On Polynomial Ideals, Their Complexity, and Applications

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Question: Does

$$\begin{aligned} -x^3 + y &= 0\,, \\ x^2 y - z &= 0 \end{aligned}$$

imply that

$$xy^3 - z^2 + y^5 - z^3 = 0 \;\; ?$$

3 different (?) Formalisms

1. finitely presented commutative semigroups
2. polynomial ideals
3. Petri nets, vector addition systems

1. Finitely pres. comm. semigroups

Given:

1. finite set Σ of symbols, commutative
2. finite set Γ of congruences:

$$l_i \equiv r_i; \;\; i = 1, \ldots, w; \; l_i, r_i \in \Sigma^*$$

3. two words $u, v \in \Sigma^*$.

Problem:

$$\text{Is } u \equiv v \; (\Gamma) \; ?$$

Example:

$$\begin{aligned} a^2 &\equiv b \\ b^2 c &\equiv c \end{aligned}$$

Then we have:

$$\begin{aligned} a^4 c &\equiv c \\ a^4 &\not\equiv \lambda \end{aligned}$$

2. Polynomial ideals

Given:

A finite set of polynomials

$$p_1, \ldots, p_w \in \mathbb{Q}[x_1, \ldots, x_n]$$

and a test polynomial p.

The ideal

$$\langle p_1, \ldots, p_w \rangle$$

generated by the p_i is the set of all polynomials q which can be written

$$q = \sum_{i=1}^{w} g_i p_i$$

with polynomials $g_i \in \mathbb{Q}[x_1, \ldots, x_n]$.

The Polynomial Ideal Membership Problem — PIMP

Let polynomials $p, p_1, \ldots, p_w \in \mathbb{Q}[x_1, \ldots, x_n]$ be given.

Decision problem:

Is $p \in \langle p_1, \ldots, p_w \rangle$?

Representation problem:

Determine $g_i \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $p = \sum_{i=1}^{w} g_i p_i$.

A small example

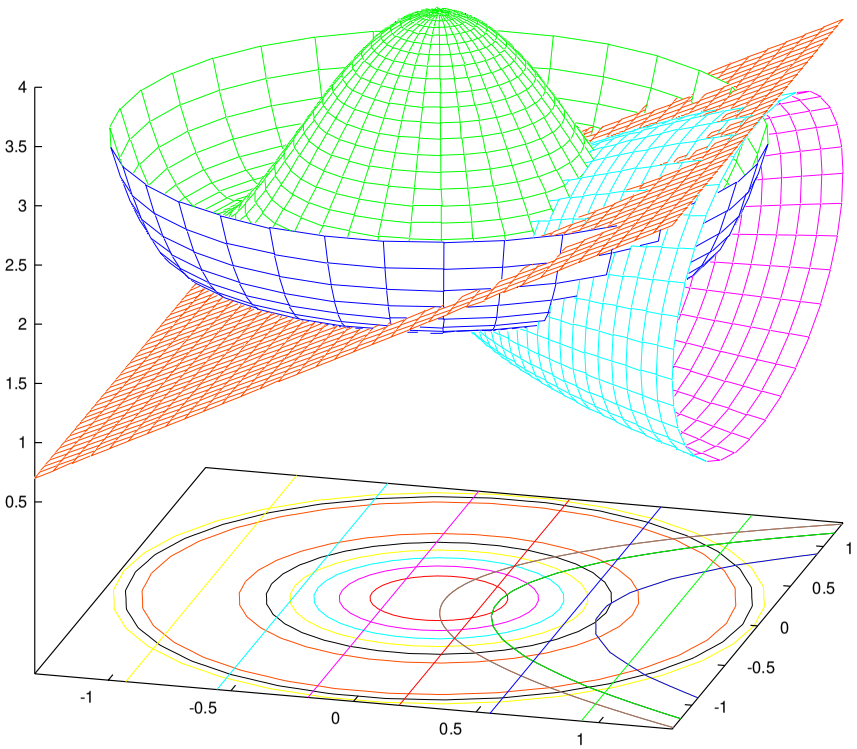
Consider the ideal generated in $\mathbb{R}^3$ by the polynomials

$$p_1: \quad z^2 - 8z - 13/10x + y^2 + 16,$$

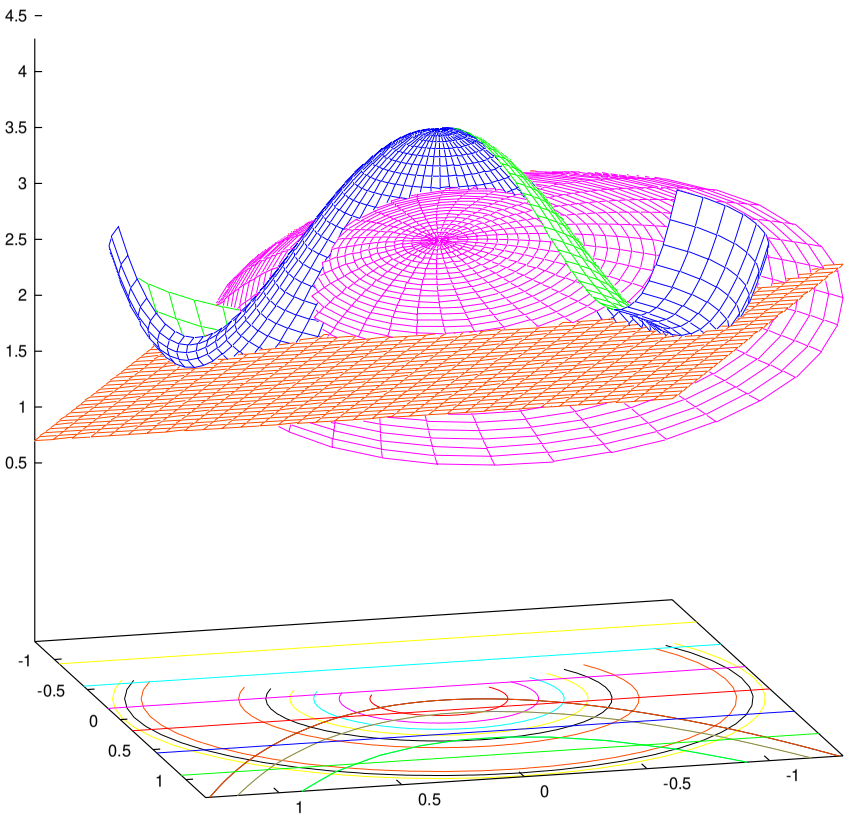
$$p_2: \quad z - 2x^4 - 4y^2x^2 + 4x^2 - 2y^4 + 4y^2 - 5, \text{ and}$$

