Asymptotic enumeration, global structure and constrained evolution

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Abstract

Determining the cardinality of a class of discrete objects and the structure of a typical element within this class is relatively easy whenever the objects can be generated by a sequence of independent events. If this is not the case the problem usually becomes much harder. In this paper we survey results and methods for such classes of objects. Our focus lies on triangle–free graphs and partially ordered sets.

1 Introduction

Given a finite set, one of the most elementary questions is to determine its cardinality. In general, counting is easy in situations where one can generate all objects in the set by a sequence of *independent* decisions such that two different sequences never produce the same object – as then the cardinality is precisely the product of the number of possibilities one had at each step. However, the required independence is often not given, for one usually finds that depending on how earlier decisions are taken later decisions may be forced.

We will be concerned with *asymptotic enumeration*, by which we mean obtaining good approximations for the cardinality of a given set. Here there is a way to circumvent the lack of independence: within the set in question, find a subset which can be enumerated by independent decisions, and show that this subset contains *almost all* of the objects of the larger set.

This strategy is very attractive since it not only estimates the cardinality of the larger set, but at the same time proves a result about the *global structure* of the larger set: draw

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an object uniformly at random from the set, then with high probability it satisfies the conditions that define the subset.

Results on the global structure of a set state what properties we can expect from a *typical* element of the set – it describes the structure that might not be common to all elements but certainly to the overwhelming majority. As an illustration, take two results, which we will consider in much more detail in this article: 'almost all triangle-free graphs are bipartite', or 'almost all partially ordered sets have height 3'. These statements are likely to surprise someone who works with odd cycles or linear orders. In other words, the degree of acceptance or understanding that a result of the above type receives will depend entirely on the subjective view of the audience, namely on whether the audience really regards all instances of the given universe as equally likely or interesting. However, there is a way to accommodate different points of view, and this brings us to the following.

The evolution of a random object (chosen from a certain class) usually denotes the following 'experiment': choose a parameter of the objects in the class under consideration, for example the number of edges (in a triangle–free graph) or the number of comparable pairs (in a poset). Now consider only one 'slice' of the class, namely only those objects where the parameter is equal to a fixed value x and pose the following questions: What does a typical object from this slice look like? How many objects are there in this slice? And how do the answers to the first two questions change when x changes?

So far we have illustrated some of the close ties between the areas of asymptotic enumeration, global structure and evolution. The theories and models in these areas have proved to be a great success, provoking the development of new techniques on the boundary between combinatorics and probability theory such as large deviation inequalities in general and martingale inequalities in particular. However, most of these methods rely heavily on the fact that the random structures are generated by a series of independent events. The aim of this article is to survey results and methods for classes where this independence is not given. We shall concentrate on classes of graphs which do not contain a fixed subgraph H (here in particular the case where H is a triangle) and partially ordered sets. For a far more detailed account on many aspects of this article we refer the reader to [40].

We conclude this section with a few remarks concerning terminology and notation. For a positive integer n we let $[n] := \{1, \ldots, n\}$. The logarithm to base 2 will be denoted by log. Often we will be somewhat sloppy and disregard rounding of real numbers.

For a graph $G = (V, E)$ and a vertex v, the *neighbourhood* $\Gamma(v)$ is the set of all neighbours of v. For $S \subseteq V$, $\Gamma(S)$ is the set of all vertices that are not contained in S but have at least one neighbour in S.

For a *partially ordered set* $P = (X, P)$ (often abbreviated as *poset*) and two points $x, y \in X$ we write $x \leq y$ if $(x, y) \in P$, and $x < y$ if $x \leq y$ and $x \neq y$. If $x < y$ then we say that x, y form a *comparable pair* or, in abuse of notation, a *relation*. Moreover we say that x is *covered* by y (also (x, y) is a *cover relation*) if $x < y$ and there is no point z for which $x < z$ and $z < y$ holds. With a poset $P = (X, P)$ we associate the *cover graph*. Its vertex set is given by X, its edges are formed by the cover pairs in P.

2 Asymptotic enumeration

Counting graphs. Many interesting families of graphs are defined by requiring that every member of the family must or, to the contrary, must not have one or several particular substructures. Let us start with the case where one subgraph is forbidden and denote by $\mathcal{F}orb(H)$ the family of all graphs that do not contain H as a (weak) subgraph. (Graphs in $\mathcal{F}orb(H)$ are sometimes called *H-free*.) Denote by $\mathcal{F}orb_n(H)$ the family of those graphs in $\mathcal{F}orb(H)$ with vertex set $[n]$. In this context, two natural questions arise. For a given graph H, *how many edges can a graph in $\mathcal{F}orb_n(H)$ possibly have?* And: *how many graphs belong to $\mathcal{F}orb_n(H)$?*

The answer to the first question, i.e. the maximum number of edges a graph G with n vertices can have without containing H as a subgraph, is denoted by $ex_n(H)$. A graph G with the maximum number of edges is called *extremal*. Turán's theorem [42, 43], which is widely considered to be the starting point of extremal graph theory, states that

$$ex_n(K_\ell) = \Big(1 - \frac{1}{\ell - 1}\Big) \cdot \frac{n^2}{2} + O(n)$$

and that the complete $(\ell-1)$-partite graph with all parts as equal as possible is the unique H-free extremal graph.

Generalizing this, Erdős, Stone, and Simonovits [20, 19] discovered that it is in fact the chromatic number of H which determines the magnitude of $ex_n(H)$:

$$ex_n(H) = \Big(1 - \frac{1}{\chi(H) - 1}\Big) \cdot \frac{n^2}{2} + o(n^2). \tag{1}$$

Moving on to the second question, observe that trivially if G is H-free, then so is each of its subgraphs and hence there are at least $2^{ex_n(H)}$ graphs in $\mathcal{F}orb_n(H)$. Quite surprisingly, this crude lower bound is already tight. In 1986, Erdős, Frankl and Rödl [11] proved that

$$|\mathcal{F}orb_n(H)| = 2^{(1-\frac{1}{\chi(H)-1})\cdot\frac{n^2}{2}+o(n^2)}, \tag{2}$$

therefore for any graph H with $\chi(H) \geq 3$

$$|\mathcal{F}orb_n(H)| = 2^{ex_n(H)\cdot(1+o(1))}. \tag{3}$$

Analogous results for the case when H is bipartite are not known. Partial results for the case of $H = C_{2k}$ were given by Kleitman and Wilson [26] and Kreuter [28].

Now we focus our attention on families of graphs, which do not contain a certain graph as an *induced subgraph*: denote by $\mathcal{F}orb^*(H)$ the family of all graphs that do not contain H as an induced subgraph and let $\mathcal{F}orb_n^*(H)$ be the subfamily formed by all graphs with vertex set $[n]$.

Here the situation immediately becomes more difficult, which is already indicated by the fact that the above definition of an extremal graph makes not much sense in this setting: $G = K_n$ would be the obvious (and unique) extremal graph from $\mathcal{F}orb_n^*(H)$ – unless H is a clique itself.

