On the Spanning Trees of Weighted Graphs*†

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Abstract

Given a weighted graph, let $W_1, W_2, W_3, \ldots$ denote the increasing sequence of all possible distinct spanning tree weights. Settling a conjecture due to Kano, we prove that every spanning tree of weight W_1 is at most $k-1$ edge swaps away from some spanning tree of weight W_k. Three other conjectures posed by Kano are proven for two special classes of graphs. Finally, we consider the algorithmic complexity of generating a spanning tree of weight W_k.

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1 Introduction

The minimum spanning tree problem is a classic problem in computer science for which a large number of sequential, parallel and distributed algorithms have been devised. To the graph theorist, however, this body of work has provided little beyond the underlying proof of correctness of the original greedy algorithms due to Kruskal and Prim [Kr][Pr]. Evidently, the advances have occurred in the areas of data structures, parallel processing techniques and distributed protocols. In contrast, this paper attacks questions arising from a generalization of the minimum spanning tree concept that requires additional insight at the graph-theoretic level.

If one considers partitioning the spanning trees of a weighted graph into weight classes, a number of natural questions arise with regard to relationships between the classes. In a recent paper, Kano [Kan] posed four conjectures that were motivated by the previous work of Kawamoto, Kajitani and Shinoda [KKS]. Our main result is a proof that every minimum spanning tree is at most $k-1$ edge swaps away from some representative of the kth weight class, settling the first of Kano's conjectures. With regard to the three remaining conjectures, we offer a stronger unified conjecture and prove that it holds for two non-trivial families of graphs.

We also consider the algorithmic complexity of generating a representative of the kth weight class. For fixed k, we obtain a polynomial time algorithm. When k is part of the input, the associated decision problem (KMST) is seen to be $\mathcal{NP}$-hard using a reduction due to Johnson and Kashdan [JK] for the related kth best spanning tree problem (KBST). Lawler [La] gave a simple branch-and-bound algorithm for KBST with a running time that is pseudo-polynomial in k. The existence of such an algorithm provides a hopeful sign that a similar result might be possible for KMST. We do not achieve this goal, but by extending a determinant method due to Okada and Onodera [OO], we obtain an algorithm for which the running time is pseudo-polynomial in the edge weights as well as k.

2 Preliminaries

Let $G=(V,E)$ be a connected undirected graph with a rational weight $w(e)$ assigned to each edge $e \in E$. Let e^- and e^+ denote the endpoints of edge e. Parallel edges and self-loops are permissible. By a cycle, we always mean a simple cycle without vertex repetition. For any subset E' of E define the weight of E', denoted $w(E')$, as the sum of the weights of the edges of E', that is,

$$w(E') = \sum_{e \in E'} w(e).$$

A *spanning tree* P of G is any subset of E for which the graph (V,P) is acyclic and connected; in order to satisfy both of these properties simultaneously it is necessary that $|P| = |V|-1$. In order to discuss the partition of the spanning trees of G into weight classes, we will make use of the following notation.

$$\begin{aligned}
T(G) &= \{P \mid P \text{ is a spanning tree of } G\} \\
W(G) &= \{w(P) \mid P \in T(G)\} \\
N(G) &= |W(G)| \\
W_i(G) &= \text{the } i\text{th smallest element of } W(G), 1 \le i \le N(G) \\
T_i(G) &= \{P \in T(G) \mid w(P) = W_i(G)\} \\
\operatorname{ord}(G,P) &= i, \text{ with } P \in T_i(G) \\
\sigma(G,P) &= |T_{\operatorname{ord}(G,P)}(G)|
\end{aligned}$$

A spanning tree of weight $W_1(G)$ is a *minimum spanning tree* of G. In general, a spanning tree of weight $W_k(G)$ will be referred to as a kth *minimal spanning tree* (k-MST) of G. Note that some previous authors have preferred to formulate their results in terms of kth *maximal* spanning trees [Kan][KKS], an equivalent concept.

Let P and Q be spanning trees of G. Then $|P \setminus Q|$ is the *distance* between P and Q and will be denoted by $d(P,Q)$. For any nonnegative integer k, let $L_k(G,P)$ represent the set of all spanning trees Q of G such that $d(P,Q) \le k$. Notice that $d(P,Q) = d(Q,P)$ so that $Q \in L_k(G,P)$ if and only if $P \in L_k(G,Q)$. For every $e \notin P$, let $Cyc(P,e)$ denote the fundamental cycle of G defined by e with respect to P. Given distinct edges a, b such that $a \in Cyc(P,b)$, the spanning tree $P \setminus \{a\} \cup \{b\}$ is defined to be a single *edge swap* away from P. For any two spanning trees P and Q, we note that the length of the shortest sequence of edge swaps required to transform P into Q is precisely $d(P,Q)$.

We will have occasion to make use of several well-known facts about 1-MSTs. The validity of each of these statements follows easily from the proof of correctness of Kruskal's greedy algorithm for computing a 1-MST [Kr]. In Kruskal's algorithm, the edges are first sorted in ascending order by weight (within a set of edges of equal weight, the order is arbitrary). The algorithm then runs through the sorted sequence of edges, adding each edge to a set T (initially empty) if and only if the resulting set of edges remains acyclic. The output of the algorithm is T.

Fact 1 *A spanning tree P is a 1-MST of G if and only if it is a 1-MST in $L_1(G,P)$.*

Fact 2 *The unique heaviest edge in some cycle cannot belong to any 1-MST. A heaviest edge in some cycle cannot belong to every 1-MST.*

Fact 3 *The unique lightest edge in some cutset must belong to every 1-MST. A lightest edge in some cutset must belong to some 1-MST.*

Fact 4 *Every 1-MST contains the same distribution of edge weights.*

Fact 5 *Every 1-MST can be generated by Kruskal's algorithm.*

Given Fact 1, it is straightforward to verify that for any i-MST P of G, $L_{i-1}(G,P)$ contains a 1-MST since, for $i > 1$, there is always an edge swap strictly reducing the weight of the spanning tree. However, going in the other direction turns out to be much more difficult. Kawamoto, Kajitani and Shinoda [KKS] studied 2-MSTs and proved that for every 1-MST P there is a 2-MST Q such that $d(P, Q) = 1$. This led Kano to pose the following conjecture, for which the result of Kawamoto et al. corresponds to the special case $i = 2$.

