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# An Optimal Greedy Algorithm for Wavelength Allocation in Directed Tree Networks

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ABSTRACT. The problem of allocating wavelengths in optical networks with directed tree topology has recently received a substantial amount of attention. Wavelengths must be assigned to connection requests which are represented as directed paths, and it is required that paths receive different wavelengths if they share a directed link. The goal is to minimize the number of different wavelengths used. The best previously known approximation algorithm for this $\mathcal{NP}$–hard problem routes any set of paths with $\frac{7}{4}L$ wavelengths, where $L$ is the maximum load on a directed link. We give an improved algorithm that uses at most $\frac{5}{3}L$ wavelengths. This is provably the best ratio that can be achieved by any local greedy algorithm for this problem.

## 1. Introduction

Optical networks provide substantially increased transmission rates as compared to conventional networks. Laser beams that are transmitted through optical fiber links achieve rates in the range of gigabits-per-second. In addition, multiple connections can share links if they are assigned different wavelengths (*wavelength-division multiplexing*). Wavelengths are a limited resource, however, and solutions to the problem of efficient wavelength allocation have substantial importance for the future development of this technology. The competing *space-division multiplexing* technology employs parallel single-wavelength links at an additional production cost but with the advantage of avoiding all wavelength assignment problems: different connections use different links and can be assigned the same wavelength without causing conflicts. If the wavelength allocation problem for wavelength-division multiplexing technology does not find a satisfactory solution, space-division multiplexing might be selected as the standard technology for high performance networks of the future.

It is not surprising that the wavelength allocation problem in optical networks has received considerable interest lately [**7, 6, 3, 5**]. Since the topologies of existing wide-area networks are mainly trees and trees of rings, wavelength allocation in tree networks has been the focus of attention. In a tree network $T$, a communication request (i.e., a source-destination pair) can be viewed as a path in $T$. Given a set

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$S$ of communication requests, wavelengths must be assigned to the corresponding paths such that paths that use the same link are assigned different wavelengths. The goal is to minimize the number of required wavelengths. We refer to wavelengths as colors, and we view the wavelength assignment problem as a path coloring problem.

Whereas Raghavan and Upfal considered undirected networks [**7**], recent work has focused on directed trees. Here, each bidirectional link of the optical network is represented by two directed edges with opposite directions, and each connection request corresponds to a directed path. The *load* $L(e)$ on a directed edge $e$ is the number of paths through that edge, and $L$ denotes the maximum load on any edge. Mihail, Kaklamanis, and Rao gave a polynomial-time algorithm that colors a given set of directed paths with $\frac{15}{8}L$ colors [**6**]. This was improved to $\frac{7}{4}L$ colors independently by Kaklamanis and Persiano [**3**] and by Kumar and Schwabe [**5**]. All these algorithms are *greedy* in the sense that they consider the nodes of the tree in depth-first order, extend an existing partial coloring by choosing colors for the uncolored paths touching the current node, and never re-color a path that was colored at a previous node. In addition, they are *local* greedy algorithms, because the only information about the paths touching the current node that is taken into account for the extension of the existing partial coloring is which of the edges incident to the current node they use.

It is known that the exact optimization problem is $\mathcal{NP}$–hard even for binary trees [**1**], and that no local greedy algorithm can go below $\frac{5}{3}L$ [**2**]. Interestingly, there is a set of connection requests in a binary tree such that $L = 3$ and *every* algorithm requires 5 wavelengths [**2**]. Kumar and Schwabe present a family of arbitrarily large instances for which $\frac{5}{4}L$ colors are necessary [**5**]. In the present paper, we present a polynomial-time approximation algorithm that colors a given set of directed paths in a tree with at most $\lceil \frac{5}{3}L \rceil$ colors. This is the best one can hope for using the (local) greedy approach. Our algorithm combines some of the techniques introduced in [**6, 3, 5**].

## 2. Outline of Algorithm

As in the previous approaches in [**6, 3, 5**], the nodes of the given tree are processed in depth-first order, starting at an arbitrary leaf node. We assume that $L = 3\ell$ is a multiple of 3. Initially, the paths touching the leaf node can easily be colored using $L$ colors. When a node $v$ is processed, all paths touching its parent (in the depth-first search tree) or any other node with smaller dfs-number have already been colored, and this coloring is now extended to include all paths touching $v$. During the whole process, the following two invariants are maintained:

**Invariant 1:** The number of colors used for coloring paths is at most $\frac{5}{3}L = 5\ell$ altogether.

**Invariant 2:** The number of colors used for coloring paths that go through a pair of directed edges with opposite directions corresponding to a single link is at most $\frac{4}{3}L = 4\ell$.