In 1992, Prömel and Steger suggested the following definition. Let $ex_n^*(H)$ denote the maximum number of edges a graph $G = ([n], E)$ can have such that there exists a graph $G_0 = ([n], E_0)$ with $E \cap E_0 = \emptyset$ so that for all $E' \subseteq E$ the graph $G' = ([n], E_0 \cup E')$ always lies in $\mathcal{F}orb_n^*(H)$. In this context G_0 is called the *supplemental* graph to the *extremal* graph G.

This now gives us a more meaningful definition of an extremal graph, but in order to obtain results analogous to (1) and (3) one first needs a parameter that can play the role of χ. The definition of this new parameter is more easily understood if one first observes that the chromatic number $\chi(H)$ and the clique covering number $\sigma(H)$ can be defined as follows:

- $\chi(H)$ is the largest integer k such that no $(k-1)$-partite graph contains H as an induced subgraph, and
- $\sigma(H)$ is the largest integer k such that no $(k-1)$-partite graph, in which *all* classes are replaced by a clique, contains H as an induced subgraph.

As a generalization of both define

- $\tau(H)$ to be the largest integer k for which there exists an integer $\tilde{k}$ with $0 \le \tilde{k} \le k-1$, such that no $(k-1)$-partite graph, in which $\tilde{k}$ classes are replaced by a clique, contains H as an induced subgraph.

Now the results by Prömel and Steger [31, 32] stand as

$$ex_n^*(H) = \left(1 - \frac{1}{\tau(H)-1}\right) \cdot \frac{n^2}{2} + o(n^2),$$

and

$$|\mathcal{F}orb_n^*(H)| = 2^{\left(1-\frac{1}{\tau(H)-1}\right)\cdot\frac{n^2}{2}+o(n^2)}. \tag{4}$$

Thus for any graph H with $\tau(H) \geq 3$

$$|\mathcal{F}orb_n^*(H)| = 2^{ex_n^*(H)\cdot(1+o(1))}. \tag{5}$$

Let us consider the situation in more general terms. A *graph property* $\mathcal{P}$ is a family of finite graphs that is closed under isomorphism. Denote by $\mathcal{P}_n$ the set of those graphs in $\mathcal{P}$ with vertex set $[n]$. $\mathcal{P}$ is called *hereditary* if it is also closed under taking *induced* subgraphs. Trivially, every hereditary property $\mathcal{P}$ can be described in terms of forbidden induced subgraphs: from the set of all graphs not contained in $\mathcal{P}$ it suffices to forbid exactly those that are minimal (w.r.t. taking induced subgraphs).

Scheinerman and Zito [39] were able to show that for every hereditary property the rate of growth, i.e. the sequence

$$\Big(|\mathcal{P}_n|\Big)_{n=1}^{\infty},$$

belongs to one of the following categories: constant, polynomially, exponentially, factorially, or superfactorially. We will be interested in properties that are very rich, by which we mean that their rate of growth is close to maximal. Following the terminology used by Bollobás [4], we define the *logarithmic density* of $\mathcal{P}$ to be

$$c_n(\mathcal{P}) := \frac{\log|\mathcal{P}_n|}{\binom{n}{2}}, \quad \text{in other words,} \quad 2^{c_n(\mathcal{P})\cdot\binom{n}{2}} = |\mathcal{P}_n|, \tag{6}$$

and denote the *asymptotic logarithmic density* of $\mathcal{P}$ by

$$\mathrm{c}(\mathcal{P}) := \lim_{n\to\infty} c_n(\mathcal{P}), \tag{7}$$

provided that the limit exists. When expressed in this form, the statements (2) and (4) stand as

$$\mathrm{c}(\mathcal{F}orb(H)) = 1 - \frac{1}{\chi(H)-1} \quad \text{and} \quad \mathrm{c}(\mathcal{F}orb^*(H)) = 1 - \frac{1}{\tau(H)-1}$$

respectively.

In order to compute $\mathrm{c}(\mathcal{P})$ we need to generalize the parameter $\tau(H)$ to $\tau(\mathcal{H})$ where $\mathcal{H}$ is a family of finite graphs: it is defined as the largest integer k for which there exists an integer $\tilde{k}$ with $0 \leq \tilde{k} \leq k-1$, such that no $(k-1)$-partite graph, in which $\tilde{k}$ classes are replaced by a clique, contains *any graph* from $\mathcal{H}$ as an induced subgraph.

For an arbitrary hereditary property $\mathcal{P}$ now let $\mathcal{H}$ be the family of all graphs not contained in $\mathcal{P}$. Bollobás and Thomason [5] (also see Alekseev [1] for an earlier and slightly weaker version) proved that the asymptotic logarithmic density exists and is determined by

$$\mathrm{c}(\mathcal{P}) = 1 - \frac{1}{\tau(\mathcal{H})-1}. \tag{8}$$

Counting posets. Now we move from counting graphs to counting partially ordered sets. While enumerating families of graphs only becomes difficult when some substructure is forbidden, counting *all* partially ordered sets on n points is already difficult *as such.* The reason being that once we have decided that the relations $x < y$ and $y < z$ are to be present, we have no choice but to insert the relation $x < z$ in order to guarantee transitivity.

Let $\mathcal{P}$ be the family of all labeled posets, and denote by $\mathcal{P}_n$ the set of those posets in $\mathcal{P}$ with point set $[n]$. Again, just as with hereditary graph properties, we will be interested in finding an estimate for the cardinality of $\mathcal{P}_n$. Again we will content ourselves with determining the logarithm of $|\mathcal{P}_n|$, however this time with a different normalizing factor. Let

$$c_n(\mathcal{P}) := \frac{\log |\mathcal{P}_n|}{n^2}, \quad \text{in other words,} \quad 2^{c_n(\mathcal{P})\cdot n^2} = |\mathcal{P}_n|, \tag{9}$$

and define

$$\mathrm{c}(\mathcal{P}) := \lim_{n\to\infty} c_n(\mathcal{P}), \tag{10}$$

provided the limit exists.

We first establish an easy lower bound on $|\mathcal{P}_n|$. It will suffice to consider posets with height 2, so fix two antichains X and Y, each on $\frac{n}{2}$ points. In this case there are no problems with transitivity, so decide *independently* for each pair $(x, y) \in X \times Y$ whether or not $x < y$ should hold. Hence

$$|\mathcal{P}_n| \;\geq\; 2^{\frac{n^2}{4}}.$$

Notice that here we didn't even care for the $\binom{n}{n/2}$ ways in which we could have chosen the points which go to, say, X.

Upper bounds are much harder to obtain. In 1970, Kleitman and Rothschild [23] first gave the following bound:

$$|\mathcal{P}_n| \;\leq\; 2^{n^2/4+O(n^{3/2}\log n)}, \tag{11}$$

hence

$$\mathrm{c}(\mathcal{P}) = \frac{1}{4}.$$

A few years later [24], they were able to compute the exponent much more precisely:

$$\log |\mathcal{P}_n| \;=\; \frac{n^2}{4} + \frac{3n}{2} + O(\log n). \tag{12}$$

The technique used by Kleitman and Rothschild in the proofs of these two papers has become known as *the Kleitman–Rothschild method.* We shall illustrate it in the next chapter with a few examples.