Conjecture 1 *If P is a 1-MST of G then $L_{i-1}(G, P)$ contains an i-MST, $1 \leq i \leq N(G)$.*

Kano was able to prove that Conjecture 1 holds for $i = 3$ and 4. We will show that Conjecture 1 is, indeed, a theorem. Kano also proposed the following three conjectures, proving them for values of i less than or equal to 3, 4, and 3, respectively.

Conjecture 2 *If P is an i-MST in $L_i(G, P)$ then P is an i-MST of G.*

Conjecture 3 *If P is an i-MST of G then P is an i-MST in $L_{i-1}(G, P)$.*

Conjecture 4 *Let $\Gamma(i, j)$ denote the graph with vertex set $T_i(G)$ and an edge between each pair of i-MSTs P, Q such that $d(P, Q) \leq j$. Then $\Gamma(i, i)$ is connected.*

Finally, we propose the following additional conjecture. In Section 4, it will be proven that Conjecture 5 implies Conjectures 2, 3 and 4.

Conjecture 5 *If P is a j-MST of G then $L_{i-1}(G, P)$ contains an i-MST, $1 \leq j < i \leq N(G)$.*

3 Proof of Conjecture 1

The main result of this section is a proof of Conjecture 1.

Lemma 3.1 *Let P and Q be spanning trees of a given graph G. Then for each edge $p \in P \setminus Q$ there is an edge $q \in Q \setminus P$ such that $p \in Cyc(P, q)$ and $q \in Cyc(Q, p)$.*

Proof: Let P, Q and p be as defined above. Removing p from P partitions the vertices of G into two classes. Let $q \in Q \setminus P$ be any edge in $Cyc(Q, p)$ whose endpoints are in different classes. Such an edge must exist. The edges p and q satisfy the claim of the lemma. $\square$

Definition 3.1 *Let the graph $G = (V, E)$ be given, and let C and D denote disjoint subsets of E. Then the graph obtained from G by* contracting *the edges of C and* discarding *the edges of D will be denoted $G[C, D]$. Notice that there is a 1-1 correspondence between the edges of $G[C, D]$ and those edges of G that do not belong to $C \cup D$. We identify the pairs of edges determined by this correspondence, inheriting edge weights in the weighted case.*

On occasion it will be useful to extend this notation to refer to a set of edges as opposed to a graph. In particular, if P is some spanning tree of a graph $G = (V, E)$ and C, D denote disjoint subsets of E, then $P[C, D]$ denotes the set of all edges in $G[C, D]$ that correspond to edges of P in G. As a further convenience, we will allow set notation (braces) to be omitted when referring to singleton sets C, D. Note that if P is a spanning tree of G and c, d are two distinct edges, then $P[c, d]$ is a spanning tree of $G[c, d]$ if and only if $c \in P$ and $d \notin P$, or $c \notin P$ and $d \in Cyc(P, c)$.

Lemma 3.2 *Let P be a spanning tree of a weighted graph $G = (V, E)$, and let $e \in E$. If $e \notin P$, let $G' = G[\emptyset, e]$ and $P' = P$. Otherwise, let $G' = G[e, \emptyset]$ and $P' = P[e, \emptyset]$. In either case, the following statements hold:*

1. *P' is a spanning tree of G'.*
2. $\mathrm{ord}(G', P') \leq \mathrm{ord}(G, P)$.
3. $\sigma(G', P') \leq \sigma(G, P)$.

Proof: The above statements follow easily once we exhibit an injective map Φ from $T(G')$ to $T(G)$ that takes P' to P and also satisfies a "constant displacement" property, namely, there exists a real value Δ such that for all spanning trees $R' \in T(G')$,

$$w(\Phi(R')) = w(R') + \Delta. \tag{1}$$

For the case $e \notin P$, take $\Phi(R')$ to be simply R' so that equation 1 holds with $\Delta = 0$. For $e \in P$, let $\Phi(R')$ be the unique spanning tree $R \in T(G)$ such that $R[e, \emptyset] = R'$ and set $\Delta = w(e)$. □

Lemma 3.3 *Let P be a 1-MST of a given weighted graph $G = (V, E)$, and let $e \in E$. If $e \in P$, let $P' = P[e, \emptyset]$. If $e \notin P$, let $P' = P[e, h]$ where h is a heaviest edge in $Cyc(P, e) \setminus \{e\}$. In either case, P' is a 1-MST of $G[e, \emptyset]$.*

Proof: If $e \in P$ then Lemma 3.2 applies. Assume that $e \notin P$. Appealing to Fact 5, consider an execution of Kruskal's algorithm on input G that generates the 1-MST P. As each edge $p \in P$ gets added to P, Kruskal's algorithm running on input $G[e, \emptyset]$ can correctly select p with one exception: The edge that first puts e^- and e^+ (the two endpoints of e) into the same component must be omitted. This edge must have weight $w(h)$ since it is a heaviest edge on the path from e^- to e^+ in P. □

Definition 3.2 *A graph $G = (V, E)$ will be called a* bispanning graph *if E is the union of two (edge) disjoint spanning trees P and Q. Such a bispanning graph will be denoted by the triple (V, P, Q).*

The following result was obtained previously by Kano using Hall's Theorem [Kan]. Here we provide an alternative proof that is explicitly constructive and introduces some of the ideas that will be used to prove Conjecture 1.