$$p_3: \quad z - x - 3\,.$$

Graph of functions in example



. . . and another view

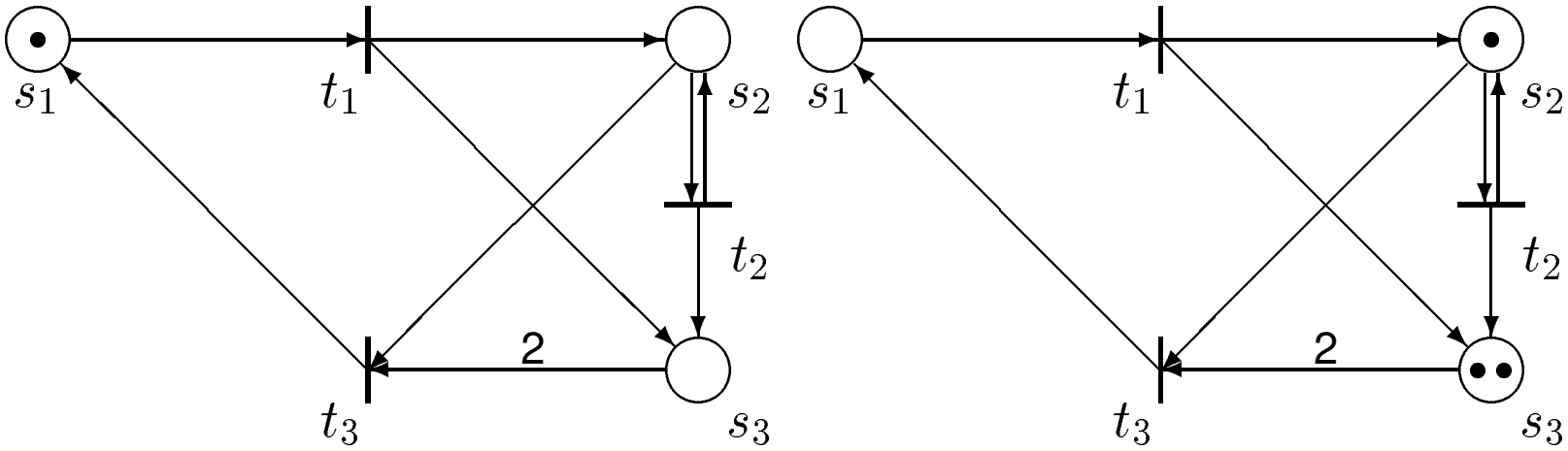


Hilbert's Nullstellensatz

Theorem: *Let k be an algebraically closed field, and let $p, p_1, \ldots, p_w \in k[x_1, \ldots, x_n]$. If p vanishes on all the common zeroes of the p_i, then there is an $e \in \mathbb{N}$ such that*

$$p^e \in \langle p_1, \ldots, p_w \rangle.$$

3. Petri nets and VAS



marking: number of tokens on places

firing of transition: change of marking

reachability set: set of reachable markings

Reversible PNs correspond to finitely presented commutative semigroups:

Symbols: s_1, s_2, s_3

congruences:			polynomials:		
s_1	$\equiv$	$s_2 s_3$	p_1	$=$	$s_2 s_3 - s_1$
s_2	$\equiv$	$s_2 s_3$	p_2	$=$	$s_2 s_3 - s_2$
$s_2 s_3^2$	$\equiv$	s_1	p_3	$=$	$s_1 - s_2 s_3^2$

Standard (Gröbner) Bases

Admissible term ordering:

(i) $x_{\pi(1)} \succ x_{\pi(2)} \succ \ldots \succ x_{\pi(n)} \succ 1$

(ii) Let m, m_1, m_2 be monomials with $m_1 \prec m_2$. Then

$$m m_1 \prec m m_2 \ .$$

Examples:

1. lex: $x_1^2 \succ x_1 x_2^3 x_3^5$
2. grevlex: $x_2^3 \succ x_1$ und $x_1 x_2 x_3 \succ x_1 x_3^2$

Definition: Let I be an ideal in $\mathbb{Q}[x] = \mathbb{Q}[x_1, \ldots, x_n]$ and $\prec$ an admissible term ordering. A set $\{g_1, \ldots, g_r\}$ of polynomials in $\mathbb{Q}[x]$ is a Standard or Gröbner basis for I (wrt. $\prec$) if the following hold:

1. $\{g_1, \ldots, g_r\}$ is a basis of I;
2. $\{LT(g_1), \ldots, LT(g_r)\}$ is a basis of the leading term ideal of I, *i.e.,* the smallest ideal containing the leading term $LT(f)$ of every $f \in I$.

Example

Consider the basis

$$\{x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1\} ,$$

with the inverse lex term ordering.

Mathematica computes the following Gröbner basis:

After changing the basis slightly:

$$\{x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1\}$$

a different symbolic algebra system produces:

$$y^{12} - y^{10} + 3z^3y^8 - 5z^2y^8 + 2y^8 - 10z^4y^6 + 20z^2y^6 - 10y^6 - 7z^6y^4 + 30z^4y^4 - 6z^3y^4 - 30z^2y^4 + 13y^4 - 5z^8y^2 + 20z^6y^2 - 30z^4y^2 + 20z^2y^2 - 5y^2 - z^{10} + z^9 + 5z^8 - 13z^6 + 10z^4 + 3z^3 - 5z^2$$

$$z^{11}x + 4z^{10}x + z^9x - 10z^8x - 4z^7x + 8z^6x - 3z^3y^{10} - 2z^2y^{10} + 4zy^{10} + 4y^{10} + z^5y^8 - 2z^4y^8 - 2z^3y^8 - 6z^6y^6 + 10z^5y^6 + 20z^4y^6 - 16z^3y^6 - 24z^2y^6 + 8zy^6 + 8y^6 + 2z^8y^4 + 21z^7y^4 + 17z^6y^4 - 78z^5y^4 - 64z^4y^4 + 90z^3y^4 + 76z^2y^4 - 32zy^4 - 32y^4 + 17z^9y^2 + 24z^8y^2 - 68z^7y^2 - 80z^6y^2 + 106z^5y^2 + 108z^4y^2 - 77z^3y^2 - 70z^2y^2 + 20zy^2 + 20y^2 + 7z^{11} + 7z^{10} - 31z^9 - 42z^8 + 55z^7 + 77z^6 - 39z^5 - 62z^4 + 8z^3 + 20z^2$$

$$zy^2x - y^2x - 1/8z^{10}x - 7/24z^9x + 1/2z^8x + 7/8z^7x - 23/24z^6x + 1/2z^4x - 1/2z^3x - zx + x + 3/8z^2y^{10} - 3/8zy^{10} - 7/24y^{10} - 1/8z^4y^8 + 11/24z^3y^8 - 3/8z^2y^8 + 1/4zy^8 - 1/4y^8 + 3/4z^5y^6 - 5/2z^4y^6 + 5/6z^3y^6 + 31/12z^2y^6 - zy^6 - 1/3y^6 - 1/4z^7y^4 - 53/24z^6y^4 + 11/6z^5y^4 + 113/12z^4y^4 - 55/8z^3y^4 - 29/4z^2y^4 + 15/4zy^4 + 19/12y^4 - 17/8z^8y^2 + 13/24z^7y^2 + 239/24z^6y^2 - 59/12z^5y^2 - 151/12z^4y^2 + 85/12z^3y^2 + 43/8z^2y^2 - 29/8zy^2 + 7/24y^2 - 7/8z^{10} + 7/12z^9 +$$

$$31/8z^8 - 11/12z^7 - 103/12z^6 + 17/6z^5 + 127/24z^4 - 3/2z^3 - 17/24z^2 + z - 1$$