3 Global structure

Almost all graphs. One of the most interesting aspects of the above results as expressed in (3) or (5) is that not only do they compute the asymptotic number of elements but they also show that a substantial proportion of the class $\mathcal{P}$ in question are from a well–structured subclass $\mathcal{P}'$, namely the class of all subgraphs of the extremal graph combined with the supplemental graph. These graphs will obviously be very similar in structure, and hence we are tempted to believe that this structure is not only representative for $\mathcal{P}'$ but also for $\mathcal{P}$.

Naturally the subject of counting elements in a set and determining the structure of a 'typical' element are closely related: if we know that a typical element has a certain structure, then this means that most elements in $\mathcal{P}$ have this structure, hence it suffices to enumerate the subclass of elements with exactly this structure – and enumerating them might actually be a lot easier because of their structure. Or, vice versa, if we find a subclass $\mathcal{P}' \subseteq \mathcal{P}$ which we can prove to be 'big', then the structure defining $\mathcal{P}'$ might well be that of a typical element of $\mathcal{P}$. This structure is what we would like to call the *global structure* of $\mathcal{P}$. Obviously here the terms *typical* and *big* are still far too vague and we shall make them precise shortly.

One way to prove that an element typically possesses a certain structure is to choose an element at random from the set in question and to show that with very high probability it has the required structure – this is the underlying theme of the theory of random graphs. It first appeared as a tool for a probabilistic construction, where Erdős [10] proved that for every n there exists a graph on n vertices which has neither a clique nor an independent set on $2\log n$ vertices, and in fact the proof shows that *almost all* graphs on n vertices satisfy this condition. Since this is the first result of this type that we are mentioning, let us state it again with more precision: Denote by $\mathcal{G}_n$ the set of all labeled graphs with vertex set $[n]$ and let $\mathcal{G}'_n$ be the set of those graphs in $\mathcal{G}_n$ that have neither a clique nor an independent set of order $2\log n$. Then

$$\frac{|\mathcal{G}'_n|}{|\mathcal{G}_n|} \longrightarrow 1 \qquad \text{as } n \to \infty.$$

Equivalently, introducing the language of classical random graph theory, let $G_{n,p}$ be the random graph obtained by starting with the empty graph on the vertex set $[n]$ and then inserting each edge *independently* with probability p. Obviously for a fixed graph G the chance that $G_{n,\frac{1}{2}} = G$ is the same for all graphs G, namely $2^{-\binom{n}{2}}$, which is the reason why $G_{n,\frac{1}{2}}$ is referred to as the *uniform model*. So Erdős' result stands as

$$\mathrm{Prob}\Big[\ \omega(G_{n,\frac{1}{2}}) < 2\log n \text{ and } \alpha(G_{n,\frac{1}{2}}) < 2\log n\Big] \longrightarrow 1 \qquad \text{as } n \to \infty.$$

We will come back to the random graph $G_{n,p}$ later, in the context of evolution. As explained in the introduction, the $G_{n,p}$ model has proved to be a great success (for an overview we refer the reader to the monograph by Bollobás [2] or the survey article by Karoński [22]), which, to a large extent, is based on the fact that $G_{n,p}$ can be generated by a series of independent events such as inserting an edge with a certain probability. Unfortunately this independence no longer exists when seeking random elements from classes such as $\mathcal{F}orb_n(H)$ or $\mathcal{F}orb^*_n(H)$, hence most of the above mentioned methods remain ineffective here.

Nevertheless there are results about the global structure of these classes. The starting point here is the class $\mathcal{F}orb_n(K_3)$. In 1976, Erdős, Kleitman and Rothschild [12] proved the following theorem.

Theorem 3.1 *Almost all triangle–free graphs are 2-colourable.*

Before we present a proof of this theorem, let us mention that it was generalized in two ways: For any constant ℓ, Lamken and Rothschild [29] showed that almost all graphs in $\mathcal{F}orb_n(C_{2\ell+1})$ are 2-colourable, and Kolaitis, Prömel and Rothschild [27] proved that almost all graphs in $\mathcal{F}orb_n(K_{\ell+1})$ are ℓ-colourable. A further extension, re–uniting the latter two results, was obtained by Prömel and Steger [33] who characterized the family of forbidden graphs H for which such a statement holds. Denote by $\mathcal{C}ol_n(\ell)$ the family of all labeled ℓ-colourable graphs with vertex set $[n]$. Then for a graph H with $\chi(H) = \ell+1 \geq 3$

$$|\mathcal{F}orb_n(H)| = (1 + o(1)) \cdot |\mathcal{C}ol_n(\ell)|$$

if and only if H contains an edge whose deletion from H causes $\chi(H)$ to decrease.

As stated at the end of the previous section, we are now going to prove Theorem 3.1 in order to illustrate the underlying idea of the Kleitman–Rothschild method. The proof consists of three parts:

I. Define sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ such that

$$\mathcal{F}orb_n(K_3) \subseteq \mathcal{C}ol_n(2) \cup \mathcal{A}(n) \cup \mathcal{B}(n) \cup \mathcal{C}(n).$$

II. Bound the cardinalities of the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ in terms of those of $\mathcal{F}orb_{n-x}(K_3)$ for appropriate $x \in \mathbb{N}$.

III. Show inductively that each of the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ is negligible compared to $\mathcal{C}ol_n(2)$.

Figure 1 outlines the desired partition of $\mathcal{F}orb_n(K_3)$ and indicates how the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ are defined. Of course, the drawing is not according to scale.

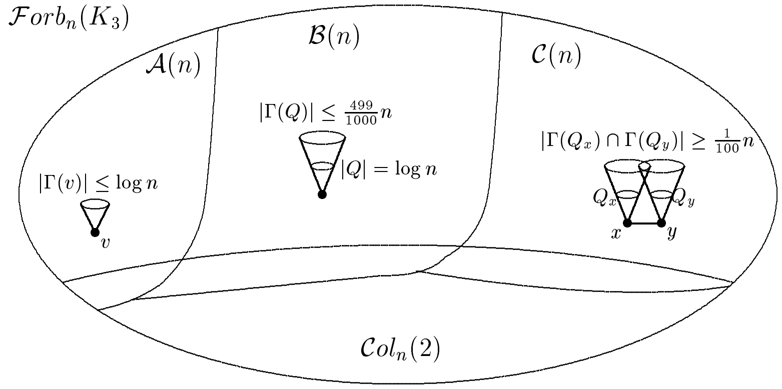

Figure 1: Partition of $\mathcal{F}orb_n(K_3)$

Part I. The three sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ will reflect properties which almost surely do not hold in a (random) bipartite graph on n vertices. As we shall see later in the proof, the colour classes in a random bipartite graph are of approximately the same size. For our purposes, it therefore suffices to view a random bipartite graph as a bipartite graph with colour classes as equal as possible, in which the edges are inserted randomly with probability 1/2. It should be intuitively clear that with very high probability in such a graph every vertex has degree at least, say, $\log n$, and that every set of size $\log n$ has at least, say, $(\frac{1}{2} - \frac{1}{1000})n$ neighbours. The definition of $\mathcal{A}(n)$ and $\mathcal{B}(n)$ captures these two properties, while the definition of $\mathcal{C}(n)$ reflects the property that in bipartite graphs all edges $\{x, y\}$ satisfy $\Gamma(\Gamma(x)) \cap \Gamma(\Gamma(y)) = \emptyset$. Let