Theorem 1 *Let P be a 1-MST and Q be an arbitrary spanning tree of a given weighted graph $G = (V, E)$. Then there exists a bijection Φ from $P \setminus Q$ to $Q \setminus P$ such that for every edge $e \in P \setminus Q$, $\Phi(e) \in Cyc(Q, e)$ and $w(\Phi(e)) \geq w(e)$.*

Proof: It is sufficient to determine a bijection between the disjoint spanning trees $P' = P \setminus Q$ and $Q' = Q \setminus P$ of the bispanning graph $G' = G[P \cap Q, E \setminus P \setminus Q]$. This is due to the observation that $Cyc(Q', e) \subseteq Cyc(Q, e)$ for all $e \in P'$.

Let p be a heaviest edge in P' and let q be any edge in Q' for which $p \in Cyc(P', q)$ and $q \in Cyc(Q', p)$. Lemma 3.1 guarantees that we can find such a q. As P' is a 1-MST, we must have $w(q) \geq w(p)$. Let $\Phi(p) = q$.

To determine the next component of the bijection, repeat this procedure on the bispanning graph $G'[q, p]$ with disjoint spanning trees $P'[q, p]$ and $Q'[q, p]$. $P'[q, p]$ is a 1-MST of $G'[q, p]$ by Lemma 3.3. Furthermore, for all $e \in P'[q, p]$,

$$Cyc(Q'[q, p], e) = Cyc(Q', e) \setminus \{q\} \subseteq Cyc(Q', e) \subseteq Cyc(Q, e)$$

so that subsequent assignments to Φ will be guaranteed to satisfy the condition $\Phi(\cdot) \in Cyc(Q, \cdot)$. $\square$

Lemma 3.4 *Conjecture 1 holds if and only if there is no weighted bispanning graph $B = (V, P, Q)$ such that $d(P, Q) \geq \mathrm{ord}(B, Q) > \mathrm{ord}(B, P) = 1$ and $\sigma(B, Q) = 1$.*

Proof: The "only if" directon is easy. To establish the "if" direction, we will prove the contrapositive. Given a counterexample (G, P, i) to Conjecture 1, let Q be a closest i-MST to P. Then repeated application of Lemma 3.2 proves that the bispanning graph $G' = G[P \cap Q, E \setminus P \setminus Q]$ with disjoint spanning trees $P' = P \setminus Q$ and $Q' = Q \setminus P$ satisfies $d(P', Q') \geq \mathrm{ord}(G', Q') > \mathrm{ord}(G', P') = 1$. Note that Q' must be a closest spanning tree of weight $w(Q')$ to P' since Q was chosen closest to P.

Thus, it is sufficient to prove that $\sigma(G', Q') = 1$. If not, there must be a spanning tree Q'' of G' such that $w(Q'') = w(Q')$ and $Q'' \neq Q'$. Then $d(P', Q'') \geq d(P', Q')$ by the definition of Q. On the other hand, $d(P', Q'') < d(P', Q')$ since Q'' must have at least one edge in common with P', whereas Q' has none. Thus, Q' must have unique weight in $T(G')$, as required. $\square$

Theorem 2 *There is no weighted bispanning graph $B = (V, P, Q)$ such that $d(P, Q) \geq \mathrm{ord}(B, Q) > \mathrm{ord}(B, P) = 1$ and $\sigma(B, Q) = 1$. Hence, Conjecture 1 holds by Lemma 3.4.*

Proof: Assume the theorem is false and let $B = (V, P, Q)$ be a counterexample with smallest possible $|V|$. We will establish a contradiction by exhibiting a smaller counterexample B'.

Let p be a heaviest edge in P and let q be any edge in Q for which $p \in Cyc(P, q)$ and $q \in Cyc(Q, p)$. Such an edge q exists by Lemma 3.1. Since P is a 1-MST and $\sigma(B, Q) = 1$, we must have $w(q) > w(p)$. Therefore, q is the unique heaviest edge in $Cyc(P, q)$ and does not belong to any 1-MST of B by Fact 2.

Now consider the smaller bispanning graph $B' = B[q,p]$ with disjoint spanning trees $P' = P[q,p]$ and $Q' = Q[q,p]$. Clearly, $d(P',Q') = d(P,Q) - 1$. Furthermore, $\mathrm{ord}(B',P') = 1$ and $\sigma(B',Q') = 1$ by Lemmas 3.3 and 3.2, respectively. Since Q' is unique and since P' and Q' cannot be empty, this implies $\mathrm{ord}(B',Q') > \mathrm{ord}(B',P')$. In order to show that B' is a counterexample to the theorem, it is sufficient to prove that $\mathrm{ord}(B',Q') < \mathrm{ord}(B,Q)$. Lemma 3.2 gives us only $\mathrm{ord}(B',Q') \leq \mathrm{ord}(B,Q)$, but in the present case that argument can be strengthened to yield the desired strict inequality. Namely, let Φ be the injective map taking $R' \in T(B')$ to $R \in T(B)$ such that $q \in R$ and $R' = R[q,p]$, and observe that Φ does not map any spanning tree of B' into $T_1(B)$. $\square$

4 Results on Conjectures 2 to 5

The main result of this section is that a proof of Conjecture 5 would confirm each of Kano's three remaining open conjectures. We will actually prove a somewhat stronger result, namely that if Conjecture 5 holds on all weighted graphs with at most n vertices, then Conjectures 2, 3 and 4 hold for all values of i less than or equal to $n-1$, n and $n-1$, respectively. Finally, we will prove that Conjecture 5 holds on two special classes of graphs, one of which was previously considered by Kano.