Initially, when only the paths touching one leaf node are colored, the invariants hold. We will show how a given coloring can be extended in polynomial time to include the paths touching an additional node without violating the invariants. It is not difficult to generalize our approach to deal with the cases $L = 3\ell + 1$ (using $4\ell + 2$ colors per link and $5\ell + 2$ colors altogether) and $L = 3\ell + 2$ (using $4\ell + 4$ colors per link and $5\ell + 4$ colors altogether). Hence, we obtain the following result:

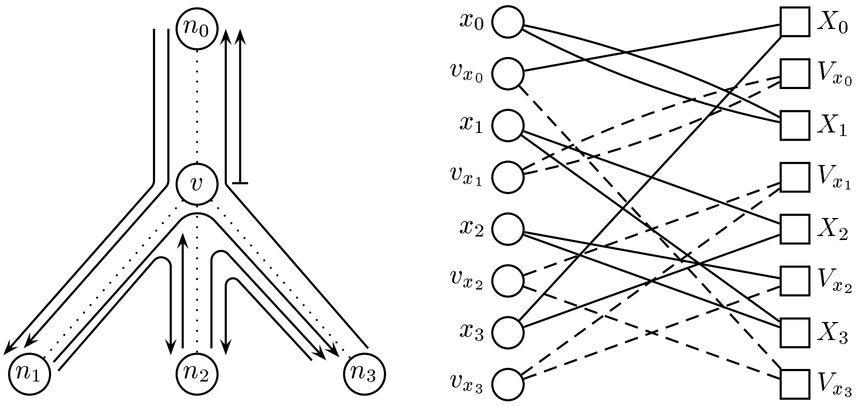



FIGURE 1. Construction of the bipartite graph $G_v$

THEOREM 2.1. *Given a set of directed paths in a tree such that the load on each directed edge is at most $L$, a valid coloring for the paths using at most $\lceil \frac{5}{3}L \rceil$ colors can be obtained in polynomial time.*

We assume that each directed edge of the tree has load exactly $L$. (Otherwise, extra paths can be added to the given instance.) The problem of extending an existing partial coloring to include the paths touching a new node $v$ is reduced to the edge-coloring problem in a bipartite graph $G_v$. Denote by $n_0$ the parent of $v$ and by $n_1, n_2, \ldots, n_k$ the children of $v$. Note that all paths touching $n_0$ have already been colored, and that the goal is to color all paths touching $v$ but not $n_0$ without violating the invariants. The bipartite graph $G_v$ has left and right vertex set $\bigcup_{i=0}^{k}\{x_i, v_{x_i}\}$ and $\bigcup_{i=0}^{k}\{X_i, V_{x_i}\}$, respectively. Each path touching $v$ contributes one edge to $G_v$ as follows:

- A path coming from a node $n_i$ and heading for a node $n_j$ contributes an edge $(x_i, X_j)$.
- A path coming from a node $n_i$ and terminating at node $v$ contributes an edge $(x_i, V_{x_i})$.
- A path starting at node $v$ and heading for a node $n_i$ contributes an edge $(v_{x_i}, X_i)$.

Note that all vertices $x_i$ and $X_i$ have degree $L$, whereas the vertices $v_{x_i}$ and $V_{x_i}$ may have smaller degree. In order to make $G_v$ $L$-regular, dummy edges are added between $v_{x_i}$ vertices on the left side and $V_{x_j}$ vertices on the right side, if necessary. (More precisely, for every edge $(x_i, X_j)$ a dummy edge $(V_{x_i}, v_{x_j})$ is added.) Figure 1 illustrates this construction, with dashed lines indicating dummy edges. Note that vertices in the same row of $G_v$ (i.e., vertices that are opposite each other) can not be adjacent.

It is easy to see that two paths touching $v$ must be assigned different colors if and only if the corresponding edges in $G_v$ share a vertex. Hence, any valid edge-coloring of $G_v$ constitutes a valid coloring for the paths touching $v$. As the paths touching $n_0$ have been colored at a previous step, the edges incident to $x_0$ and $X_0$ have already received a color. We refer to these edges as *pre-colored* edges. The colors that appear on pre-colored edges of $G_v$ are called *single* colors if they appear only once, and *double* colors if they appear twice (i.e., they are shared by $x_0$ and $X_0$). Denote by $S$ the number of single colors, and by $D$ the number of double colors. Invariant 2 ensures that $S + D \leq \frac{4}{3}L$. Since we assume that every edge has load $L$, we have $S + 2D = 2L$ and, consequently, $D \geq \frac{2}{3}L$.