$$y^4x + z^3x - x + y^4 + 2z^2y^2 - 2y^2 + z^4 - 2z^2 + 1$$

$$z^4x^2 + z^3x^2 - 2z^2x^2 - 1/6z^{10}x - 2/3z^9x - 1/4z^8x + 7/6z^7x - 1/12z^6x - z^5x + z^4x + 1/2z^2y^{10} + 1/3zy^{10} - 5/12y^{10} - 1/6z^4y^8 + 1/3z^3y^8 + 1/4z^2y^8 + 1/2y^8 + z^5y^6 - 5/3z^4y^6 - 17/6z^3y^6 + 17/6z^2y^6 + 2/3zy^6 - 1/3y^6 - 1/3z^7y^4 - 7/2z^6y^4 - 3z^5y^4 + 131/12z^4y^4 + 23/4z^3y^4 - 13z^2y^4 - 8/3zy^4 + 29/6y^4 - 17/6z^8y^2 - 4z^7y^2 + 119/12z^6y^2 + 17/2z^5y^2 - 16z^4y^2 - 14/3z^3y^2 + 12z^2y^2 + 5/3zy^2 - 55/12y^2 - 7/6z^{10} - 7/6z^9 + 55/12z^8 + 21/4z^7 - 31/4z^6 - 15/2z^5 + 83/12z^4 + 41/12z^3 - 31/12z^2$$

$$y^2x^2 + z^2x^2 - x^2 - y^4 - z^3 + 1$$

$$x^3 + y^2 + z^2 - 1$$

$$y^{15}+5z^2y^{12}-6y^{12}+10z^4y^9-20z^2y^9+10y^9-3z^3y^8+3y^8+10z^6y^6-30z^4y^6+30z^2y^6-10y^6-3z^6y^4+6z^3y^4-3y^4+5z^8y^3-20z^6y^3+30z^4y^3-20z^2y^3+5y^3+z^{10}-z^9-5z^8+13z^6-10z^4-3z^3+5z^2$$

$$z^{21}x+9z^{20}x+42z^{19}x+117z^{18}x+195z^{17}x+123z^{16}x-232z^{15}x-678z^{14}x-660z^{13}x+45z^{12}x+783z^{11}x+744z^{10}x+115z^9x-300z^8x-240z^7x-64z^6x+z^{12}y^{14}-6z^{11}y^{14}-48z^{10}y^{14}-136z^9y^{14}-219z^8y^{14}-210z^7y^{14}-57z^6y^{14}+198z^5y^{14}+408z^4y^{14}+422z^3y^{14}+264z^2y^{14}+96zy^{14}+16y^{14}-10z^{12}y^{13}-54z^{11}y^{13}-144z^{10}y^{13}-256z^9y^{13}-342z^8y^{13}-336z^7y^{13}-168z^6y^{13}+132z^5y^{13}+384z^4y^{13}+418z^3y^{13}+264z^2y^{13}+96zy^{13}+16y^{13}-6z^{13}y^{12}-26z^{12}y^{12}-54z^{11}y^{12}-84z^{10}y^{12}-150z^9y^{12}-276z^8y^{12}-366z^7y^{12}-254z^6y^{12}+66z^5y^{12}+360z^4y^{12}+414z^3y^{12}+264z^2y^{12}+96zy^{12}+16y^{12}+2z^{14}y^{11}-36z^{13}y^{11}-235z^{12}y^{11}-590z^{11}y^{11}-750z^{10}y^{11}-308z^9y^{11}+711z^8y^{11}+1770z^7y^{11}+2019z^6y^{11}+922z^5y^{11}-792z^4y^{11}-1642z^3y^{11}-1240z^2y^{11}-480zy^{11}-80y^{11}-z^{15}y^{10}-44z^{14}y^{10}-222z^{13}y^{10}-523z^{12}y^{10}-743z^{11}y^{10}-684z^{10}y^{10}-223z^9y^{10}+795z^8y^{10}+2058z^7y^{10}+2449z^6y^{10}+1232z^5y^{10}-672z^4y^{10}-1622z^3y^{10}-1240z^2y^{10}-480zy^{10}-80y^{10}-20z^{15}y^9-76z^{14}y^9-90z^{13}y^9-18z^{12}y^9-138z^{11}y^9-684z^{10}y^9-1018z^9y^9-88z^8y^9+1806z^7y^9+2738z^6y^9+1542z^5y^9-552z^4y^9-1602z^3y^9-1240z^2y^9-480zy^9-80y^9+z^{16}y^8-64z^{15}y^8-373z^{14}y^8-856z^{13}y^8-877z^{12}y^8+88z^{11}y^8+1680z^{10}y^8+2968z^9y^8+2781z^8y^8+314z^7y^8-2900z^6y^8-3648z^5y^8-1472z^4y^8+718z^3y^8+1080z^2y^8+480zy^8+80y^8-2z^{17}y^7-64z^{16}y^7-308z^{15}y^7-630z^{14}y^7-622z^{13}y^7-156z^{12}y^7+558z^{11}y^7+1686z^{10}y^7+2916z^9y^7+2318z^8y^7-1112z^7y^7-4496z^6y^7-4116z^5y^7-656z^4y^7+1916z^3y^7+1872z^2y^7+768zy^7+128y^7+z^{18}y^6-16z^{17}y^6-38z^{16}y^6+132z^{15}y^6+541z^{14}y^6+414z^{13}y^6-902z^{12}y^6-1874z^{11}y^6-26z^{10}y^6+3544z^9y^6+4061z^8y^6-270z^7y^6-4787z^6y^6-4584z^5y^6-848z^4y^6+1884z^3y^6+1872z^2y^6+768zy^6+128y^6-44z^{17}y^5-232z^{16}y^5-444z^{15}y^5-176z^{14}y^5+748z^{13}y^5+1712z^{12}y^5+2096z^{11}y^5+1356z^{10}y^5-1244z^9y^5-4336z^8y^5-4180z^7y^5-324z^6y^5+2748z^5y^5+2320z^4y^5+512z^3y^5-288z^2y^5-192zy^5-32y^5-z^{19}y^4-36z^{18}y^4-158z^{17}y^4-265z^{16}y^4-160z^{15}y^4-32z^{14}y^4-25z^{13}y^4+770z^{12}y^4+2683z^{11}y^4+3037z^{10}y^4-565z^9y^4-5025z^8y^4-4922z^7y^4-497z^6y^4+2820z^5y^4+2368z^4y^4+520z^3y^4-288z^2y^4-192zy^4-32y^4+2z^{20}y^3+4z^{19}y^3+45z^{18}y^3+286z^{17}y^3+642z^{16}y^3+202z^{15}y^3-1587z^{14}y^3-2686z^{13}y^3-240z^{12}y^3+4030z^{11}y^3+4238z^{10}y^3-1108z^9y^3-5533z^8y^3-3486z^7y^3+1823z^6y^3+3882z^5y^3+1768z^4y^3-642z^3y^3-1080z^2y^3-480zy^3-80y^3-10z^{19}y^2-44z^{18}y^2-40z^{17}y^2+148z^{16}y^2+460z^{15}y^2+544z^{14}y^2+254z^{13}y^2-318z^{12}y^2-1258z^{11}y^2-2018z^{10}y^2-1108z^9y^2+1306z^8y^2+2552z^7y^2+1334z^6y^2-418z^5y^2-872z^4y^2-432z^3y^2-$$