Lemma 4.1 *Conjecture 5 holds for every $i \leq i^*$ if and only if there is no weighted bispanning graph $B = (V,P,Q)$ such that $d(P,Q) \geq \mathrm{ord}(B,Q) > \mathrm{ord}(B,P)$, $\mathrm{ord}(B,Q) \leq i^*$ and $\sigma(B,Q) = 1$.*

Proof: The proof is similar to that given for Lemma 3.4. As in Lemma 3.4, the "only if" direction is completely straightforward and left to the reader. For the other direction, assume that (G,P,i) is a counterexample to Conjecture 5, and let Q be a closest i-MST to P. Then $d(P,Q) \geq i = \mathrm{ord}(G,Q) > \mathrm{ord}(G,P)$. Consider $G' = G[P \cap Q, E \setminus P \setminus Q]$, and let $P' = P \setminus Q$ and $Q' = Q \setminus P$. By Lemma 3.2, we have

$$d(P',Q') = d(P,Q) \geq i \geq \mathrm{ord}(G',Q') > \mathrm{ord}(G',P').$$

It is also clear, as in the proof of Lemma 3.4, that, if $\sigma(G',Q') > 1$, then Q was not chosen closest to P. $\square$

Lemma 4.2 *If Conjecture 2 fails for $i = i^*$, then Conjecture 5 fails for some $i \leq i^*$.*

Proof: Let (G,R,i) be a counterexample to Conjecture 2, that is, assume that R is an i-MST in $L_i(G,R)$ but not an i-MST of G. As stated in Fact 1, there is always a weight reducing edge swap for a spanning tree which is not of minimum weight. Hence, $L_{i-1}(G,R)$ contains a 1-MST of G, and there must be a least integer $j \leq i$ such that $L_i(G,R)$ does not contain a j-MST. For the same reason, there must also be a k-MST P, $k < j$, such that $d(P,R) \leq i - j + 1$. Let Q be a closest j-MST to P. Since Q is not contained in $L_i(G,R)$ we have

$$d(P,Q) \geq i + 1 - d(P,R) \geq j.$$

Now consider the bispanning graph $G' = G[P \cap Q, E \setminus P \setminus Q]$ with disjoint spanning trees $P' = P \setminus Q$ and $Q' = Q \setminus P$. Clearly, $d(P,Q) = d(P', Q')$, and by repeated application of Lemma 3.2, $\mathrm{ord}(G', Q') > \mathrm{ord}(G', P')$. Because the weight difference between P and Q is the same as that between P' and Q', we have

$$d(P', Q') \geq j \geq \mathrm{ord}(G', Q') > \mathrm{ord}(G', P').$$

Finally, the fact that Q was chosen closest to P implies that $\sigma(G', Q') = 1$. Now Lemma 4.1 can be applied to complete the proof. □

Lemma 4.3 *Conjecture 3 holds for every $i \leq i^*$ if and only if there is no weighted bispanning graph $B = (V, P, Q)$ such that $d(P,Q) \geq \mathrm{ord}(B,P) > \mathrm{ord}(B,Q)$, $\mathrm{ord}(B,P) \leq i^*$ and $\sigma(B,Q) = 1$.*

Proof: The proof is similar to that given for Lemmas 3.4 and 4.1. Once again, the "only if" direction is straightforward and left to the reader. For the other direction, assume that (G, P, i) is a counterexample to Conjecture 3, let j be any integer less than i such that $L_{i-1}(G, P)$ does not contain a j-MST, and let Q be a closest j-MST to P. Then $d(P,Q) \geq i = \mathrm{ord}(G,P) > \mathrm{ord}(G,Q)$. Consider $G' = G[P \cap Q, E \setminus P \setminus Q]$, and let $P' = P \setminus Q$ and $Q' = Q \setminus P$. By Lemma 3.2, we have

$$d(P', Q') = d(P,Q) \geq i \geq \mathrm{ord}(G', P') > \mathrm{ord}(G', Q').$$

Finally, the fact that Q was chosen closest to P implies that $\sigma(G', Q') = 1$. □

Lemma 4.4 *Conjecture 4 holds for every $i \leq i^*$ if and only if there is no weighted bispanning graph $B = (V, P, Q)$ such that $d(P,Q) > \mathrm{ord}(B,P) = \mathrm{ord}(B,Q)$, $\mathrm{ord}(B,P) \leq i^*$ and $\sigma(B,Q) = 2$.*

Proof: Once again, the proof is similar to that given for Lemma 3.4, and the "only if" direction is left as an exercise. Assume that (G, i) is a counterexample to Conjecture 4, and let P, Q be a closest pair of i-MSTs belonging to different connected components of $\Gamma(i, i)$. Then $d(P,Q) > \mathrm{ord}(G,Q) = \mathrm{ord}(G,P) = i$. Let G', P', Q' be defined as in the previous Lemma. Again, by Lemma 3.2,

$$d(P', Q') > \mathrm{ord}(G', Q') = \mathrm{ord}(G', P').$$

If $\sigma(G', Q') > 2$ then there is a spanning tree R' of G' which has the same weight as P' (and Q') but which is different from P' and Q'. This implies that P and Q cannot be a closest unconnected pair in $\Gamma(i, i)$. □

Lemma 4.5 *If there exists a weighted bispanning graph $B = (V, P, Q)$ with $\mathrm{ord}(B,P) > 1$ and $\sigma(B,Q) = 1$, then there exists a weighted bispanning graph $B' = (V', P', Q')$ with $|V'| = |V| - 1$, $\mathrm{ord}(B', P') < \mathrm{ord}(B,P)$, $\mathrm{ord}(B', Q') \leq \mathrm{ord}(B,Q)$, $\sigma(B', Q') = 1$ and $w(P) - w(P') > w(Q) - w(Q')$.*

Proof: Since $\mathrm{ord}(B,P) > 1$, Fact 1 implies that there exist edges $p \in P$ and $q \in Q$ such that $p \in Cyc(P,q)$ and $w(p) > w(q)$. Consider the bispanning graph $B' = B[q,p] = (V',P',Q')$ where $P' = P[q,p]$ and $Q' = Q[q,p]$. Note that $|V'| = |V| - 1$ and $w(P) - w(P') = w(p) > w(q) = w(Q) - w(Q')$. By Lemma 3.2, $\mathrm{ord}(B',Q') \leq \mathrm{ord}(B,Q)$, $\sigma(B',Q') = 1$ and $\mathrm{ord}(B',P') \leq \mathrm{ord}(B, P \setminus \{p\} \cup \{q\})$. Now $\mathrm{ord}(B, P \setminus \{p\} \cup \{q\}) < \mathrm{ord}(B,P)$ since $w(p) > w(q)$. Hence, $\mathrm{ord}(B',P') < \mathrm{ord}(B,P)$, which completes the proof. □

