;X,\rho)| \le \nu^{\tilde{r}/20}\binom{\sigma^2 q}{\tilde{r}}$$

for $m \ge \rho^{-6}n^{5/3}$.

Proof We intend to apply Lemma 5.18. To this aim we identify $B \equiv \tilde{Q}$, $B^* \equiv \tilde{Q}^*$, with $|B^*| \ge (1-6\nu^{1/8})|B|$, and $A_{u,1},\dots,A_{u,h_\nu} \equiv (S_{u,w,y})_{y\in Y_{u,w}}$ for $u \in \tilde{Q}^*$ (cf. Definition 5.13). Note that $S_{u,w,y} \subseteq SQC(w,\{u\},v) \setminus X$. Hence it suffices to bound the number of non-t_ρ-spreading sets $R \in \binom{\tilde{Q}}{\tilde{r}}$.

Let us check the preconditions of Lemma 5.18. As before, let $\sigma = (\mu)^{\ominus 5}$. For condition (i) observe that

$$\nu^3\frac{m^3}{n^5} \le \sigma^4\frac{m^5}{n^8},$$

if $m \ge n^{3/2}$. Hence in this case we obtain by (16) that $h_\nu \le d_\mu$.

Condition (ii) is satisfied for all $m^3 \ge \rho^{-6}n^5$ since then

$$\begin{aligned}\frac{m^4}{n^6} \le \mu\sigma\rho^4\nu^4\frac{m^7}{n^{11}} \quad &\Rightarrow \quad \rho^{-1}\frac{n^2}{m}\sigma\frac{m^5}{n^8} \le \nu\rho^3\frac{m^3}{n^5}\sigma^2\mu\frac{m}{n}\nu^3\frac{m^3}{n^5}\\ &\Rightarrow \quad 2t_\rho\cdot d_\mu \le 6\nu^{1/8}\cdot h_\rho\cdot\sigma^2 q_\mu\cdot h_\nu.\end{aligned}$$

Condition (iii) is satisfied for all $m^3 \ge \rho^{-6}n^5$ due to

$$\mu^2\sigma^2\frac{m^2}{n^3} \ge 40\rho^{-4}\frac{n^7}{m^4} \Rightarrow \frac{1}{9}\mu^2\sigma^2\frac{m^2}{n^3} \ge 4\rho^{-1}\frac{n^2}{m}\rho^{-3}\frac{n^5}{m^3} \Rightarrow \tilde{r} \ge 4\frac{t_\rho}{h_\rho}.$$

Since all preconditions are satisfied, Lemma 5.18 gives

$$|\bar{\mathcal{R}}^5(w,Q,v;X,\mu)| \le (48\nu^{1/8})^{\tilde{r}/2}\binom{\sigma^2 q}{q_\mu} \le \nu^{\tilde{r}/20}\binom{\sigma^2 q}{q_\mu},$$

and the proof is complete. □

As we want to find many square candidate covers we are interested in vertices $v \in V_1$ which satisfy the following property:

$$(S_{\rho,\mu}) \quad \forall Q \subseteq \Gamma_3(v), |Q| = q_\mu : \; Q \in \mathcal{SC}_{3,2}(v; \rho, \mu).$$

For this property we again define a corresponding set of bad graphs where we abbreviate the property $(N^{3,4}_{\rho,\mu}[V_4[\neg(Q_\rho, T_\rho)])$ by $(N^*_{\rho,\mu})$

$$\begin{aligned}\mathcal{B}^S(n, m; \varepsilon, \rho, \mu) \;\; &:= \;\; \{G \in \mathcal{S}^{QTN}(n, m; \varepsilon, \rho, (\mu)^{\ominus 5}) \mid \\ &\quad\;\; |V_1[(\neg S_{\rho,\mu}, N^*_{\rho,\mu}, D)]| \geq (\mu)^{\oplus} n\}.\end{aligned} \tag{18}$$

The following lemma shows that there are very few such bad graphs. A routine application of Lemma 3.5 will suffice for that.