$$80z^2y^2 - 6z^{20}y - 20z^{19}y + 2z^{18}y + 94z^{17}y + 106z^{16}y - 42z^{15}y + 112z^{14}y + 784z^{13}y + 746z^{12}y - 1040z^{11}y - 2768z^{10}y - 1716z^9y + 1318z^8y + 2770z^7y + 1442z^6y - 398z^5y - 872z^4y - 432z^3y - 80z^2y + z^{22} + 6z^{21} + 37z^{20} + 132z^{19} + 191z^{18} - 156z^{17} - 927z^{16} - 972z^{15} + 822z^{14} + 2938z^{13} + 2097z^{12} - 1862z^{11} - 4500z^{10} - 2504z^9 + 1617z^8 + 3228z^7 + 1614z^6 - 378z^5 - 872z^4 - 432z^3 - 80z^2$$

$$z^2yx - 2zyx + yx - 44425/69984z^{20}x - 163535/34992z^{19}x - 639107/34992z^{18}x - 2698855/69984z^{17}x - 1286021/34992z^{16}x + 2619449/69984z^{15}x + 443767/2916z^{14}x + 1277093/7776z^{13}x - 436135/11664z^{12}x - 2790745/11664z^{11}x - 125515/729z^{10}x + 1600469/23328z^9x + 4427383/34992z^8x + 946153/34992z^7x - 1980847/69984z^6x + 1/4z^5x - 1/2z^4x + 1/4z^3x - z^2x + 2zx - x - 44425/69984z^{11}y^{14} + 339305/69984z^{10}y^{14} + 1628711/69984z^9y^{14} + 111115/2592z^8y^{14} + 457223/11664z^7y^{14} + 72931/11664z^6y^{14} - 59791/1296z^5y^{14} - 168541/1944z^4y^{14} - 1797889/23328z^3y^{14} - 196225/8748z^2y^{14} + 158203/17496zy^{14} + 250613/34992y^{14} + 222125/34992z^{11}y^{13} + 208925/8748z^{10}y^{13} + 785005/17496z^9y^{13} + 1011689/17496z^8y^{13} + 1980241/34992z^7y^{13} + 891121/34992z^6y^{13} - 354533/8748z^5y^{13} - 3162047/34992z^4y^{13} - 1351039/17496z^3y^{13} - 830827/34992z^2y^{13} + 171325/17496zy^{13} + 237491/34992y^{13} + 44425/11664z^{12}y^{12} + 89815/8748z^{11}y^{12} + 455137/34992z^{10}y^{12} + 631759/34992z^9y^{12} + 389551/8748z^8y^{12} + 158494/2187z^7y^{12} + 860033/17496z^6y^{12} - 68578/2187z^5y^{12} - 3310039/34992z^4y^{12} - 2749969/34992z^3y^{12} - 101393/4374z^2y^{12} + 158203/17496zy^{12} + 250613/34992y^{12} - 44425/34992z^{13}y^{11} + 872405/34992z^{12}y^{11} + 7686377/69984z^{11}y^{11} + 11626231/69984z^{10}y^{11} + 4404103/69984z^9y^{11} - 420709/2592z^8y^{11} - 1417253/3888z^7y^{11} - 4737577/11664z^6y^{11} - 32731/243z^5y^{11} + 5470091/17496z^4y^{11} + 29810057/69984z^3y^{11} + 287521/1944z^2y^{11} - 795389/17496zy^{11} - 1248691/34992y^{11} + 44425/69984z^{14}y^{10} + 209105/7776z^{13}y^{10} + 2242763/23328z^{12}y^{10} + 4995035/34992z^{11}y^{10} + 2600449/23328z^{10}y^{10} + 717805/23328z^9y^{10} - 9140507/69984z^8y^{10} - 14566699/34992z^7y^{10} - 36482669/69984z^6y^{10} - 215965/1296z^5y^{10} + 11468003/34992z^4y^{10} + 29956543/69984z^3y^{10} + 598613/3888z^2y^{10} - 869747/17496zy^{10} - 1174333/34992y^{10} + 222125/17496z^{14}y^9 + 40025/1458z^{13}y^9 - 31325/11664z^{12}y^9 - 133003/5832z^{11}y^9 + 435073/3888z^{10}y^9 + 354365/1296z^9y^9 + 336577/4374z^8y^9 - 972461/2187z^7y^9 - 1454837/2187z^6y^9 - 430759/1944z^5y^9 + 4099939/11664z^4y^9 + 560825/1296z^3y^9 + 293839/1944z^2y^9 - 777893/17496zy^9 -$$

$$1266187/34992y^9 - 44425/69984z^{15}y^8 + 323995/7776z^{14}y^8 + 5923523/34992z^{13}y^8 +$$
$$952247/4374z^{12}y^8 - 149135/7776z^{11}y^8 - 26096155/69984z^{10}y^8 - 20168129/34992z^9y^8 -$$
$$19074349/34992z^8y^8 - 1947773/69984z^7y^8 + 29182693/34992z^6y^8 + 63744949/69984z^5y^8 +$$
$$165673/17496z^4y^8 - 489187/972z^3y^8 - 7674947/34992z^2y^8 + 817259/17496zy^8 + 1226821/34992y^8 +$$
$$44425/34992z^{16}y^7 + 449615/11664z^{15}y^7 + 4580449/34992z^{14}y^7 + 2445551/17496z^{13}y^7 -$$
$$84311/11664z^{12}y^7 - 54575/432z^{11}y^7 - 915223/3888z^{10}y^7 - 3314095/5832z^9y^7 -$$
$$6944063/11664z^8y^7 + 1333129/4374z^7y^7 + 1811375/1458z^6y^7 + 30909727/34992z^5y^7 -$$
$$2537123/8748z^4y^7 - 1660150/2187z^3y^7 - 2559353/8748z^2y^7 + 696235/8748zy^7 + 939029/17496y^7 -$$
$$44425/69984z^{17}y^6 + 261185/23328z^{16}y^6 + 456911/69984z^{15}y^6 - 7506655/69984z^{14}y^6 -$$
$$6334411/34992z^{13}y^6 + 5389457/34992z^{12}y^6 + 13651993/23328z^{11}y^6 + 11243399/69984z^{10}y^6 -$$
$$34018261/34992z^9y^6 - 26889559/23328z^8y^6 + 1149559/4374z^7y^6 + 52849991/34992z^6y^6 +$$
$$11403491/11664z^5y^6 - 1946329/5832z^4y^6 - 52987825/69984z^3y^6 - 1278583/4374z^2y^6 +$$
$$613129/8748zy^6 + 1022135/17496y^6 + 488675/17496z^{16}y^5 + 592115/5832z^{15}y^5 + 80563/972z^{14}y^5 -$$
$$343673/2187z^{13}y^5 - 6493751/17496z^{12}y^5 - 6239263/17496z^{11}y^5 - 742009/2916z^{10}y^5 +$$
$$1044131/5832z^9y^5 + 2290939/2187z^8y^5 + 6131711/5832z^7y^5 - 5292641/17496z^6y^5 -$$
$$18535223/17496z^5y^5 - 1604699/4374z^4y^5 + 56383/216z^3y^5 + 2626225/17496z^2y^5 -$$
$$43378/2187zy^5 - 29413/2187y^5 + 44425/69984z^{18}y^4 + 1526545/69984z^{17}y^4 + 1489043/23328z^{16}y^4 +$$
$$661867/17496z^{15}y^4 - 1622543/34992z^{14}y^4 + 545069/34992z^{13}y^4 + 112471/8748z^{12}y^4 -$$
$$3012031/5832z^{11}y^4 - 30731213/34992z^{10}y^4 + 7525711/69984z^9y^4 + 103064435/69984z^8y^4 +$$
$$82980083/69984z^7y^4 - 28598633/69984z^6y^4 - 12665851/11664z^5y^4 - 234601/648z^4y^4 +$$
$$18247517/69984z^3y^4 + 674599/4374z^2y^4 - 54313/2187zy^4 - 47891/4374y^4 - 44425/34992z^{19}y^3 -$$
$$5365/11664z^{18}y^3 - 614141/23328z^{17}y^3 - 9605029/69984z^{16}y^3 - 10602997/69984z^{15}y^3 +$$
$$20587543/69984z^{14}y^3 + 27215087/34992z^{13}y^3 + 6637291/34992z^{12}y^3 - 82735585/69984z^{11}y^3 -$$
$$89174777/69984z^{10}y^3 + 9389951/17496z^9y^3 + 42424103/23328z^8y^3 + 24933521/34992z^7y^3 -$$
$$37215199/34992z^6y^3 - 4967245/4374z^5y^3 - 343051/17496z^4y^3 + 36291181/69984z^3y^3 +$$
$$3891055/17496z^2y^3 - 699161/17496zy^3 - 1344919/34992y^3 + 222125/34992z^{18}y^2 + 2525/144z^{17}y^2 -$$
$$376315/34992z^{16}y^2 - 43247/432z^{15}y^2 - 4391987/34992z^{14}y^2 - 718225/34992z^{13}y^2 +$$
$$2475415/34992z^{12}y^2 + 836129/4374z^{11}y^2 + 2579827/5832z^{10}y^2 + 2744041/8748z^9y^2 -$$
$$5028437/11664z^8y^2 - 6578429/8748z^7y^2 - 521381/3888z^6y^2 + 6628477/17496z^5y^2 +$$