Lemma 4.6 *If Conjecture 5 fails for some i, then Conjecture 5 fails on some weighted bispanning graph with at most $i+1$ vertices.*

Proof: If Conjecture 5 fails for some i, then Lemma 4.1 implies that there exists a weighted bispanning graph $B = (V,P,Q)$ with $d(P,Q) \geq \mathrm{ord}(B,Q) > \mathrm{ord}(B,P)$, $\mathrm{ord}(B,Q) \leq i$ and $\sigma(B,Q) = 1$. If $d(P,Q) \leq i$, there is nothing left to prove, since the number of vertices in B is equal to $d(P,Q)+1$. Now suppose that $d(P,Q) > i$, so that $d(P,Q)$ is strictly greater than $\mathrm{ord}(B,Q)$. If $\mathrm{ord}(B,P) = 1$, we have a contradiction to Theorem 2. Hence, we may assume that $\mathrm{ord}(B,P) > 1$, and Lemma 4.5 can be applied to obtain a weighted bispanning graph $B' = (V',P',Q')$ with $d(P',Q') = d(P,Q) - 1$, $d(P',Q') \geq \mathrm{ord}(B',Q') > \mathrm{ord}(B',P')$, $\mathrm{ord}(B',Q') \leq i$ and $\sigma(B',Q') = 1$. The preceding argument can now be repeated, with B' playing the role of B. This process is guaranteed to terminate, either by arriving at a contradiction to Theorem 2, or by obtaining a counterexample to Conjecture 5 of the desired form. □

Lemma 4.7 *If Conjecture 3 fails for some i, then Conjecture 5 fails on a weighted bispanning graph with at most i vertices.*

Proof: If Conjecture 3 fails for $i = i^*$, then Lemma 4.3 implies that there exists a weighted bispanning graph $B = (V,P,Q)$ with $d(P,Q) \geq \mathrm{ord}(B,P) > \mathrm{ord}(B,Q)$, $\sigma(B,Q) = 1$ and $\mathrm{ord}(B,P) \leq i^*$. This is impossible if $d(P,Q) = 1$, so $d(P,Q) > 1$. By applying Lemma 4.5, we can obtain a weighted bispanning graph $B' = (V',P',Q')$ with $|V'| = |V| - 1$, $\mathrm{ord}(B',P') < \mathrm{ord}(B,P)$ and $\mathrm{ord}(B',Q') \leq \mathrm{ord}(B,Q)$. Also, $\sigma(B',Q') = 1$, which implies that $\mathrm{ord}(B',P') \neq \mathrm{ord}(B',Q')$.

If $\mathrm{ord}(B',Q') > \mathrm{ord}(B',P')$ then Conjecture 5 fails on B' with $i = i^* - 1$ since $d(P',Q') \geq \mathrm{ord}(B',Q')$. Otherwise, $d(P',Q') \geq \mathrm{ord}(B',P') > \mathrm{ord}(B',Q')$ and we have obtained another counterexample to Conjecture 3, this time for some i strictly less than i^*. Hence, we can repeat the preceding argument with B' playing the role of B. This process is guaranteed to terminate, either by reaching a contradiction with $d(P,Q) = 1$, or by obtaining a counterexample to Conjecture 5 of the desired form. □

Lemma 4.8 *If Conjecture 4 fails for some i, then Conjecture 5 fails on a weighted bispanning graph with at most $i+1$ vertices.*

Proof: If Conjecture 4 fails for $i = i^*$, then Lemma 4.4 implies that there exists a weighted bispanning graph $B = (V,P,Q)$ with $d(P,Q) > \mathrm{ord}(B,P) = \mathrm{ord}(B,Q)$, $\sigma(B,Q) = 2$ and $\mathrm{ord}(B,P) \leq i^*$. This is impossible if $d(P,Q) = 1$, so $d(P,Q) > 1$. By applying Lemma 4.5, we can obtain a weighted bispanning graph $B' = (V', P', Q')$ with $|V'| = |V| - 1$, $i^* \geq \text{ord}(B, P) \geq \text{ord}(B', Q') > \text{ord}(B', P')$ and $\sigma(B', Q') = 1$. Thus, B' represents a counterexample to Conjecture 5 for some $i \leq i^*$, and Lemma 4.6 implies the claim. □

Theorem 3 *If Conjecture 5 holds on all weighted bispanning graphs with at most n vertices then Conjectures 2, 3 and 4 must hold for all values of i less than or equal to $n-1$, n and $n-1$, respectively.*

Proof: These results follow immediately from Lemmas 4.2, 4.6, 4.7 and 4.8. □

Corollary 3.1 *Conjecture 5 implies Conjectures 2, 3 and 4.*

Thus far, we have been able to prove Conjecture 5 only for certain special classes of graphs. We will now examine two such families that were obtained by restricting the graph structure, in the first instance, and the set of allowable edge weights, in the second instance.

Definition 4.1 *Let P and Q be spanning trees of a given graph G. Then P and Q are related by a* parallel swap *if and only if there exists a bijection Φ from $P \setminus Q = \{e_1, \ldots, e_t\}$ to $Q \setminus P = \{\Phi(e_1), \ldots, \Phi(e_t)\}$ such that each of the 2^t sets of $|P|$ edges containing $P \cap Q$ and exactly one edge from each pair $\{e_i, \Phi(e_i)\}, 1 \leq i \leq t$, is a spanning tree of G.*

As an aside, it is possible to prove that spanning trees P and Q are related by a parallel swap if and only if Lemma 3.1 is satisfied by a unique $q \in Q \setminus P$ for each $p \in P \setminus Q$. This means that one may readily determine whether or not two particular spanning trees are related by a parallel swap. However, we do not suggest that the class of bispanning graphs composed of trees related by parallel swaps is of any independent interest.