$\mathcal{A}(n)$ denote the set of all graphs in $\mathcal{F}orb_n(K_3)$ which contain a vertex v such that $|\Gamma(v)| \le \log n$,

$\mathcal{B}(n)$ denote the set of all graphs in $\mathcal{F}orb_n(K_3) \setminus \mathcal{A}(n)$ which contain a vertex v and a set $Q \subseteq \Gamma(v)$ of size $\log n$ such that $|\Gamma(Q)| \le (\frac{1}{2} - \frac{1}{1000})n$, and

$\mathcal{C}(n)$ denote the set of all graphs in $\mathcal{F}orb_n(K_3) \setminus [\mathcal{A}(n) \cup \mathcal{B}(n)]$ which contain an edge $\{x, y\}$ and sets $Q_x \subseteq \Gamma(x)$ and $Q_y \subseteq \Gamma(y)$ of size $|Q_x| = |Q_y| = \log n$ such that $|\Gamma(Q_x) \cap \Gamma(Q_y)| \ge \frac{1}{100} n$.

It is not difficult to show that these sets have the required properties.

Lemma 3.2 $$\mathcal{F}orb_n(K_3) \subseteq Col_n(2) \cup \mathcal{A}(n) \cup \mathcal{B}(n) \cup \mathcal{C}(n).$$

Proof. Let $G = ([n], E)$ be an arbitrary but fixed graph in $\mathcal{F}orb_n(K_3) \setminus [\mathcal{A}(n) \cup \mathcal{B}(n) \cup \mathcal{C}(n)]$. We need to show that $G \in Col_n(2)$. To see this fix for every vertex $v \in [n]$ a set

$Q_v \subseteq \Gamma(v)$ of size $\log n$ and let $R_v = \Gamma(Q_v)$. Observe that by the definition of the set $\mathcal{A}(n)$ such sets Q_v exist and that by the definition of $\mathcal{B}(n)$ the sets R_v satisfy $|R_v| \geq (\frac{1}{2} - \frac{1}{1000})n$. Furthermore, by the definition of the set $\mathcal{C}(n)$, we have $|R_x \cap R_y| \leq \frac{1}{100}n$ for all $\{x, y\} \in E$. One easily checks that these last two bounds imply that G can neither contain a C_5 nor a C_7 or C_9. Indeed, suppose there were an odd cycle $v_1, \ldots, v_{2\ell+1}$ (where $\ell = 2, 3, 4$) and let the vertices in R_{v_1} form something like the 'left half' of $[n]$. Then R_{v_2} must lie mostly to the right, R_{v_3} mostly to the left and so on, but the odd parity of the cycle yields a contradiction.

Choose now an arbitrary edge $\{x, y\} \in E$. As G contains no C_5 we conclude that R_x and R_y are stable sets and that $R_x \cap R_y = \emptyset$. Let $S = [n] \setminus (Q_x \cup Q_y \cup R_x \cup R_y)$ and

$$S_x = \{v \in S |\ R_v \cap R_x \neq \emptyset\} \qquad \text{and} \qquad S_y = \{v \in S |\ R_v \cap R_y \neq \emptyset\}.$$

Observe that certainly $S_x \cup S_y = S$ and that, as G contains no C_7 or C_9, we also know that $R_x \cup S_x$ and $R_y \cup S_y$ are stable and $S_x \cap S_y = \emptyset$. Now we have a proper two colouring of G with colour classes $R_x \cup S_x \cup Q_y$ and $R_y \cup S_y \cup Q_x$. □

Part II. In this part we bound the cardinalities of the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ in terms of those of $\mathcal{F}orb_{n-x}(K_3)$ for an appropriate $x \in \mathbb{N}$. This is done by first choosing an appropriate subset of the vertex set, a K_3-free graph on the remaining vertices and then connecting the vertices of the special set with the remaining vertices.

Lemma 3.3 *For all sufficiently large n*

$$\begin{aligned}
\log \frac{|\mathcal{A}(n)|}{|\mathcal{F}orb_{n-1}(K_3)|} &\leq 2(\log n)^2 && (13)\\
\log \frac{|\mathcal{B}(n)|}{|\mathcal{F}orb_{n-\log n}(K_3)|} &\leq \left(\frac{1}{2} - \frac{1}{2000}\right) n \log n && (14)\\
\log \frac{|\mathcal{C}(n)|}{|\mathcal{F}orb_{n-2}(K_3)|} &\leq \left(1 - \frac{1}{2000}\right) n. && (15)
\end{aligned}$$

Proof. Here we only prove (14) to give an illustration of the method. For the proof of (13) and (15) one can proceed very similarly.

Construct all graphs in $\mathcal{B}(n)$ as follows. First choose the set Q and a triangle-free graph on $[n] \setminus Q$ (in at most $\binom{n}{\log n} \cdot |\mathcal{F}orb_{n-\log n}(K_3)|$ ways). Then choose the set $R = \Gamma(Q)$ (less than 2^n ways) and connect Q to the set R (at most $2^{|Q|\cdot|R|} \leq 2^{\log n \cdot (\frac{1}{2} - \frac{1}{1000})n}$ ways). Together this gives

$$\begin{aligned}
\log \frac{|\mathcal{B}(n)|}{|\mathcal{F}orb_{n-\log n}(K_3)|} &\leq (\log n)^2 + n + \left(\frac{1}{2} - \frac{1}{1000}\right) n \log n\\
&\leq \left(\frac{1}{2} - \frac{1}{2000}\right) n \log n,
\end{aligned}$$

for n sufficiently large.

Observe that we did not use the fact that the set Q is a subset of the neighbourhood of some vertex v here. That is, we could also have let $\mathcal{B}(n)$ denote the (larger) set of all graphs in $\mathcal{F}orb_n(K_3)$ which contain a set $Q \subseteq \Gamma(v)$ of size $\log n$ such that $|\Gamma(Q)| \le (\frac{1}{2} - \frac{1}{1000})n$. $\Box$

Part III. To show inductively that the sets $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ are negligible compared to $\mathcal{C}ol_n(2)$ we first establish bounds on the growth rate of the set $\mathcal{C}ol_n(2)$.

Lemma 3.4 $|\mathcal{C}ol_n(2)| \ge 2^{\frac{1}{2}(n-1)}|\mathcal{C}ol_{n-1}(2)|$ *for all* $n \in \mathbb{N}$.