Lemma 5.20 ($\mathcal{B}^S$ is small) *Let $(\rho)^{\oplus} \lll \sigma = (\mu)^{\ominus 5}$. Then*

$$|\mathcal{B}^S(n, m; \varepsilon, \rho, \mu)| \leq ((\mu)^{\oplus})^m \binom{n^2}{m}^{10}.$$

Proof We define the neighbourhood function

$$\mathcal{N}(v) := \{X \in \binom{V_3}{d_v} \mid \exists Q \subseteq X, |Q| = q_\mu : \; Q \in \overline{\mathcal{SC}}_{3,2}(v, \rho, \mu)\}$$

By Lemma 5.15 we know that $|\mathcal{SC}_{3,2}(v; \rho, \mu)| \leq \mu^{2q} \binom{n}{q_\mu}$ for $v \in V_1[(N^{3,4}_{\rho,\mu}[V_4[\neg(Q_\rho, T_\rho)]]$, and hence

$$|\mathcal{N}(v)| \leq \mu^{2q} \binom{n}{q_\mu} \binom{n}{d_v - q_\mu} \leq \mu^{d_v} \binom{n}{d_v},$$

and the claim follows by Lemma 3.5. □

For future reference we introduce the set

$$\begin{aligned}\mathcal{S}^{QTNS}(n, m; \varepsilon, \rho, \mu) \;\; &:= \;\; \mathcal{S}^{QTN}(n, m; \varepsilon, \rho, (\mu)^{\ominus 5}) \setminus \\ &\quad\;\; (\mathcal{B}^N_{1,3,4}(n, m; \varepsilon, \rho, \mu) \cup \mathcal{B}^S(n, m; \varepsilon, \rho, \mu)).\end{aligned}$$

By (15) and Lemma 5.20 we have that

$$|\mathcal{S}^{QTNS}(n, m; \varepsilon, \rho, \mu)| \geq (1 - 2((\mu)^{\oplus})^m) \binom{n^2}{m}^{10} \tag{19}$$

if $(\rho)^{\oplus} \lll (\mu)^{\ominus 5}$. Also note that for graphs $G \in \mathcal{S}^{QTNS}(n, m; \varepsilon, \rho, \mu)$ we have that

$$|V_1[(N^*_{\rho,\mu}, S_{\rho,\mu}, Q_\rho, D)]| \geq (1 - (\mu)^{\oplus 2} - (\mu)^{\oplus} - 3(\rho)^{\oplus}) n \tag{20}$$

if $(\rho)^{\oplus 3} \leq 1 - (\mu)^{\oplus}$ by (11),(14) and (18).

6 Cover families

The remainder of the proof is completely analogous to the proof for the analogous result for the K_4, given in [4]. We will therefore state most of the lemmas without proof and refer the reader to [4] for the technical details. An abstract, unified and complete presentation of the proofs for both cases $H = K_4$ and $H = K_5$ can be found in [18].

Motivation Although we have already defined square covers we have not yet proved their occurrence in typical graphs. Instead we have shown that there are many square candidate covers Q and that sets of size $\Theta(m^2/n^3)$ inside Q close $\Theta(n^2/m)$ square candidates. Observe that a typical vertex $w \in V_2$ has $\Theta(m^2/n^3)$ neighbours in Q for $|Q| = \Theta(m/n)$. If these neighbours indeed close $\Theta(n^2/m)$ square candidates and this is true for most vertices in V_2, this suffices to show that Q is a square cover.

Unfortunately, by considering one square candidate cover alone we do not achieve a sufficiently small probability for bad graphs. This is due to the fact that by Lemma 5.19 we only gain a factor $\tilde{r} = \Theta(m^2/n^3)$ in the exponent for a single bad neighbourhood of size $\Theta(m^2/n^3)$ inside a specific square candidate cover Q. Observe that in the previous proofs we have needed $\Theta(m/n)$ in the exponent to obtain sufficiently small probabilities. This technical difficulty is overcome by considering a partition of V_3 into $\Theta(n^2/m)$ square candidate covers Q. By applying the above arguments to all these sets at once, we gain a factor of $\Theta(n^2/m \cdot m^2/n^3) = \Theta(m/n)$ in the exponent, which will suffice to complete the proof.

In the sequel we introduce such partitions of V_3.

Definition 6.1 (Cover families) *Let $\rho, \mu > 0$, $\tilde{p} := p_\mu/2$ and let $X_1, \ldots, X_{\tilde{p}} \subseteq V_5$. We say that the set $P^* = \{v_1, \ldots, v_{\tilde{p}}\} \subseteq V_1$ induces a (ρ, μ)-*cover family* $Q_1, \ldots, Q_{\tilde{p}} \subseteq V_3$ which is resistant to $(X_l)_{l=1,\ldots,\tilde{p}}$ if the following conditions are satisfied for all $l = 1, \ldots, \tilde{p}$:*

(i) The sets $Q_1, \ldots, Q_{\tilde{p}}$ are pairwise disjoint,

(ii) $Q_l \subseteq \Gamma_2(v_l)$, $|Q_l| = q_\mu$,

(iii) Q_l is an X_l-resistant (ρ, μ)-square candidate cover of V_2,

(iv) $\tilde{Q}_l$, which is given by Definition 5.13, is a μ-multicover of V_2.