Theorem 4 *Restricted to bispanning graphs $B = (V, P, Q)$ for which P and Q are related by a parallel swap, Conjecture 5 holds.*

Proof: Assume there exists a counterexample $B = (V, P, Q)$. Let $\{e_i, \Phi(e_i)\}$, for $i = 1, \ldots, t$, be the pairing of edges for the parallel swap. Assume without loss of generality that

$$w(\Phi(e_i)) - w(e_i) \leq w(\Phi(e_{i+1})) - w(e_{i+1}), \text{ for } i = 1, \ldots, t-1,$$

implying that, if we execute the edge swaps $\{e_i, \Phi(e_i)\}$ in the order $i = 1, \ldots, t$, the sequence of weights of the resulting spanning trees first decreases and then increases. Also, this sequence may contain no duplicate entries since otherwise we could omit the swaps between the two equal weight spanning trees and obtain a spanning tree Q' different from Q with $w(Q') = w(Q)$, contradicting the fact that Q is unique in its weight class. Thus, the length of the swap sequence (which is equal to $d(P, Q)$) can be at most $\text{ord}(B, Q) - 1$, proving the theorem. □

It is straightforward to check that every bispanning graph $B = (V, P, Q)$ with $|V| \leq 3$ admits a parallel swap between P and Q. For $|V| = 4$, only the bispanning graph in

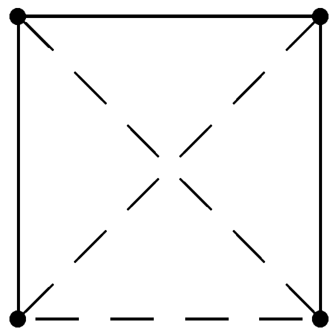

Figure 1: The smallest bispanning graph without a parallel swap.

Figure 1 does not have this property, but we can show by case analysis that Conjecture 5 holds for this particular graph as well. Using Theorem 3, these observations re-establish the results obtained by Kano with respect to the small cases of Conjectures 2 through 4.

Another interesting class of weighted graphs for which we can prove Conjecture 5 are those with edge weights drawn from an arithmetic sequence of length 3. Weighted graphs of this sort arise in the analysis of LCR networks [Kaj], and Kano was able to prove Conjecture 1 for such graphs.

Lemma 4.9 *For a fixed choice of k, let A denote an unordered k-tuple of edge weights, and let S denote a set of unordered k-tuples of edge weights. Further assume that every weighted bispanning graph $B' = (V', P', Q')$ for which $|V'| = k+1$, $\sigma(B', Q') = 1$, and the edge weights of Q' (P') are given by A has edge weights of P' (Q') given by some k-tuple in S. Then every weighted bispanning graph $B = (V, P, Q)$ for which $|V| > k+1$, $\sigma(B, Q) = 1$, and some subset of k edges of Q (P) has edge weights given by A must have a subset of k edges of P (Q) with edge weights given by some k-tuple in S.*

Proof: We will first prove the "unparenthesized" version of the lemma. Let $B = (V, P, Q)$ denote a weighted bispanning graph for which $|V| > k+1$, some subset Q' of k edges of Q has edge weights given by A, and no subset of k edges of P has edge weights given by a k-tuple in S. Now consider any weighted bispanning graph $B' = (V', P', Q')$ with $|V'| = k+1$ that can be obtained from B by contracting every edge in $Q \setminus Q'$ and discarding edges from P so that the set of remaining edges P' forms a spanning tree of the new vertex set V'. By the assumption of the lemma, $\sigma(B', Q') > 1$, that is, there exists a spanning tree Q'' of B' with $w(Q') = w(Q'')$ and $Q' \neq Q''$. But then $Q \setminus Q' \cup Q''$ is easily seen to be a spanning tree of B with the same weight as Q. Hence, $\sigma(B, Q) > 1$, resulting in a contradiction.

Now consider the parenthesized version of the lemma. Let $B = (V, P, Q)$ denote a weighted bispanning graph for which $|V| > k+1$, some subset P' of k edges of P has edge weights given by A, and no subset of k edges of Q has edge weights given by a k-tuple in S. Let $B' = (V', P', Q')$ denote any weighted bispanning graph with $|V'| = k+1$ that can be obtained from B by first discarding all of the edges in $P \setminus P'$ and then

entry	if P (Q) has weights...	then Q (P) must have weights...
1	$\{0,1,2\}$	$\{\}$
2	$\{0,0,1\}$	$\{\{0,2,2\},\{2,2,2\},\{1,1,2\}^*,\{1,2,2\}^*\}$
3	$\{0,1,1\}$	$\{\{2,2,2\},\{1,2,2\}^*\}$
4	$\{0,0,2\}$	$\{\{1,2,2\}\}$
5	$\{0,2,2\}$	$\{\{0,0,1\}\}$
6	$\{1,1,2\}$	$\{\{0,0,0\},\{0,0,1\}^*\}$
7	$\{1,2,2\}$	$\{\{0,0,0\},\{0,0,2\},\{0,0,1\}^*,\{0,1,1\}^*\}$

Table 1: Admissible combinations of edge weights.

successively contracting edges of Q as long as the resulting graph contains no cycle in P'. Now proceed exactly as in the preceding version of the lemma. □

Lemma 4.10 *Let $B = (V, P, Q)$ be a weighted bispanning graph with $|V| = 4$, $\sigma(B, Q) = 1$ and edge weights drawn from $\{0, 1, 2\}$. Then for each entry in Table 1, if P (Q) has the edge weights indicated in the first column, then the edge weights of Q (P) must be given by one of the 3-tuples listed in the second column.*

Proof: This lemma can be proven by performing an exhaustive case analysis. The rather large number of cases can be reduced to a reasonable number by a judicious pruning of the search space. However, the details of such a pruning procedure are not of particular interest, and will thus be omitted. □

In the proof of the following theorem, every conclusion drawn from a reference to Table 1 implicitly invokes Lemmas 4.9 and 4.10.