Proof. This follows easily from the observation that we can obtain every bipartite graph on n vertices by first choosing a bipartite graph on $n-1$ vertices ($|\mathcal{C}ol_{n-1}(2)|$ choices), fixing an arbitrary 2-colouring of it, and connecting the n-th vertex in all possible ways to the *larger* colour class (there are at least $2^{\frac{1}{2}(n-1)}$ ways to do this). $\Box$

Proof of Theorem 3.1. Since $\mathcal{C}ol_n(2) \subseteq \mathcal{F}orb_n(K_3)$ it suffices to show that there exist constants $c \ge 0$ and $\gamma > 1$ such that

$$|\mathcal{F}orb_n(K_3)| \le (1 + c\gamma^{-n})|\mathcal{C}ol_n(2)| \tag{16}$$

holds for all $n \in \mathbb{N}$. Let $\gamma = 2^{\frac{1}{3000}}$ and choose n_0 large enough so that Lemma 3.3 holds for all $n \ge n_0$ and such that for all $n \ge n_0$:

$$\max\,\{2^{-\frac{1}{2}n+2(\log n)^2+2},\ 2^{-\frac{1}{2000}n\log n+\frac{1}{2}(\log n)^2+1},\ 2^{-\frac{1}{2000}n+3}\,\} \le \frac{1}{3}\gamma^{-n}.$$

Subsequently, choose $c \ge 1$ such that (16) holds for all $n \le n_0$. We conclude the proof by induction on n. So assume (16) holds for some $n-1 \ge n_0$. By Lemma 3.2 we know that

$$|\mathcal{F}orb_n(K_3)| \le |\mathcal{A}(n)| + |\mathcal{B}(n)| + |\mathcal{C}(n)| + |\mathcal{C}ol_n(2)|.$$

Hence, it suffices to show that the ratio of each of the set $\mathcal{A}(n)$, $\mathcal{B}(n)$ and $\mathcal{C}(n)$ with $\mathcal{C}ol_n(2)$ is at most $\frac{c}{3}\gamma^{-n}$. In each of the three cases this can be done using Lemma 3.3, induction hypothesis, and Lemma 3.4. Here we demonstrate this for the class $\mathcal{B}(n)$ only.

$$\begin{aligned}
\frac{|\mathcal{B}(n)|}{|\mathcal{C}ol_n(2)|} &= \frac{|\mathcal{B}(n)|}{|\mathcal{F}orb_{n-\log n}(K_3)|}\cdot\frac{|\mathcal{F}orb_{n-\log n}(K_3)|}{|\mathcal{C}ol_{n-\log n}(2)|}\cdot\prod_{i=1}^{\log n}\frac{|\mathcal{C}ol_{n-i}(2)|}{|\mathcal{C}ol_{n-i+1}(2)|} \\
&\le 2^{(\frac{1}{2}-\frac{1}{2000})n\log n}\cdot\underbrace{(1+c\gamma^{-n+\log n})}_{\le 2c}\cdot 2^{-\sum_{i=1}^{\log n}\frac{1}{2}(n-i)} \\
&\le 2^{-\frac{1}{2000}n\log n+\frac{1}{2}(\log n)^2+1}\cdot c \le \frac{c}{3}\gamma^{-n}.
\end{aligned}$$

□

Almost all posets. Having seen the Kleitman–Rothschild method at work in some detail, we now return to its origins, namely to the proofs of (11) and (12). We follow the terminology used in [7] and define the following family of posets. A poset $P = (X, P)$ is a *k-layer poset*, if there exists a partition of its point set $X = X_1 \cup \cdots \cup X_k$ into k disjoint antichains (*layers*) such that

$$\begin{aligned} x < y \text{ with } x \in X_i \text{ and } y \in X_j \quad &\Longrightarrow \quad i < j \\ \text{for every } i, j \text{ with } j > i + 1: \quad x \in X_i,\ y \in X_j \quad &\Longrightarrow \quad x < y. \end{aligned}$$

In rough terms, the proofs of (11) and (12) can now be described as follows. Partition $\mathcal{P}_n$ into a number of subclasses. One of these contains only 3-layer posets with point set $[n]$ that have about $n/2$ points in the middle layer X_2 and about $n/4$ points in the two outer layers X_1 and X_3. Call this class $\mathcal{Q}_n$ and prove that all classes other than $\mathcal{Q}_n$ are negligible in size. Finally obtain an approximation for $|\mathcal{P}_n|$ by enumerating $\mathcal{Q}_n$, which is easy, since one can use the special structure of layer posets. (We shall give some more details for a variant of this proof at the end of this section.) Obviously this strategy not only proves (11) and (12) but at the same time the following theorem.

Theorem 3.5 ([24]) *Almost all posets are 3-layer posets.*

Admittedly the 3-layer structure seems at first glance surprisingly 'narrow minded' and the above result is hence a bit of a disappointment, since it somewhat questions the validity of the uniform model for random posets as an interesting area of investigation. As Brightwell [6] puts it: '[it] does not conform to the practising mathematician's view of what a "typical" partial order ought to look like.' This explains the development of other models of random partially ordered sets, such as random bipartite orders, random graph orders and random k-dimensional orders, for which we refer the interested reader to the survey articles by Brightwell [6], Trotter [41], and Winkler [44].

We conclude this section by pointing out how Theorem 3.5 can be established via a modification of the above proof of Theorem 3.1. For the complete proof we refer the reader to [7].

Denote by $\mathcal{A}'(n)$, $\mathcal{B}'(n)$, and $\mathcal{C}'(n)$ those posets in $\mathcal{P}_n$ whose cover graph lies in $\mathcal{A}(n)$, $\mathcal{B}(n)$, and $\mathcal{C}(n)$ respectively. Define two more classes. $\mathcal{D}'(n)$ contains those posets in $\mathcal{P}_n$ whose cover graph is (i) connected and bipartite with colour classes of roughly the same size, and (ii) has the additional property that there exist vertices u, v, w in the same colour class that do *not* have a common neighbour. Finally $\mathcal{L}(n)$ is defined as the family of all 2-layer, 3-layer and 4-layer posets in $\mathcal{P}_n$.

We claim that

$$\mathcal{P}_n \subseteq \mathcal{L}(n) \cup \mathcal{A}'(n) \cup \mathcal{B}'(n) \cup \mathcal{C}'(n) \cup \mathcal{D}'(n). \tag{17}$$

Since every cover graph is triangle–free, the proof of Lemma 3.2 shows that the cover graph of a poset in $\mathcal{P}_n \setminus [\mathcal{A}'(n) \cup \mathcal{B}'(n) \cup \mathcal{C}'(n)]$ satisfies the above property (i). Hence the following lemma suffices to establish (17).

Lemma 3.6 *If the cover graph of $P \in \mathcal{P}_n$ is bipartite but doesn't have property (ii), then $P \in \mathcal{L}(n)$.*

Proof. Denote by A and B the colour classes of the cover graph of P. If A is an antichain in P, we are done, as then every element of B is either below or above A, but not both, and every pair u, v with u below A and v above it, must have a common neighbour in A and thus $u < v$. By symmetry, the same applies if B is an antichain.