Furthermore, one can assume that $D$ is *exactly* $\frac{2}{3}L$. The reason is that, if $D$ is greater than $\frac{2}{3}L$, one can simply "split" an appropriate number of double colors by assigning one of the two pre-colored edges colored with the same double color a new color for the duration of this edge-coloring phase. A series of color exchanges can then re-establish the original colors on the pre-colored edges without violating the invariants.

Assuming that $S = \frac{2}{3}L$ and $D = \frac{2}{3}L$, we show that $G_v$ can be edge-colored using at most $\frac{1}{3}L$ new colors (i.e., colors that do not appear on the pre-colored edges) such that no row of $G_v$ sees more than $\frac{4}{3}L$ colors. (We say that a row "sees" a color $c$ if an edge incident to a vertex of that row is colored with $c$. Similarly, we say that a vertex "sees" $c$ if an incident edge is colored $c$.) This implies that both invariants hold at the end of the current edge-coloring phase.

Since the bipartite graph $G_v$ is $L$-regular, its edges can be partitioned into $L$ disjoint perfect matchings. Each matching is classified according to the colors on its two pre-colored edges [**3**]. Matchings with two single colors are called SS-matchings. Matchings with a single color and a double color are called ST-matchings. Matchings with two different double colors are called TT-matchings. A matching with the same double color on both pre-colored edges is a PP-matching. In addition, we ensure that the number of PP-matchings is maximal, i.e., the subgraph obtained as the union of all SS-, ST-, and TT-matchings does not contain a PP-matching. This can be done in polynomial time with standard techniques.

We obtain 3-regular subgraphs, each containing two single colors and four (not necessarily distinct) double colors, by partitioning the $L$ matchings into groups of 3 matchings in an appropriate way. The uncolored edges of each such subgraph are then colored using at most one new color and reusing some of the old colors. Hence, at most $D/2 = \frac{1}{3}L$ new colors are used altogether, as required.

In addition, when we color each 3-regular subgraph $H$ we make sure that no row of $H$ (except the top row with vertices $x_0$ and $X_0$) sees more than 4 colors. Hence, no row sees more than $\frac{4}{3}L$ colors altogether. The details of the partitioning and edge-coloring are presented in Section 3. An alternative way to maintain Invariant 2 is to allow more than 4 colors per row in a subgraph, but to make sure that whenever a row sees a new color it does not see one of the old colors. This alternative algorithm is described in more detail in [**4**].

## 3. Partitioning and Coloring Details

Kumar and Schwabe introduce a very helpful concept that allows to color a special kind of 3-regular subgraph of $G_v$ using at most one new color such that every row except the top row sees at most 4 colors [**5**]. Precisely speaking, they consider 3-regular subgraphs $H$ of $G_v$ with the property that, of the six pre-colored edges of $H$, two are colored with the same *preserved* double color $d$, two are colored with (possibly distinct) double colors, and the remaining two are colored with single colors $s$ and $s'$. In addition, they assume that the edges colored $s$ and $s'$ are not incident to the same vertex out of $\{x_0, X_0\}$. We refer to such subgraphs of $G_v$ as *KS-subgraphs*. Furthermore, they call a subgraph $H_1$ of $G_v$ a *gadget* if $x_0$ and $X_0$

have degree 3 in $H_1$ while all other vertices have degree 2. Their important result concerning KS-subgraphs is stated in the following lemma.

LEMMA 3.1. [**5**] *Let $H$ be a KS-subgraph of $G_v$ with preserved double color $d$ and single colors $s$ and $s'$. If $H$ can be partitioned into a gadget $H_1$ and a matching $H_2$ on all vertices other than $x_0$ and $X_0$, then $H$ can be colored using colors $d$, $s$, $s'$, and at most one new color such that no row of $H$ except the top row sees more than 4 colors.*

Kumar and Schwabe obtain their $\frac{7}{4}L$ result by extracting and coloring KS-subgraphs until the ratio of remaining edges with double colors to edges with single colors drops from $1:1$ (they assume $\frac{3}{2}L$ colors per link, and hence $S = 2D$) to $1:2$ or lower. They have to treat the remaining edges in a rather complicated way, however, and require 3 colors in a row for every two matchings considered in the worst case.

We improve on the $\frac{7}{4}L$ result by Kumar and Schwabe by showing how the entire graph $G_v$ can be partitioned into 3-regular subgraphs each of which can be colored with one new color and with at most 4 colors per row.