$$7556995/34992z^4y^2 - 1000777/34992z^3y^2 - 1187455/34992z^2y^2 + 44425/11664z^{19}y +$$
$$225985/34992z^{18}y - 95083/5832z^{17}y - 1486403/34992z^{16}y + 76165/4374z^{15}y + 1918787/34992z^{14}y -$$
$$918205/5832z^{13}y - 666692/2187z^{12}y + 2133181/11664z^{11}y + 27617131/34992z^{10}y +$$
$$7490587/17496z^9y - 19862143/34992z^8y - 351121/432z^7y - 256829/2187z^6y + 30862/81z^5y +$$
$$1904011/8748z^4y - 1005151/34992z^3y - 1218073/34992z^2y + 2zy - y - 44425/69984z^{21} -$$
$$193795/69984z^{20} - 637177/34992z^{19} - 440941/8748z^{18} - 1109995/69984z^{17} + 13565093/69984z^{16} +$$
$$22302653/69984z^{15} - 7954033/69984z^{14} - 56030677/69984z^{13} - 21454273/34992z^{12} +$$
$$10510465/17496z^{11} + 30624971/23328z^{10} + 326326/729z^9 - 7179065/8748z^8 - 63266461/69984z^7 -$$
$$335959/4374z^6 + 27268337/69984z^5 + 7320799/34992z^4 - 607117/34992z^3 - 1309927/34992z^2 - 2z + 1$$

$$y^4x + z^3x - x + y^6 + 2z^2y^3 - 2y^3 + z^4 - 2z^2 + 1$$

$$z^7x^2 + 3z^6x^2 + 5z^5x^2 - 5z^3x^2 - 4z^2x^2 + 18zy^3x - 18y^3x - 27zy^2x + 27y^2x - 62197/3888z^{20}x -$$
$$477679/3888z^{19}x - 1943665/3888z^{18}x - 275920/243z^{17}x - 5079481/3888z^{16}x + 1881863/3888z^{15}x +$$
$$1277051/324z^{14}x + 585259/108z^{13}x + 1223257/1296z^{12}x - 1792241/324z^{11}x - 494006/81z^{10}x -$$
$$588079/1296z^9x + 5526469/1944z^8x + 6288353/3888z^7x - 171599/1944z^6x + 3/2z^5x + 3z^4x +$$
$$9zx - 9x - 62197/3888z^{11}y^{14} + 113819/972z^{10}y^{14} + 2422655/3888z^9y^{14} + 69253/54z^8y^{14} +$$
$$1841305/1296z^7y^{14} + 853763/1296z^6y^{14} - 24436/27z^5y^{14} - 1059965/432z^4y^{14} - 447539/162z^3y^{14} -$$
$$1500209/972z^2y^{14} - 683551/1944zy^{14} + 18625/972y^{14} + 310985/1944z^{11}y^{13} + 1268849/1944z^{10}y^{13} +$$
$$2612831/1944z^9y^{13} + 3714277/1944z^8y^{13} + 4052521/1944z^7y^{13} + 2667577/1944z^6y^{13} -$$
$$495835/972z^5y^{13} - 4658429/1944z^4y^{13} - 2664679/972z^3y^{13} - 379001/243z^2y^{13} - 680149/1944zy^{13} +$$
$$16195/972y^{13} + 62197/648z^{12}y^{12} + 562279/1944z^{11}y^{12} + 102851/243z^{10}y^{12} + 1169275/1944z^9y^{12} +$$
$$322844/243z^8y^{12} + 4410193/1944z^7y^{12} + 3911539/1944z^6y^{12} - 126505/1944z^5y^{12} -$$
$$2273447/972z^4y^{12} - 5320567/1944z^3y^{12} - 1513331/972z^2y^{12} - 694243/1944zy^{12} + 22513/972y^{12} -$$
$$62197/1944z^{13}y^{11} + 150205/243z^{12}y^{11} + 11520437/3888z^{11}y^{11} + 5006425/972z^{10}y^{11} +$$
$$12994009/3888z^9y^{11} - 610735/216z^8y^{11} - 1438795/144z^7y^{11} - 17560223/1296z^6y^{11} -$$
$$436225/54z^5y^{11} + 18625895/3888z^4y^{11} + 11704235/972z^3y^{11} + 2534063/324z^2y^{11} +$$
$$3392969/1944zy^{11} - 82919/972y^{11} + 62197/3888z^{14}y^{10} + 442429/648z^{13}y^{10} + 3421945/1296z^{12}y^{10} +$$