So suppose there exists $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $a_1 < a_2$ and $b_1 < b_2$. Before we proceed let us first pause to observe that the fact that P does not have property (ii) implies that neither A not B can contain three elements x_1, x_2, x_3 such that $x_1 < x_2 < x_3$. Hence, the four sets $A_1 := \{a \in A \mid a < a_2\}$, $A_2 := \{a \in A \mid a_1 < a\}$, $B_1 := \{b \in B \mid b < b_2\}$, and $B_2 := \{b \in B \mid b_1 < b\}$ are all antichains and satisfy $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$. As P doesn't have property (ii) we also know that for all $a \in A$ the elements a, a_1, and a_2 have a common neighbour $b \in B$ in the cover graph. As $a_1 < a_2$, there is only one way, how b can be related to a_1 and a_2. Namely, $a_1 < b < a_2$. For a and b there are two choices: either $a < b$, in which case $a \in A_1$, or $b < a$, in which case $a \in A_2$. That is, we have $A = A_1 \cup A_2$ (and by symmetry also $B = B_1 \cup B_2$). Now there are two cases: either $b \in B_1$ or $b \in B_2$. Assume that $b \in B_1$ (the case that $b \in B_2$ is similar, and is left to the reader) and consider two arbitrary vertices $a' \in A_1$ and $a'' \in A_2$. Again using that P doesn't have property (ii) we know that there exits a vertex $b' \in B$ such that b' is a common neighbour of a_1, a', and a'' in the cover graph. As a_1 and a' are unrelated and $a_1 < a''$, there is only one way in which this can be: $a_1 < b' < a''$ and $a' < b'$. Thus, in particular, every element in A_1 is below every element of A_2. As, similarly, every element in B_1 is below every element in B_2 the assumption that $b \in B_1$ implies also that b' has to belong to B_1 as well. Let us restate this fact: for every pair of vertices $a' \in A_1$ and $a'' \in A_2$ there exists a vertex $b' \in B_1$ such that a' is covered by b' and b' is covered by a''. – This implies that no vertex in $b'' \in B_1$ can be below any vertex in A_1 (since otherwise b'' would be below some $b' \in B_1$), that no vertex in B_2 can be below any vertex $a'' \in A_2$, (since otherwise a'' could not cover any vertex in B_1), and that every vertex in A_1 is below every vertex in B_2. Therefore A_1, B_1, A_2, B_2 form a 4-layering of P. $\square$

Having established (17), the proof of Theorem 3.5 can then be completed by showing that the classes $\mathcal{A}'(n)$, $\mathcal{B}'(n)$, $\mathcal{C}'(n)$, and $\mathcal{D}'(n)$ are negligible in size and that within $\mathcal{L}(n)$ the proportion of 2-layer and 4-layer posets is much smaller than that of 3-layer posets.

4 Constrained Evolution

Evolving graphs. In a seminal series of papers Erdős und Rényi [13, 14, 15, 16, 17, 18] investigated the evolution of random graphs, thereby marking the starting point of the theory of random graphs. For many graph properties Erdős und Rényi noticed so-called phase transition phenomena: *Until p reaches a certain threshold, $G_{n,p}$ will almost surely not have the property in question, whereas beyond this threshold suddenly it will almost surely possess it.* Here *almost surely* stands for "with probability tending to 1, as $n \to \infty$". This area has been and still is a flourishing field of research. Again we refer the reader for more details and further references to [2, 22].

Our attention will be focused on what could be called *constrained* evolution, by which we mean the evolution of a family of objects that is given by structural side–constraints. Consider for example the family of triangle-free graphs. We have already seen that almost all triangle-free graphs are bipartite. But is this also true if we focus our attention on an 'early' evolutionary stage? Let us choose our random graph uniformly, not from $\mathcal{F}orb_n(K_3)$ but from $\mathcal{F}orb_{n,m}(K_3)$, the set of all triangle–free graphs with vertex set $[n]$ that have m edges. As already pointed out before, due to the lack of independence most of the methods used to obtain results about $G_{n,p}$ are ineffective when one considers random graphs from $\mathcal{F}orb_n(H)$. Nevertheless, Prömel and Steger [36] could prove the following theorem.

Theorem 4.1 *There exist constants $c_1, c_2, c_3 > 0$ such that*

$$\text{Prob}\Big[\ G \textit{ is bipartite} \mid G \in \mathcal{F}orb_{n,m}(K_3)\Big] = 1 - o(1), \qquad \textit{for all } m \geq c_1 n^{7/4} \log n, \quad (18)$$

$$\text{Prob}\Big[\ G \textit{ is bipartite} \mid G \in \mathcal{F}orb_{n,m}(K_3)\Big] = o(1), \qquad \textit{for all } c_2 n \leq m \leq c_3 n^{3/2}. \quad (19)$$

Classical results about the evolution of $G_{n,p}$ state that $p = 1/n$ marks the threshold of any cycle in $G_{n,p}$, from which one deduces that there exists a constant c_4 such that

$$\text{Prob}\Big[\ G \text{ is bipartite} \mid G \in \mathcal{F}orb_{n,m}(K_3)\Big] = 1 - o(1), \qquad \text{for all } m \leq c_4 n.$$

Interestingly, this indicates that the evolution of the random triangle-free graph exhibits *two* phase transitions with respect to being bipartite: first it is almost surely bipartite, then it is not, and then it is once again. The question remains open in the interval $n^{3/2} \ll m \ll n^{7/4} \log n$. Here Luczak [30] has recently made substantial progress by showing that in this range with high probability a random graph is nearly bipartite (i.e. it can be made bipartite by removing very few edges).

The statement (18) is again proven by using the Kleitman–Rothschild method. Therefore we don't include the proof here and instead concentrate on (19).

Proof of (19) in Theorem 4.1. Consider a random graph $G_{n,m}$ in the $G(n,m)$-model, i.e. a graph randomly chosen from the set of all graphs on n vertices with m edges. Then, obviously

$$|\mathcal{F}orb_{n,m}(K_3)| = \mathrm{Prob}\,[G_{n,m} \text{ is triangle-free}\,] \cdot \binom{\binom{n}{2}}{m}.$$

As the number of bipartite graphs on n vertices with m edges can be estimated fairly easily (see below), all we need are good bounds for the probability that a random graph $G_{n,m}$ is triangle-free.

In 1987 Janson, Łuczak, Ruciński [21], improving a result of Bollobás [3], fully determined the order of magnitude of the logarithm of the probability that a random graph $G_{n,p}$ is H-free. The proof of their theorem is fundamentally based on the independence of appropriately defined events, a fact which is not true in the $G(n,m)$-model. The following theorem, proven in [35], shows that nevertheless a similar result holds for the $G(n,m)$-model.