If color $a$ appears on an edge $e$ incident to $x_0$ in $G_v$, we denote the other endpoint of $e$ by $r(a)$. Similarly, if color $b$ appears on an edge $f$ incident to $X_0$, we denote the other endpoint of $f$ by $l(b)$. First, we investigate under which condition a KS-subgraph can be partitioned into gadget and matching.

LEMMA 3.2. *Let $H$ be a KS-subgraph of $G_v$, and let $d$ be its preserved double color. If $H$ does not contain a PP-matching, it can be partitioned into a gadget and a matching on all vertices other than $x_0$ and $X_0$.*

PROOF. Denote the colors on edges incident to $x_0$ by $d$, $d_1$ and $s$, and the colors on edges incident to $X_0$ by $d$, $d_2$ and $s'$. Note that $d_1$ may be equal to $d_2$. Remove vertices $x_0$ and $X_0$ from $H$. Introduce dummy edges $(l(d), r(d))$, $(l(s'), r(s))$ and $(l(d_2), r(d_1))$. The obtained graph $H'$ is again 3-regular, and its edges can be split into three perfect matchings $M_1$, $M_2$ and $M_3$. If each matching contains exactly one dummy edge, that matching containing the dummy edge $(l(d), r(d))$ immediately gives a PP-matching in $H$. Otherwise, one of the three matchings, say $M_1$, does not contain a dummy edge. In this case, $H$ can be partitioned into the gadget $H \setminus M_1$ and the matching $M_1$. □

Computing a maximal number of PP-matchings in advance ensures that KS-subgraphs obtained as the union of three non-PP-matchings later on can be colored according to Lemma 3.1. In Section 3.1, we show how KS-subgraphs and other 3-regular subgraphs whose coloring is straightforward can be extracted from $G_v$ until a configuration out of a limited set of alternatives is obtained. In Section 3.2, we show how to deal with these alternatives whose partitioning is more involved. For the coloring of certain cycle covers and gadgets we employ two technical lemmas which are presented in Section 3.3.

**3.1. Partitioning into 3-regular graphs (simple cases).** ST- and TT-matchings are viewed as a collection of chains and cycles. Given a matching $M$, we refer to the color of the edge incident to $x_0$ as the *left* color of $M$, and to the color of the edge incident to $X_0$ as the *right* color of $M$. A sequence $\langle M_0, M_1, \ldots, M_{l-1}\rangle$ of $l > 1$ TT-matchings is a *cycle* of length $l$ if the right color of $M_i$ is equal to the left color of $M_{i+1 \bmod l}$ for all $0 \leq i \leq l-1$. A sequence $\langle M_0, M_1, \ldots, M_{l-1}\rangle$

of $l > 1$ matchings is a *chain* of length $l$ if $M_0$ and $M_{l-1}$ are ST-matchings, $M_1, M_2, \ldots, M_{l-2}$ are TT-matchings, and the right color of $M_i$ is equal to the left color of $M_{i+1}$ for all $0 \leq i \leq l-2$. The terms *$l$-cycle* and *$l$-chain* refer to cycles and chains of length $l$, respectively.

The set of all matchings can now be grouped into PP-matchings, SS-matchings, chains, and cycles. Since we have $S = D$, the ratio of the number of edges with double colors to the number of edges with single colors is $2:1$ altogether. The only groups of matchings that have a smaller ratio than that are SS-matchings and 2-chains. Therefore, whenever there is a group whose number of double-color edges is more than twice the number of its single color edges, there must be a sufficient number of SS-matchings and 2-chains to outweigh this imbalance. As we extract only 3-regular subgraphs with four double-color edges and two single-color edges, the ratio of double-color edges to single-color edges in the resulting graph is still $2:1$.

Let us introduce some notation. We use letters $a$, $a'$, $b$, $b', \ldots$ to denote double colors, and letters $s$, $s'$, $t$, $t', \ldots$ to denote single colors. For instance, an SS-matching with left color $s$ and right color $s'$, a 2-chain, a 3-chain, and a 5-cycle are represented as follows:

$$\begin{matrix} s \\ s' \end{matrix} \qquad \begin{matrix} s & a \\ a & s' \end{matrix} \qquad \begin{matrix} s & a & b \\ a & b & s' \end{matrix} \qquad \begin{matrix} a & b & c & d & e \\ b & c & d & e & a \end{matrix}$$

The following groups of matchings can easily be combined with SS-matchings and 2-chains in order to obtain KS-subgraphs.