$$17348077/3888z^{11}y^{10} + 5615981/1296z^{10}y^{10} + 752717/324z^{9}y^{10} - 4695277/1944z^{8}y^{10} - 5435419/486z^{7}y^{10} - 65292989/3888z^{6}y^{10} - 810656/81z^{5}y^{10} + 17408485/3888z^{4}y^{10} + 23167685/1944z^{3}y^{10} + 1279303/162z^{2}y^{10} + 3393455/1944zy^{10} - 79517/972y^{10} + 310985/972z^{14}y^{9} + 85697/108z^{13}y^{9} + 42407/216z^{12}y^{9} - 103399/216z^{11}y^{9} + 216385/81z^{10}y^{9} + 4993523/648z^{9}y^{9} + 4383565/972z^{8}y^{9} - 18382733/1944z^{7}y^{9} - 38206769/1944z^{6}y^{9} - 7934489/648z^{5}y^{9} + 1361629/324z^{4}y^{9} + 7696165/648z^{3}y^{9} + 2558201/324z^{2}y^{9} + 3481421/1944zy^{9} - 119855/972y^{9} - 62197/3888z^{15}y^{8} + 677117/648z^{14}y^{8} + 4468807/972z^{13}y^{8} + 6790313/972z^{12}y^{8} + 1245179/648z^{11}y^{8} - 33102991/3888z^{10}y^{8} - 66936577/3888z^{9}y^{8} - 9515029/486z^{8}y^{8} - 14650573/1944z^{7}y^{8} + 69509327/3888z^{6}y^{8} + 13836461/486z^{5}y^{8} + 19730045/1944z^{4}y^{8} - 230074/27z^{3}y^{8} - 3896771/486z^{2}y^{8} - 3283619/1944zy^{8} + 34805/972y^{8} + 62197/1944z^{16}y^{7} + 318035/324z^{15}y^{7} + 7021567/1944z^{14}y^{7} + 9150649/1944z^{13}y^{7} + 945481/648z^{12}y^{7} - 274297/108z^{11}y^{7} - 20197/3z^{10}y^{7} - 5368237/324z^{9}y^{7} - 13276025/648z^{8}y^{7} + 461875/972z^{7}y^{7} + 19909601/648z^{6}y^{7} + 31341785/972z^{5}y^{7} + 4016549/972z^{4}y^{7} - 16177499/972z^{3}y^{7} - 6190703/486z^{2}y^{7} - 2719867/972zy^{7} + 69883/486y^{7} - 62197/3888z^{17}y^{6} + 19949/72z^{16}y^{6} + 979745/3888z^{15}y^{6} - 5079635/1944z^{14}y^{6} - 20912123/3888z^{13}y^{6} + 8030785/3888z^{12}y^{6} + 19665773/1296z^{11}y^{6} + 35006123/3888z^{10}y^{6} - 81555203/3888z^{9}y^{6} - 1916497/54z^{8}y^{6} - 21773417/3888z^{7}y^{6} + 135926551/3888z^{6}y^{6} + 1287217/36z^{5}y^{6} + 5916637/1296z^{4}y^{6} - 32085167/1944z^{3}y^{6} - 3112483/243z^{2}y^{6} - 2784991/972zy^{6} + 99529/486y^{6} + 684167/972z^{16}y^{5} + 75122/27z^{15}y^{5} + 485837/162z^{14}y^{5} - 1406837/486z^{13}y^{5} - 2468096/243z^{12}y^{5} - 12022495/972z^{11}y^{5} - 1161359/108z^{10}y^{5} + 225721/324z^{9}y^{5} + 12742687/486z^{8}y^{5} + 5653507/162z^{7}y^{5} + 4520299/972z^{6}y^{5} - 23357087/972z^{5}y^{5} - 16545389/972z^{4}y^{5} + 35243/108z^{3}y^{5} + 1603811/486z^{2}y^{5} + 150658/243zy^{5} + 9277/243y^{5} + 62197/3888z^{18}y^{4} + 1078499/1944z^{17}y^{4} + 771331/432z^{16}y^{4} + 5993371/3888z^{15}y^{4} - 147232/243z^{14}y^{4} + 124853/486z^{13}y^{4} + 1373297/3888z^{12}y^{4} - 5566013/432z^{11}y^{4} - 25434103/972z^{10}y^{4} - 12031801/1944z^{9}y^{4} + 132624275/3888z^{8}y^{4} + 19792573/486z^{7}y^{4} + 15863623/3888z^{6}y^{4} - 887429/36z^{5}y^{4} - 22074205/1296z^{4}y^{4} - 65795/1944z^{3}y^{4} + 1690805/486z^{2}y^{4} + 330233/486zy^{4} - 4817/243y^{4} - 62197/1944z^{19}y^{3} - 1175/54z^{18}y^{3} - 290107/432z^{17}y^{3} - 892627/243z^{16}y^{3} - 19475671/3888z^{15}y^{3} + 1379251/243z^{14}y^{3} + 82420621/3888z^{13}y^{3} + 45846113/3888z^{12}y^{3} - 97789063/3888z^{11}y^{3} - 154747355/3888z^{10}y^{3} - 1148833/3888z^{9}y^{3} + 9568087/216z^{8}y^{3} + 125604427/3888z^{7}y^{3} - 56776217/3888z^{6}y^{3} - 15699683/486z^{5}y^{3} - 45915757/3888z^{4}y^{3} + 8058817/972z^{3}y^{3} + 7925977/972z^{2}y^{3} + 3522245/1944zy^{3} -$$

$$149015/972y^3 + 310985/1944z^{18}y^2 + 39911/81z^{17}y^2 - 210943/1944z^{16}y^2 - 821969/324z^{15}y^2 - 3862621/972z^{14}y^2 - 3672235/1944z^{13}y^2 + 2003287/1944z^{12}y^2 + 4967135/972z^{11}y^2 + 8306009/648z^{10}y^2 + 23634265/1944z^9y^2 - 1395853/216z^8y^2 - 40032863/1944z^7y^2 - 1675367/162z^6y^2 + 5383087/972z^5y^2 + 13455157/1944z^4y^2 + 3326843/1944z^3y^2 - 33347/972z^2y^2 + 27zy^2 - 27y^2 + 62197/648z^{19}y + 46961/243z^{18}y - 37357/108z^{17}y - 2278337/1944z^{16}y + 12601/243z^{15}y + 2618267/1944z^{14}y - 2286719/648z^{13}y - 16970189/1944z^{12}y + 359333/216z^{11}y + 38981425/1944z^{10}y + 4210861/243z^9y - 15481069/1944z^8y - 4854419/216z^7y - 2558140/243z^6y + 3457595/648z^5y + 13502785/1944z^4y + 3592685/1944z^3y - 117425/972z^2y - 62197/3888z^{21} - 18193/243z^{20} - 939481/1944z^{19} - 2773717/1944z^{18} - 423455/486z^{17} + 17694533/3888z^{16} + 9179513/972z^{15} + 324893/972z^{14} - 38272073/1944z^{13} - 84649877/3888z^{12} + 14325983/1944z^{11} + 44942477/1296z^{10} + 29531183/1296z^9 - 11709841/972z^8 - 50399609/1944z^7 - 42613627/3888z^6 + 10608523/1944z^5 + 13367677/1944z^4 + 3763271/1944z^3 - 162623/972z^2 - 9z + 9$$