Theorem 4.2 *For every graph H with chromatic number $\chi(H) \geq 3$ there exist positive constants c_1 and c_2 such that for all $n \in \mathbb{N}$ and all $m \leq ex(n,H)$*

$$e^{-c_1\, g(n,m)} \leq \mathrm{Prob}\,[G_{n,m} \textit{ is } H\textit{-free}] \leq e^{-c_2\, g(n,m)},$$

where

$$g(n,m) = \min\{n^{|V(L)|} \cdot (m\tbinom{n}{2}^{-1})^{|E(L)|} \mid L \subseteq H, |E(L)| > 0\}.$$

□

Observe that, trivially, for all $m > ex(n,H)$ we have $\mathrm{Prob}\,[G_{n,m}$ is H-free$] = 0$. Denote by $S_{n,m}$ the number of bipartite graphs on n vertices with exactly m edges. We need the following trivial bound on $S_{n,m}$. (Much better bounds can be found in [34].)

$$S_{n,m} \leq 2^n \cdot \binom{\frac{1}{4}n^2}{m}, \tag{20}$$

since there are exactly 2^n ways to choose a two-colouring and at most $\binom{\frac{1}{4}n^2}{m}$ ways to choose m edges respecting a given two-colouring.

To complete the proof of (19), now observe that by Theorem 4.2 there exists a constant $c > 0$ such that

$$|\mathcal{F}orb_{n,m}(K_3)| \geq e^{-c\, m^3 n^{-3}} \cdot \binom{\binom{n}{2}}{m} \qquad \text{for all } m \leq n^{3/2}.$$

It is easy to see that

$$\binom{\binom{n}{2}}{m} \geq e^{\frac{m}{3}} \binom{\frac{1}{4}n^2}{m}.$$

Using $c_2 n \le m \le c_3 n^{3/2}$ this shows that for an appropriate choice of c_2 and c_3

$$\frac{S_{n,m}}{|\mathcal{F}orb_{n,m}(K_3)|} \le 2^n e^{cm^3 n^{-3} - \frac{m}{3}} = o(1).$$

$\square$

Evolving posets. Similarly to our observations concerning triangle–free graphs with a given number of edges, we now view the situation of partially ordered sets from the same angle. Almost all n-point posets are 3-layer posets. But what does a typical poset on n points *with a given number of comparable pairs* look like? And: *how many such posets are there?*

Let us introduce some notation. Denote by $\mathcal{P}_{n,d}$ the family of all labeled posets with point set $[n]$ and $d \cdot n^2$ relations, and let

$$\mathrm{c}(d) := \lim_{n\to\infty} \frac{\log |\mathcal{P}_{n,d}|}{n^2},$$

provided the limit exists. Recall from (11) that for any $d \in (0, \frac{1}{2})$,

$$\mathrm{c}(d) \ \le \ \frac{1}{4}. \tag{21}$$

In order to compute $\mathrm{c}(d)$ we will construct for every $0 < d < \frac{1}{2}$ a family $\mathcal{P}'$ of posets, such that

$$\mathcal{P}'_n \subseteq \mathcal{P}_{n,d}$$

and

$$\lim_{n\to\infty} \frac{\log |\mathcal{P}'_n|}{n^2} = \lim_{n\to\infty} \frac{\log |\mathcal{P}_{n,d}|}{n^2} = \mathrm{c}(d).$$

A family $\mathcal{P}'$ satisfying the above properties will be called *d-significant* or, if the value of d is either irrelevant or obvious from the context, simply *significant.* It is important to note that the methods used to determine $\mathrm{c}(d)$ don't seem to be strong enough to give results on the global structure of $\mathcal{P}_{n,d}$.

To construct such a family $\mathcal{P}'$, we introduce the concept of a configuration. A k-layer poset is said to have *configuration* $(\lambda_1, \ldots, \lambda_k; p)$ if

$$|X_i| = \lambda_i \cdot n \qquad \text{for all } i \in [k]$$

and the number of relations between X_i and X_{i+1} is equal to

$$p \cdot |X_i| \cdot |X_{i+1}|$$

for all $i \in [k-1]$. A *configuration* is called *d-significant* if the family of all posets with this configuration is d-significant.

Strangely enough, the question of determining $\mathrm{c}(d)$ seems to have first arisen in quite a different context: In 1978, Dhar [8] suggested that partial orders can represent the states of a certain model of lattice gas with energy proportional to the number of comparable pairs in the order.

Dhar was able to show that the function $\mathrm{c}(d)/d$ is monotone nonincreasing while the function $\mathrm{c}(d)/(1-d)$ is monotone nondecreasing, which implies that $\mathrm{c}(d)$ is continuous. Moreover he proved that

$$\left(\frac{1}{2}-x, \frac{1}{2}, x;\ \frac{1}{2}\right) \text{ is } d\text{-significant for } d \in \left[\frac{1}{8}, \frac{3}{16}\right]$$
$$\text{where} \quad x := \frac{1}{4} - \sqrt{\frac{3}{16} - d},$$

and therefore

$$\text{for } \frac{1}{8} \leq d \leq \frac{3}{16} \quad : \qquad \mathrm{c}(d) \equiv \frac{1}{4}. \tag{22}$$

Dhar [8, 9] also obtained several upper bounds for $\mathrm{c}(d)$, from which he concluded that

$$\text{whenever either } d < 0.077 \text{ or } d > 0.35: \quad \mathrm{c}(d) < \frac{1}{4},$$

which implies that $\mathrm{c}(d)$ is nonanalytic. He conjectured that for each d there is a d-significant configuration. In this case, so he predicted, there would be intervals where p as a function of d would remain constant and $\mathrm{c}(d)$ would be a linear function in d. As we shall see, he was quite right.

In 1979, Kleitman and Rothschild [25] considered the evolution process in the range of $0 < d \leq \frac{1}{8}$. In this range the family of bipartite posets consisting of two disjoint antichains of cardinality $\frac{n}{2}$ and dn^2 relations between the antichains is d-significant (in other words the configuration $(\frac{1}{2}, \frac{1}{2}; 4d)$ is d-significant) and thus

$$\text{for } 0 < d \leq \frac{1}{8}: \qquad \mathrm{c}(d) = \frac{1}{4} \cdot H(4 \cdot d), \tag{23}$$

where

$$H(x) = -x \log x - (1-x)\log(1-x)$$

denotes the entropy function. Figure 2 illustrates both (22) and (23). The paper by Kleitman and Rothschild concludes with the following remark.

> Above $\frac{3}{16}n^2$ one might guess that the dominant states have three [layers] with equal top and bottom population at least until $d = \frac{2}{9}n^2$ at which point all three [layers] would be equal in size. Eventually four and higher [layer posets] would

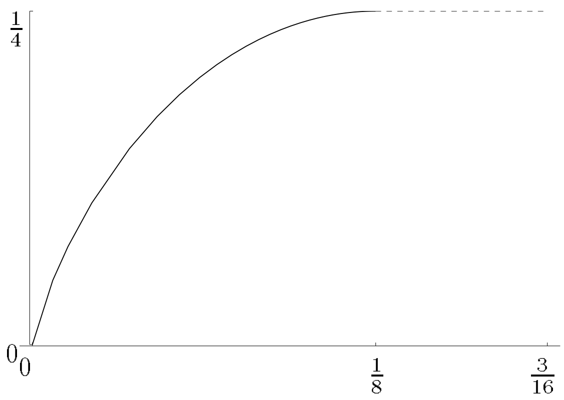

Figure 2: c(d) in the range $(0, \frac{3}{16})$

dominate. Just where and what happens there is not obvious. Presumably phase transitions occur at each such point. One not only lacks a method of proof, but even sensible conjectures in this higher range.