The following lemma gives a sufficient condition for cover families to contain many square covers.

Lemma 6.2 (Cover families contain many square covers) *Consider a (ρ, μ)-cover family $Q_1, \ldots, Q_{\tilde{p}} \subseteq V_3$ induced by $P^* = \{v_1, \ldots, v_{\tilde{p}}\} \subseteq V_1$ with forbidden vertex sets $X_1, \ldots, X_{\tilde{p}} \subseteq V_5$. A vertex $w \in V_2$ is called* non-spreading *if there exist at least $\mu\tilde{p}$ sets Q_i such that*

$$w \in \tilde{T}(Q_i) \ \wedge \ \exists R \subseteq \Gamma_3(w) \cap \tilde{Q}_i, |R| = \tilde{r}/2 : \ R \in \bar{\mathcal{R}}^5(w, Q_i, v_i; X_i, \rho).$$

If there are at most μn non-spreading vertices, then

$$|\{i \in \{1, \ldots, \tilde{p}\} \mid Q_i \textit{ is an } X_i\textit{-resistant } (\rho, (\mu)^\oplus)\textit{-square cover via } v_i\}| \geq (1 - \sqrt{(\mu)^\oplus})\tilde{p}.$$

Proof Analogous to Lemma 5.19 in [4]. □

The following definition and lemma show that most graphs contain few non-spreading vertices and, consequently, we are able to construct many square cover, once we have found many cover families.

Definition 6.3 (Too many non-spreading vertices)

$$\begin{aligned}\mathcal{B}^R(n, m; \varepsilon, \rho, \mu) \quad := \quad & \{G \in \mathcal{S}_5(n, m; \varepsilon) \mid \exists X_1, \ldots, X_{\tilde{p}} \subseteq V_5 \ \exists P^* = \{v_1, \ldots, v_{\tilde{p}}\} \subseteq V_1 \\ & \exists \ (\rho, \mu)\textit{-cover family } Q_1, \ldots, Q_{\tilde{p}} \subseteq V_2 \textit{ induced by } \{v_1, \ldots, v_{\tilde{p}}\} \\ & \textit{which is resistant to } (X_l)_{l=1,\ldots,\tilde{p}}\textit{:} \\ & \textit{there exist at least } \mu n \textit{ non-spreading vertices } w \in V_2\}\end{aligned}$$

Lemma 6.4 ($\mathcal{B}^R$ is small) *If $(\rho)^{\oplus} \lll (\mu)^{\ominus 5}$, and $m \geq \rho^{-6} n^{5/3}$, then*

$$|\mathcal{B}^R(n,m;\varepsilon,\rho,\mu)| \leq ((\mu)^{\oplus})^m \binom{n^2}{m}^{10}.$$

Proof Analogous to Lemma 7.7 in [4] (cf. Lemma 5.19). □

The following lemma shows how we can construct many cover families.

Lemma 6.5 (Covers induce square cover families) *Let $(\rho)^{\oplus} \lll \sigma = (\mu)^{\ominus 5}$, and let $X_1, \ldots, X_{\tilde{p}} \subseteq V_5$ satisfy $|X_l| \leq (1-2\mu)n$ for $l = 1, \ldots, \tilde{p}$. For a $(N^*_{\rho,\mu}, S_{\rho,\mu}, C_\rho, D)$-qualified μ-cover $P \subseteq V_1$ the set P^* induces a (ρ,μ)-cover family with forbidden vertex sets $X_1, \ldots, X_{\tilde{p}}$.*

Proof Let $Q_1 = W_1, \ldots, Q_{\tilde{p}} = W_{\tilde{p}}$. Due to the definition of $(S_{\rho,\mu})$ we conclude that $Q_1, \ldots, Q_{\tilde{p}} \in \mathcal{S}_{3,2}(v;\rho,\mu)$, and since $|X_l| \leq (1-2\mu)n$ even

$$Q_l \in \mathcal{S}_{3,2}(v, X_l; \rho, \mu) \text{ for all } l = 1, \ldots, \tilde{p}$$

due to Lemma 5.15, i.e., Q_l is a X_l-resistant (ρ,μ)-square candidate cover. Finally, by the definition of (C_ρ) we conclude that the set $\tilde{Q}_l$ is a μ-multicover of V_2, as $|\tilde{Q}_l| = \sigma^2|Q|$ and $\sigma^2 \geq \rho$. □