$$zyx^2 - yx^2 + z^6x^2 + 2z^5x^2 + 3z^4x^2 - 3z^3x^2 - 3z^2x^2 - zx^2 + x^2 + 7zy^3x - 7y^3x - 12zy^2x + 12y^2x - 62093/7776z^{20}x - 15037/243z^{19}x - 1974401/7776z^{18}x - 2274307/3888z^{17}x - 2701393/3888z^{16}x + 1471927/7776z^{15}x + 160163/81z^{14}x + 820093/288z^{13}x + 1775483/2592z^{12}x - 6997327/2592z^{11}x - 8397505/2592z^{10}x - 615715/1296z^9x + 10699009/7776z^8x + 7091971/7776z^7x + 217021/7776z^6x + z^5x + 5/4z^4x + 3/4z^3x + 5zx - 5x - 62093/7776z^{11}y^{14} + 450211/7776z^{10}y^{14} + 1224587/3888z^9y^{14} + 572525/864z^8y^{14} + 61387/81z^7y^{14} + 999865/2592z^6y^{14} - 364105/864z^5y^{14} - 40125/32z^4y^{14} - 3812519/2592z^3y^{14} - 3424889/3888z^2y^{14} - 945749/3888zy^{14} - 33719/3888y^{14} + 310465/3888z^{11}y^{13} + 644123/1944z^{10}y^{13} + 2700925/3888z^9y^{13} + 3905987/3888z^8y^{13} + 4334645/3888z^7y^{13} + 1493281/1944z^6y^{13} - 380585/1944z^5y^{13} - 293434/243z^4y^{13} - 2832059/1944z^3y^{13} - 1726417/1944z^2y^{13} - 942347/3888zy^{13} - 39065/3888y^{13} + 62093/1296z^{12}y^{12} + 287125/1944z^{11}y^{12} + 431569/1944z^{10}y^{12} + 153943/486z^9y^{12} + 2668523/3888z^8y^{12} + 4595723/3888z^7y^{12} + 530431/486z^6y^{12} + 24677/486z^5y^{12} - 281903/243z^4y^{12} - 705055/486z^3y^{12} - 3447245/3888z^2y^{12} - 955469/3888zy^{12} - 27887/3888y^{12} - 62093/3888z^{13}y^{11} + 1195327/3888z^{12}y^{11} + 11665603/7776z^{11}y^{11} + 20826185/7776z^{10}y^{11} + 7257919/3888z^9y^{11} - 1093649/864z^8y^{11} - 2191181/432z^7y^{11} - 18490621/2592z^6y^{11} - 3942919/864z^5y^{11} + 15841297/7776z^4y^{11} + 47780995/7776z^3y^{11} + 5657215/1296z^2y^{11} + 4704931/3888zy^{11} +$$

$$186577/3888y^{11}+62093/7776z^{14}y^{10}+884813/2592z^{13}y^{10}+64403/48z^{12}y^{10}+4517213/1944z^{11}y^{10}+$$
$$2017553/864z^{10}y^{10}+128209/96z^{9}y^{10}-4288277/3888z^{8}y^{10}-43983913/7776z^{7}y^{10}-$$
$$8543879/972z^{6}y^{10}-14644769/2592z^{5}y^{10}+13952111/7776z^{4}y^{10}+47175827/7776z^{3}y^{10}+$$
$$2849465/648z^{2}y^{10}+4704445/3888zy^{10}+196783/3888y^{10}+310465/1944z^{14}y^{9}+131917/324z^{13}y^{9}+$$
$$165697/1296z^{12}y^{9}-147217/648z^{11}y^{9}+426449/324z^{10}y^{9}+2550619/648z^{9}y^{9}+9873259/3888z^{8}y^{9}-$$
$$17557669/3888z^{7}y^{9}-4920281/486z^{6}y^{9}-1112293/162z^{5}y^{9}+37493/24z^{4}y^{9}+3906707/648z^{3}y^{9}+$$
$$5695771/1296z^{2}y^{9}+4786579/3888zy^{9}+124369/3888y^{9}-62093/7776z^{15}y^{8}+1350535/2592z^{14}y^{8}+$$
$$18125525/7776z^{13}y^{8}+7103509/1944z^{12}y^{8}+798427/648z^{11}y^{8}-2014181/486z^{10}y^{8}-$$
$$69123395/7776z^{9}y^{8}-10136435/972z^{8}y^{8}-17653931/3888z^{7}y^{8}+66547639/7776z^{6}y^{8}+$$
$$115182683/7776z^{5}y^{8}+24044455/3888z^{4}y^{8}-269935/72z^{3}y^{8}-4162441/972z^{2}y^{8}-4603357/3888zy^{8}-$$
$$270655/3888y^{8}+62093/3888z^{16}y^{7}+212147/432z^{15}y^{7}+3571399/1944z^{14}y^{7}+1206181/486z^{13}y^{7}+$$
$$1186199/1296z^{12}y^{7}-515045/432z^{11}y^{7}-744575/216z^{10}y^{7}-11042377/1296z^{9}y^{7}-$$
$$7025765/648z^{8}y^{7}-2247317/3888z^{7}y^{7}+2195017/144z^{6}y^{7}+66891659/3888z^{5}y^{7}+1642999/486z^{4}y^{7}-$$
$$15580525/1944z^{3}y^{7}-13532807/1944z^{2}y^{7}-3768173/1944zy^{7}-142895/1944y^{7}-62093/7776z^{17}y^{6}+$$
$$357047/2592z^{16}y^{6}+525857/3888z^{15}y^{6}-10057949/7776z^{14}y^{6}-10786067/3888z^{13}y^{6}+$$
$$6413327/7776z^{12}y^{6}+1096481/144z^{11}y^{6}+4906787/972z^{10}y^{6}-78227581/7776z^{9}y^{6}-$$
$$47738575/2592z^{8}y^{6}-16274735/3888z^{7}y^{6}+132719201/7776z^{6}y^{6}+49492879/2592z^{5}y^{6}+$$
$$1078505/288z^{4}y^{6}-61589201/7776z^{3}y^{6}-13586267/1944z^{2}y^{6}-3828923/1944zy^{6}-$$
$$89921/1944y^{6}+683023/1944z^{16}y^{5}+915737/648z^{15}y^{5}+1035847/648z^{14}y^{5}-1285447/972z^{13}y^{5}-$$
$$5011187/972z^{12}y^{5}-3184595/486z^{11}y^{5}-3796163/648z^{10}y^{5}-19495/216z^{9}y^{5}+25380091/1944z^{8}y^{5}+$$
$$11892013/648z^{7}y^{5}+7211327/1944z^{6}y^{5}-22648957/1944z^{5}y^{5}-4551529/486z^{4}y^{5}-31405/54z^{3}y^{5}+$$
$$3113069/1944z^{2}y^{5}+109258/243zy^{5}+12082/243y^{5}+62093/7776z^{18}y^{4}+2157695/7776z^{17}y^{4}+$$
$$1180093/1296z^{16}y^{4}+3247837/3888z^{15}y^{4}-115379/486z^{14}y^{4}+110635/972z^{13}y^{4}+$$
$$693821/3888z^{12}y^{4}-16622791/2592z^{11}y^{4}-105032993/7776z^{10}y^{4}-15910187/3888z^{9}y^{4}+$$
$$129546847/7776z^{8}y^{4}+83691797/3888z^{7}y^{4}+7163699/1944z^{6}y^{4}-31024403/2592z^{5}y^{4}-$$
$$24365701/2592z^{4}y^{4}-5850743/7776z^{3}y^{4}+407632/243z^{2}y^{4}+115333/243zy^{4}+6250/243y^{4}-$$
$$62093/3888z^{19}y^{3}-15511/1296z^{18}y^{3}-871439/2592z^{17}y^{3}-14439323/7776z^{16}y^{3}-$$
$$10235095/3888z^{15}y^{3}+20484593/7776z^{14}y^{3}+41801113/3888z^{13}y^{3}+51887995/7776z^{12}y^{3}-$$
$$46707793/3888z^{11}y^{3}-80508941/3888z^{10}y^{3}-13363721/7776z^{9}y^{3}+56751797/2592z^{8}y^{3}+$$