The following recently proven theorem by Prömel, Steger, and Taraz [37, 38] now settles this question and describes the evolution process in terms of significant families and the behaviour of c(d) in the full range $0 < d < \frac{1}{2}$.

Let us illustrate in a few words what happens after d passes the point $d_3 := \frac{3}{16} = 0.1875$. At this point our d-significant configuration is given by $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}; \frac{1}{2})$. As d increases, the density p of relations between consecutive layers increases, the layers X_1 and X_3 grow at the same rate while X_2 shrinks, and the function c(d) immediately begins to decrease.

When d reaches the point $d_3' := 0.2204..$, the pattern changes: p, having reached the value $0.5827..$, now remains constant. A new fourth layer X_4 begins to grow, X_1 and X_2 shrink while X_3 grows. Perhaps most surprisingly, in this range c(d) is a *linear* decreasing function. This phase ends at the point $d_4 := 0.2536..$, when $|X_1| = |X_4|$ and $|X_2| = |X_3|$. Interestingly, X_4 has now half the size of both X_1 and X_3 at the point d_3'.

We have tried to illustrate this in Figures 3, 4, and 5. After the point d_3' the evolution process repeats the same two–step manner an infinite number of times, as illustrated in Figure 6.

Theorem 4.3 *There is an infinite sequence of constants*

$$0 = d_2 < d_2' < d_3 < d_3' < d_4 < \cdots < \frac{1}{2}$$

such that the following holds.

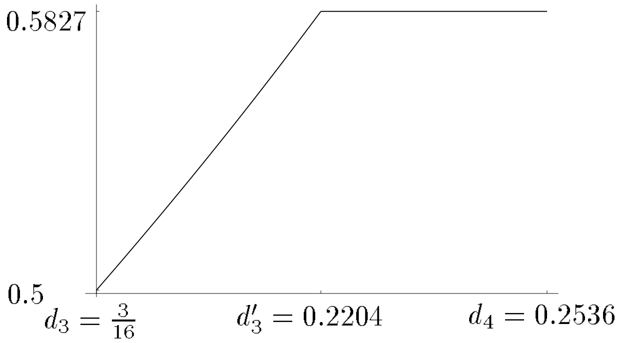

Figure 3: $p(d)$ in the range $[d_3, d_4]$.

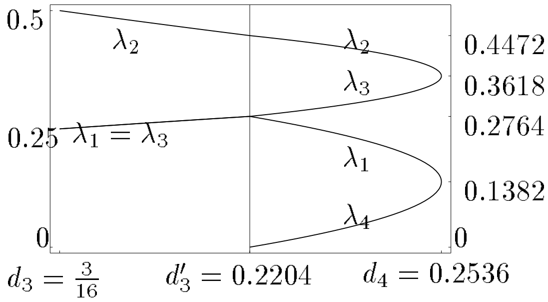

Figure 4: $\lambda_i(d)$ in the range $[d_3, d_4]$.

For every $d \in (d_k, d'_k]$ there exists a d-significant configuration $(\lambda_1, \ldots, \lambda_k; p)$. As d increases from d_k to d'_k, the density p is monotone increasing and, if $k \geq 3$, the function $\mathrm{c}(d)$ *is monotone decreasing.*

For every $d \in (d'_k, d_{k+1}]$ there exists a d-significant configuration $(\lambda_1, \ldots, \lambda_{k+1}; p)$. As d increases from d'_k to d_{k+1}, the density p remains constant, a new $(k+1)$st layer begins to grow, and the function $\mathrm{c}(d)$ *is linear in d and, if $k \geq 3$, monotone decreasing.*

As stated before, our methods don't seem to be strong enough to determine the global structure of $\mathcal{P}_{n,d}$. We conjecture that in fact *almost all* posets in $\mathcal{P}_{n,d}$ have the d-significant configuration.

In the statement of the above theorem we have been somewhat vague in that we merely stated that the constants d'_k and d_k existed. It would of course be much nicer if we could give an explicit formula – preferably not only for these constants but also for the density $p(d)$ and the layer sizes $\lambda_i(d)$ of the d-significant configuration as functions of d, and, most importantly, the function $\mathrm{c}(d)$. This is possible, but the formulae involved turn out to be not very attractive, so we don't include them here.

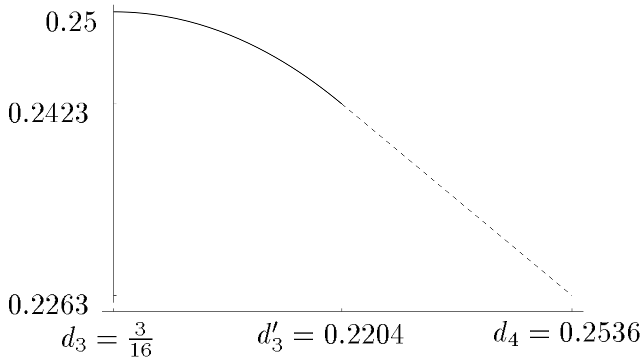

Figure 5: $c(d)$ in the range $[d_3, d_4]$.

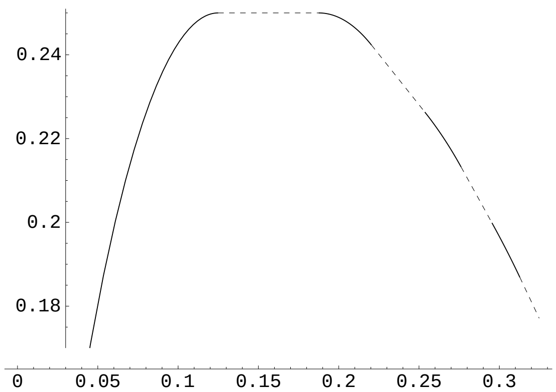

Figure 6: $c(d)$ in the range $[0.05, 0.32]$.

We conclude with a rather philosophical than mathematical discussion on two different ways of interpreting this picture. One would be to call every interval $[d'_{k-1}, d'_k]$ an evolutionary phase, since here a significant proportion of posets has k layers. The other would be to view the intervals $[d_k, d'_k]$ as the phases and the intervals $[d'_{k-1}, d_k]$ as the phase transitions. Let us give two reasons why we believe that the latter view is more appropriate. First, the d-significant configuration within the intervals $[d'_{k-1}, d_k]$ is not symmetric, it looks different when viewed from below or above. By the standards of theoretical physics, this indicates an unstable and therefore transitory state through which the system passes quickly. This leads us to the second reason: what might be held against phase transitions of the form $[d'_{k-1}, d_k]$ is that they are too 'long' – but they are only long from the point of view of d: when viewed in terms of p, they are in fact very, very short.

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