Let

$$\mathcal{S}^{QTNSR}(n,m;\varepsilon,\rho,\mu) := \mathcal{S}^{QTNS}(n,m;\varepsilon,\rho,\mu) \setminus \mathcal{B}^R(n,m;\varepsilon,\rho,\mu).$$

By (19) and Lemma 6.4 we obtain that if $(\rho)^{\oplus} \lll (\mu)^{\ominus 5}$ and $m \geq \rho^{-6} n^{5/3}$ then

$$|\mathcal{S}^{QTNSR}(n,m;\varepsilon,\rho,\mu)| \geq (1 - 3((\mu)^{\oplus})^m) \binom{n^2}{m}^{10}. \tag{21}$$

Combining Lemma 6.2 and Lemma 6.5 we are able to show that most vertices in V_1 of a graph in $\mathcal{S}^{QTNSR}(n,m;\varepsilon,\rho,\mu)$ satisfy the property

$$(R_{\rho,\mu}) \qquad \begin{array}{l} \forall X \subseteq V_5, |X| \leq (1-2\mu)n \; \exists Q_v \subseteq \Gamma_3(v): \\ \qquad Q_v \text{ is an } X\text{-resistant } (\rho,(\mu)^{\oplus})\text{-square cover via } v. \end{array}$$

This leads to the following lemma.

Lemma 6.6 (Construct square covers) *If $(\rho)^{\oplus} \lll (\mu)^{\ominus 5}$, then any graph $G \in \mathcal{S}^{QTNSR}(n,m;\varepsilon,\rho,\mu)$ contains at least $(1 - 5(\mu)^{\oplus 3})n$ vertices $v \in V_1$ which satisfy $(R_{\rho,\mu})$.*

Proof Due to (20) on page 29 we know that V_1 is $(N^*_{\rho,\mu}, S_{\rho,\mu}, Q_\rho, D)$-enhanced $2(\mu)^{\oplus 2}$-homogeneous. Now we can proceed as in the proof of Lemma 7.9 in [4] and we only give an outline of the proof idea. Firstly, we construct many sets $P^* \subseteq V_1$ which induce cover families. This is achieved using Lemma 4.11 and Lemma 6.5. Secondly, we deduce from Definition 6.3 that there are only few non-spreading vertices and by Lemma 6.2 it follows that there exist indeed many square covers. □

6.1 Clique candidates

In this section we are finally in a position to prove the existence of many K_5-candidates. Before doing so we give the following definition.

Definition 6.7 (Clique candidates) *For $v \in V_1$ and a set $A \subseteq V_2$, we define the set of clique candidates by*

$$\begin{aligned} CC_5(v,A) := \{x \in V_5 \quad | \quad & \exists w \in A\, \exists u \in \Gamma_3(v) \cap \Gamma_3(w)\, \exists y \in \Gamma_4(v) \cap \Gamma_4(w) \cup \Gamma_4(u) \\ & s.t. \{vw\}, \{ux\}, \{wx\}, \{xy\} \in E\}. \end{aligned}$$

*The vertex v is a ξ-*clique covering vertex *if $|CC_5(v, \Gamma_2(v))| \geq (1-\xi)n$.*

A vertex $v \in V_1$ is said to satisfy the *clique candidate property* (A) if

$$\begin{array}{ll} (A_{\rho,\mu}) & \forall A \subseteq \Gamma_2(v), |A| \leq \rho q \\ & \forall Q \subseteq \Gamma_3(v) \text{ s.t. } Q \text{ is a } CC_5(v,A)\text{-resistant } (\rho, (\mu)^{\oplus})\text{-square cover :} \\ & \quad (\Gamma_2(v) \setminus A) \cap S_2(Q, v, CC_5(v,A), \rho) \neq \emptyset \end{array}$$

In the following we show that most graphs in $S_5(n, m; \varepsilon)$ contain many K_5-candidates.