Theorem 5 *Restricted to weighted bispanning graphs with edge weights drawn from the set $\{a, a + d, a + 2d\}$, Conjecture 5 holds.*

Proof: Assume without loss of generality that $a = 0$ and $d = 1$, and let $B = (V, P, Q)$ be a counterexample with smallest possible $|V|$. Since B is a counterexample, we have $d(P, Q) \geq \mathrm{ord}(B, Q) > \mathrm{ord}(B, P)$, $\sigma(B, Q) = 1$. By the discussion following Theorem 4, we can assume that $|V| \geq 5$. Furthermore, $\mathrm{ord}(B, P) > 1$ by Theorem 2, and Q cannot be a maximum spanning tree since the maximization version of Fact 1 would imply that $d(P, Q) \leq N(B) - \mathrm{ord}(B, P) = \mathrm{ord}(B, Q) - \mathrm{ord}(B, P) < \mathrm{ord}(B, Q)$. Since Q is heavier than P, we can conclude that neither P nor Q is a 1-MST or an $N(B)$-MST.

We now argue that neither P nor Q can consist entirely of edges of the same weight. Since neither P nor Q is a 1-MST or an $N(B)$-MST, it is immediate that neither P nor Q can consist entirely of weight 0 or weight 2 edges. If Q consists entirely of weight 1 edges, then P cannot contain any edges of weight 1 or else $\sigma(B, Q) > 1$. Hence, P must consist entirely of weight 0 and weight 2 edges, and must contain at least one of each. Entries 4 and 5 of Table 1 then lead to a contradiction. Thus, Q must contain at least one edge that is not of weight 1, and if P consists entirely of weight 1 edges, Table 1 leads to a contradiction. Therefore, neither P nor Q can consist entirely of edges of the same weight.

By the first entry of Table 1, neither P nor Q can contain an edge of each of the three possible weights. Combining this observation with the fact that neither P nor Q can be a 1-MST or an $N(B)$-MST, we can now eliminate from consideration all of the 3-tuples in Table 1 that are marked with an asterisk. For example, if P (Q) contains no edges of weight 2 and contains at least two weight 1 edges (entry 3) then the second alternative in the rightmost column of entry 3 would imply that Q (P) contains only weight 1 and weight 2 edges, making it heaviest in B.

In carrying out the remainder of the proof, it will be useful to have a concise notation for indicating the composition of the edge weights of a particular spanning tree. Accordingly, we will say that $T \equiv 0^i 1^j 2^k$ if and only if the spanning tree T consists of i edges of weight 0, j edges of weight 1, and k edges of weight 2.

Let $k = |P| = |Q|$. In the following argument, the variables i and j will each be used to denote some integer value between 1 and $k-1$, inclusive. We now prove that either

Condition A: $P \equiv 0^i 1^{k-i}$ and $Q \equiv 0 2^{k-1}$, or

Condition B: $P \equiv 0^{k-1} 2$ and $Q \equiv 1^i 2^{k-i}$,

must hold. There are three cases to consider. (i) If $P \equiv 0^i 1^{k-i}$ then $Q \not\equiv 1^j 2^{k-j}$ since P is not a 1-MST, and so entries 2, 3 and 4 of Table 1 imply that Condition A applies. (ii) We cannot have $P \equiv 1^i 2^{k-i}$, with reasoning as follows. Since $w(Q) > w(P)$, entries 6 and 7 would imply that $Q \equiv 0^j 2^{k-j}$ with $1 < j < k$. Hence, $Q \equiv 0^{k-1} 2$, for otherwise entry 5 would imply that P contains an edge of each of the three possible weights, contradicting entry 1. But $Q \equiv 0^{k-1} 2$ implies that $w(Q) < w(P)$, a contradiction. (iii) We cannot have $P \equiv 0^i 2^{k-i}$ with $1 < i < k-1$ since then entries 4 and 5 would imply that Q contains an edge of each of the three possible weights, contradicting entry 1. $P \equiv 0 2^{k-1}$ is also impossible since then entry 5 would imply $w(P) > w(Q)$. Finally, if $P \equiv 0^{k-1} 2$ then entry 4 of Table 1 implies that Condition B applies. Thus, in all cases, either Condition A or Condition B must hold.

We now prove the theorem by showing that either of Conditions A and B leads to a contradiction. To this end, note that if either Condition A or Condition B holds, then no 1-MST contains an edge of weight 2. Furthermore, the weight of Q must exceed the weight of P by at least 3 since $k \geq 4$. Now let q be an edge of weight 2 in Q, let $p \in Cyc(P, q)$, and consider the weighted bispanning graph $B' = (V', P', Q') = B[q, p]$. Since edge q does not belong to any 1-MST of B, we can strengthen the argument of Lemma 3.2 (as in the proof of Theorem 2) to show that $\mathrm{ord}(B', Q') < \mathrm{ord}(B, Q)$. Lemma 3.2 also implies that $\sigma(B', Q') = 1$. Furthermore, $d(P', Q') = d(P, Q) - 1$, and $\mathrm{ord}(B', P') < \mathrm{ord}(B', Q')$ since $w(P') \leq w(P) \leq w(Q) - 3 < w(Q')$. Thus, B' is also a counterexample to Conjecture 5, contradicting the minimality of B. □

5 The generation problem

For fixed k, Theorem 2 yields a polynomial time algorithm for computing a k-MST. In particular, if $n = |V|$ and $m = |E|$ we have

Corollary 5.1 *Given a weighted graph G, a k-MST of G can be generated in $O((mn)^{k-1})$ time.*

Proof: First compute a 1-MST M. This can be done in $O(m + n\log n)$ time using Fibonacci heaps [FT], although for the present argument it would suffice to use one of the simpler 1-MST algorithms with a higher asymptotic running time. Then compute the weight of all spanning trees that are strictly fewer than k edge swaps away from M. There are at most

$$\sum_{i=0}^{k-1} \binom{n-1}{i}\binom{m-n+1}{i} = O((mn)^{k-1}) \tag{2}$$

such spanning trees and the weight of each tree can be computed incrementally in constant time by performing the enumeration in a depth-first manner. At any point in the computation we need to remember the k largest distinct weights found so far. This only increases the running time by a constant factor since k is fixed. □

In order to discuss complexity issues, we introduce the language problem associated with the problem of finding a k-MST.