$$68978845/3888z^7y^3 - 45834331/7776z^6y^3 - 128446523/7776z^5y^3 - 55627499/7776z^4y^3 +$$
$$27747047/7776z^3y^3 + 16868221/3888z^2y^3 + 4824487/3888zy^3 + 63133/3888y^3 + 310465/3888z^{18}y^2 +$$
$$325927/1296z^{17}y^2 - 69823/1944z^{16}y^2 - 1641901/1296z^{15}y^2 - 8062141/3888z^{14}y^2 -$$
$$2135233/1944z^{13}y^2 + 1660715/3888z^{12}y^2 + 10035251/3888z^{11}y^2 + 2132911/324z^{10}y^2 +$$
$$12725305/1944z^9y^2 - 882389/324z^8y^2 - 10152877/972z^7y^2 - 7703591/1296z^6y^2 +$$
$$8885575/3888z^5y^2 + 14077583/3888z^4y^2 + 1111699/972z^3y^2 + 269197/3888z^2y^2 + 12zy^2 - 12y^2 +$$
$$62093/1296z^{19}y + 387971/3888z^{18}y - 214169/1296z^{17}y - 2316565/3888z^{16}y - 69113/3888z^{15}y +$$
$$2592223/3888z^{14}y - 2217817/1296z^{13}y - 17385025/3888z^{12}y + 645061/1296z^{11}y +$$
$$2433767/243z^{10}y + 18230531/1944z^9y - 1576189/486z^8y - 4934963/432z^7y - 23785315/3888z^6y +$$
$$1410127/648z^5y + 3527597/972z^4y + 1169533/972z^3y + 123883/3888z^2y - zy + y - 62093/7776z^{21} -$$
$$294905/7776z^{20} - 1896895/7776z^{19} - 2836079/3888z^{18} - 1896163/3888z^{17} + 4339531/1944z^{16} +$$
$$18930365/3888z^{15} + 4109437/7776z^{14} - 75897065/7776z^{13} - 89966545/7776z^{12} +$$
$$21700007/7776z^{11} + 5657717/324z^{10} + 32836639/2592z^9 - 19506331/3888z^8 - 51629797/3888z^7 -$$
$$50374091/7776z^6 + 17195287/7776z^5 + 13970663/3888z^4 + 604571/486z^3 + 47581/3888z^2 - 4z + 4$$

$$y^3x^2 + z^2x^2 - x^2 - y^4 - z^3 + 1$$

$$x^3 + y^3 + z^2 - 1$$

Another Example

Using the basis

$$\{-x^3 + y, x^2y - z, 1 - v(xy^3 - z^2 + y^5 - z^3)\} ,$$

with any term ordering,

our favorite symbolic algebra package computes the following Gröbner basis:

$$1$$

This example highlights the so-called Rabinovich trick.

Some Literature on PIMP

— Decidability: Hilbert [1890], G. Hermann [1926]

for rings: Seidenberg [1974], Richman [1974]

— doubly exponential degree bound with coefficients in $\mathbb{Q}$: Hermann [1926], Mayr/Meyer [1982]

— exp. degree bound for special p: Brownawell[1987], Heintz et al.[1988], Berenstein/Yger [1988]

— exponential upper space bound for finitely presented commutative semigroups (fpcs): Mayr/Meyer [1982]

— exponential lower space bound for fpcs's: Mayr/Meyer [1982]

— exponential upper space bound for the general word problem, with coefficients in $\mathbb{Q}$: Mayr [1988]

— doubly exponential upper degree bound for Gröbner bases: Dubé [1990]

Complexity of the word problem for fpcs

Theorem: *Let* $d = \max\{\deg(p_i)\}$. *If*

$$p \in \langle p_1, \ldots, p_w \rangle,$$

then there are $g_1, \ldots, g_w$ *such that*

$$\text{(i)} \quad p = \sum_{i=1}^{w} g_i p_i;$$

$$\text{(ii)} \quad \deg(g_i) \leq \deg(p) + (wd)^{2^n}.$$

For a proof, see Hermann [1926], Mayr/Meyer [1982].

Korollar: *The (uniform) word problem for finitely presented commutative semigroups can be solved in exponential space (in the size of the input).*

See Mayr/Meyer [1982].

. . . and the "bad" news:

Theorem: *Each problem solvable in exponential space is log-lin reducible to the word problem for (finitely presented) commutative semigroups.*

See Mayr/Meyer [1982].

An immediate consequence is

Korollar: *The (uniform) word problem for finitely presented commutative semigroups requires exponential space in the worst case.*

Consequences

The exponential space lower bound for the word problem for commutative semigroups implies a corresponding lower bound for the (uniform) word problem for polynomial ideals since the former is a special case of the latter:

Consider u, v and the l_i and r_i as monomials. Then:

$$u \equiv v \,(\Gamma) \iff u - v \in \langle r_1 - l_1, \ldots, r_w - l_w \rangle$$

We also obtain an exponential space lower bound for the reachability problem for reversible Petri nets and hence Petri nets (resp. vector addition systems) in general.

A lower bound for Gröbner bases

The exponential space lower bound for the word problem for commutative semigroups also implies (see Huynh [1986]) the existence of polynomial ideals for which every Gröbner (Standard) basis contains at least $2^{2^{cn}}$ elements as well as elements of degree $\geq 2^{2^{cn}}$ (here, n is the size of the input, and $c > 0$ is some constant).

Upper bounds for PIMP

Hermann's degree bound implies that solving

$$p = \sum_{i=1}^{w} g_i p_i$$

(where the unknown g_i's are polynomials) can be reduced to solving a system of linear equations with scalar entries and as unknowns the coefficients of those monomials with degrees smaller than Hermann's bound. The dimension of this linear system is

$$2^{2^{cn}},$$

(n and c as above). The solvability of this system, as well as (if it is indeed solvable) a particular solution, can be determined without ever explicitly writing down the system. An efficient implementation requires space only exponential in n.

We also obtain

Theorem: *If* $p = \sum_{i=1}^{w} g_i p_i$ *then suitable polynomials* g_i *can be computed in exponential space (to write down the output generally requires much more space).*

And:

Theorem: *Given the conditions of Hilbert's Nullstellensatz, an appropriate exponent* e *can be computed in exponential space.*

Special cases of polynomial ideals

Degree bounds given by Brownawell [1986] and others imply for the special cases

1. 0-dimensional ideals, and
2. homogeneous ideals

that PIMP can be solved in PSPACE.

Using a reduction from the LBA problem we obtain

Theorem: *The word problem for polynomial ideals with a homogenous basis is PSPACE complete.*

Conclusion, open problems

There are very close connections between the complexities of various problems for commutative semigroups, Petri nets, and polynomial ideals. These sometimes produce interesting applications of results obtained in one of these areas.

Open problems:

1. Which properties of ideals make PIMP simpler?
2. Investigate polynomial ideals with integer coefficients!
3. Determine the complexity of the Petri net reachability problem ($\equiv$ coefficients in $\mathbb{N}$ or $\mathbb{Q}^+$)!