Definition 6.8 (Vertices without K_5-candidates)

$$\begin{aligned} \mathcal{B}^A(n,m;\varepsilon,\rho,\mu) \quad := \quad & \{G \in \mathcal{S}_5(n,m;\varepsilon) \mid \\ & \textit{at least } (\mu)^{\oplus 2} n \textit{ vertices } v \in V_1 \textit{ satisfy } (D) \textit{ but not } (A_{\rho,\mu})\} \end{aligned}$$

Lemma 6.9 ($\mathcal{B}^A$ is small) *If $(\rho)^{\oplus} \lll (\mu)^{\ominus 5}$ then*

$$|\mathcal{B}^A(n,m;\varepsilon,\rho,\mu)| \leq ((\mu)^{\oplus 2})^m \binom{n^2}{m}^{10}.$$

Proof Analogous to Lemma 7.12 in [4]. $\square$

Let

$$\mathcal{S}^{QTNSRA}(n,m;\varepsilon,\rho,\mu) := \mathcal{S}^{QTNSR}(n,m;\varepsilon,\rho,\mu) \setminus \mathcal{B}^A(n,m;\varepsilon,\rho,\mu).$$

It follows from (21) and Lemma 6.9 that if $(\rho)^{\oplus} \lll (\mu)^{\ominus 5}$ and $m \geq \rho^{-6} n^{5/3}$ then

$$|\mathcal{S}^{QTNSRA}(n,m;\varepsilon,\rho,\mu)| \geq (1 - 2((\mu)^{\oplus 2})^m) \binom{n^2}{m}^{10}. \tag{22}$$

Also, for a graph $G \in \mathcal{S}^{QTNSRA}(n,m;\varepsilon,\rho,\mu)$ at least $(1-6(\mu)^{\oplus 3})n \geq (2\mu)^{\oplus} n$ vertices in V_1 satisfy $(R_{\rho,\mu})$ and $(A_{\rho,\mu})$ by Lemma 6.6 and Definition 6.8. The following lemma shows that these properties suffice to prove the existence of K_5-candidates.

Lemma 6.10 (Construction of K_5-candidates) *For every vertex $v \in V_1$ which satisfies $(R_{\rho,\mu})$ and $(A_{\rho,\mu})$ we can find a set $A \subseteq \Gamma_2(v)$ such that*

$$|CC_5(v,A)| \geq (1-2\mu)n,$$

i.e., the vertex v is a 2μ-clique covering vertex.

Proof Analogous to Lemma 7.13 in [4]. □

The following lemma shows that only very few clique free graphs with many clique covering vertices exist.

Lemma 6.11 (Few clique covering vertices exist) *The number of graphs $G \in \mathcal{S}_5(n,m;\varepsilon)$ for which V_1 contains $(\xi)^{\oplus}n$ ξ-clique covering vertices v which satisfy (D) and*

$$(\neg K) \quad \Gamma_5(v) \cap CC_5(v, \Gamma_2(v)) = \emptyset$$

is bounded by

$$((\xi)^{\oplus})^m \binom{n^2}{m}^{10}.$$

Proof Analogous to Lemma 7.15 in [4]. □

Now the proof of Theorem 2.3 is complete since for every $\beta > 0$ we can set

$$\bar{\mu} = (\beta)^{\ominus 3}, \bar{\rho} = (\mu)^{\ominus 7} \text{ and } \mathcal{S} = \{\bar{\rho}, (\bar{\rho})^{\oplus}, \bar{\mu}, (\bar{\mu})^{\ominus 5}\}.$$

Then by (22)

$$|\mathcal{S}^{QTNSRA}(n,m;\varepsilon,\bar{\rho},\bar{\mu})| \geq (1 - 2((\bar{\mu})^{\oplus 2})^m)\binom{n^2}{m}^{10} \tag{23}$$

for sufficiently small ε. Combining this with Lemma 6.11 we conclude that at most $3((\bar{\mu})^{\oplus 2})^m\binom{n^2}{m} \leq \beta^m\binom{n^2}{m}$ graphs in $\mathcal{S}_5(n,m;\varepsilon)$ exist which do not contain K_5 as a subgraph.

Note that our proof actually yields a somewhat stronger result than Theorem 2.3. The condition (K) could be strengthened to

$$(\neg K') \quad |\Gamma_5(v) \cap CC_5(v, \Gamma_2(v))| \leq (1-\tau)q$$

for a suitable constant $\tau > 0$ which is large in comparison to ξ. Using this we obtain that almost all neighbours $x \in \Gamma_5(v)$ of almost all clique covering vertices are part of a subgraph K_5 together with v.

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