Definition 5.1 *Given a graph $G = (V, E)$, positive integer weights $w(e)$ for each $e \in E$, positive integers k and B. The* KMST problem *is to determine whether or not there are k spanning trees of G with distinct weights less than or equal to B.*

By omitting the word "distinct" from Definition 5.1, we obtain the definition of the kth best spanning tree problem (KBST).

Theorem 6 *KMST is $\mathcal{NP}$-hard.*

Proof: The proof is basically identical to the one given in [JK] for KBST, which is not known to be in $\mathcal{NP}$. The reduction is from HAMILTON CIRCUIT. □

Note that the $O((mn)^{k-1})$ algorithm described earlier does not solve KMST in time pseudo-polynomial in k. The running time of such an algorithm would have to be polynomial in $|V|$, k, $\log B$ and $\log w_{max}$, where w_{max} is the weight of the heaviest edge. Interestingly, a pseudo-polynomial time solution is known for KBST; the approach is to generate all spanning trees up to the kth [La]. Unfortunately, this method is not powerful enough for KMST since there may be exponentially many spanning trees belonging to classes below the kth.

We will now present a determinant-based KMST algorithm for which the running time is pseudo-polynomial in the edge weights as well as k. Let G be a graph with $n+1$ vertices $v_0, \ldots, v_n$ and m edges $e_1, \ldots, e_m$. Construct the $m \times (n+1)$ matrix A with

$$a_{ij} = \begin{cases} +1, & \text{if } e_i = (v_j, v_k) \text{ for some } k > j; \\ -1, & \text{if } e_i = (v_j, v_k) \text{ for some } k < j; \\ 0, & \text{otherwise.} \end{cases}$$

and let A_0 be the $m \times n$ matrix A with column 0 deleted. Letting A_S denote the matrix (a_{ij}) for $i \in S$ and j ranging from 1 to n, we have

$$\det A_0^T A_0 = \sum_{\substack{S \subseteq \{1,\ldots,m\} \\ |S|=n}} [\det A_S]^2$$

by the Binet-Cauchy formula (see [Kn]). Furthermore,

$$\det A_S = \begin{cases} \pm 1, & \text{if } (V,S) \text{ is a tree;} \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

so that the determinant of $A_0^T A_0$ is the number of spanning trees of G. This result is due to Okada and Onodera [OO].

Similary, we can construct the $m \times (n+1)$ matrix B with

$$b_{ij} = \begin{cases} +x^{w(e_i)}, & \text{if } e_i = (v_j, v_k) \text{ for some } k > j; \\ -x^{w(e_i)}, & \text{if } e_i = (v_j, v_k) \text{ for some } k < j; \\ 0, & \text{otherwise,} \end{cases}$$

and let B_0 be the $m \times n$ matrix B with column 0 deleted. Let $p(x)$ be the determinant of $B_0^T A_0$, and let p_k be the coefficient of x^k in $p(x)$.

$$\begin{aligned} \det B_0^T A_0 &= \det \left[(x^{w(e_1)} \cdots x^{w(e_m)}) A_0^T A_0 \right] \\ &= \sum_{\substack{S \subseteq \{1,\ldots,m\} \\ |S|=n}} [\det A_S]^2 x^{w(S)} \\ &= \sum p_k x^k \end{aligned}$$

Using equation 3 we find that p_k is the number of spanning trees of G with weight k, that is, $p(x)$ is the generating function for the number of spanning trees of G by weight.

This relationship leads us to a pair of algorithms for solving KMST. The first idea is to compute $\det B_0^T A_0$ for all integer values of x from 0 up to W_{max}, where $W_{max} = W_{N(G)}(G)$, and then interpolate to obtain $p(x)$. The interpolation is the most costly step, and it involves solving a system of $W_{max} + 1$ equations with $O(W_{max}^2)$ digit integer coefficients. This can be done in time polynomial in W_{max} by the result of Edmonds [Ed], who proved that Gaussian elimination with pivoting is in $\mathcal{P}$ for exact rational arithmetic.

A second approach is to obtain an upper bound for p_k and then compute $\det B_0^T A_0$ for a single value of x that is large (or small) enough to ensure that all of the coefficients p_k can be extracted from the final result. Letting $C = B_0^T A_0$ with $x = 1$, we can derive a suitable bound on p_k as follows.

$$\begin{aligned} p_k &\leq \det C \\ &\leq \prod_{1 \leq i \leq n} \left(\sum_{1 \leq j \leq n} c_{ij}^2 \right)^{1/2} \end{aligned}$$

The first inequality follows from the fact that $p_k \geq 0$ for all k; the second is Hadamard's inequality.

Either of these methods gives a KMST algorithm with running time pseudo-polynomial in the edge weights as well as k. At the expense of an extra polynomial factor, we can easily convert them into algorithms for generating a k-MST.

6 Open problems

We have succeeded in proving one of Kano's four conjectures. The remaining three would be confirmed by a proof of Conjecture 5. The problem of determining the existence of a pseudo-polynomial (in k alone) algorithm for generating a k-MST remains open.

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