**A cycle of even length** can be divided into pairs of matchings such that every pair of matchings has a common double color. Each pair can either be combined with an SS-matching to obtain one KS-subgraph, or with two 2-chains to obtain two KS-subgraphs. For example, a 4-cycle can be combined with two 2-chains and one SS-matching:

$$\begin{matrix} a & b & c & d \\ b & c & d & a \end{matrix} + \begin{matrix} s & e \\ e & s' \end{matrix} + \begin{matrix} t & f \\ f & t' \end{matrix} + \begin{matrix} u \\ u' \end{matrix} = \begin{matrix} u & a & b \\ u' & b & c \end{matrix} + \begin{matrix} s & e & c \\ e & s' & d \end{matrix} + \begin{matrix} t & f & d \\ f & t' & a \end{matrix}$$

Note that there is always a sufficient number of SS-matchings or 2-chains available for such reductions because the ratio of double-color edges to single-color edges is $2:1$ altogether. In particular, when no SS-matchings are left, there must be at least one 2-chain for every remaining matching of the cycle.

**A chain of odd length** can be divided into two parts. The two matchings at the beginning of the chain and the matching at the end together constitute a KS-subgraph. If the chain has length 3, we are finished. Otherwise, a sequence of TT-matchings of even length remains. This sequence has the property that any two consecutive matchings share a double color. Hence, this sequence can be handled just like a cycle of even length.

**Two chains of even length,** one of which has length $> 2$, can also be combined with SS-matchings or 2-chains. Denote the longer chain by $C_1$ and the other one by $C_2$. Combine the first two matchings of $C_1$ with the last matching of $C_2$, and the first matching of $C_2$ with the last two matchings of $C_1$. Both combinations result in KS-subgraphs. We are left with one or two sequences of even length such that consecutive matchings in each sequence share a double color. Hence, these sequences can be handled like cycles of even length.

**A 2-chain and a cycle of odd length** can be handled by combining the 2-chain with the first matching of the cycle to obtain a KS-subgraph. The remainder of the cycle is a sequence of even length with the property that consecutive matchings share a double color, and it can be handled like a cycle of even length.

**A 2-chain and a PP-matching** can be colored without any new color. Color the PP-matching with its double color and each of the matchings in the 2-chain with its single color. Since the number of colors that appear in this 3-regular subgraph is 4 altogether, the constraint on the number of colors per row is satisfied.

**Two PP-matchings and an SS-matching** can also be colored without using any new color. Color each PP-matching with its double color and the SS-matching with one of its single colors. Again, there are only 4 colors involved altogether.

Applying these reductions in arbitrary order repeatedly as long as possible, we are left with SS-matchings, at most one PP-matching, at most one chain of even length $> 2$, and a number of cycles of odd length. (If there was a 2-chain left, the fact that the ratio of double-color edges to single-color edges is $2:1$ altogether would imply that there is also at least one PP-matching, cycle, or chain of length $> 3$. Hence, the 2-chain should have been combined with one of these.) Section 3.2 shows how these remaining matchings can be colored without violating the invariants.

**3.2. Partitioning into 3-regular graphs (difficult cases).** We assume that none of the remaining cycles and chains contains parallel pre-colored edges. Otherwise, such cycles or chains can be divided into two cycles (one of which has even length and can be handled as shown in the previous section) or a cycle and a shorter chain (or SS-matching) by exchanging the parallel edges in the two matchings. For instance, if $r(b) = r(e)$ in the following example, the 6-chain can be divided into a 3-chain and a 3-cycle.

$$\begin{matrix} s & a & \mathbf{b} & c & d & \mathbf{e} \\ a & b & c & d & e & s' \end{matrix} \longrightarrow \begin{matrix} s & a & \mathbf{b} \\ a & b & s' \end{matrix} + \begin{matrix} \mathbf{e} & c & d \\ c & d & e \end{matrix}$$

First, we show how to handle two cycles of odd length.

LEMMA 3.3. *Given two cycles, each of them without parallel pre-colored edges, of length $2i+1$ $(i \geq 1)$ and $2j+1$ $(j \geq 1)$, respectively, and $i+j+1$ SS-matchings, these $n = 3(i+j+1)$ matchings can be colored using at most $n/3$ new colors such that each row of $G_v$ sees at most $\frac{4}{3}n$ colors.*

PROOF. Denote the cycles by $C_1$ and $C_2$. If a pre-colored edge of $C_1$ and a pre-colored edge of $C_2$ are parallel, a cycle $C$ of even length can be obtained by exchanging these two edges. $C$ can then be colored as required using the method of the previous section. Hence, we assume that there are no parallel pre-colored edges in $C_1 \cup C_2$.

Pick an arbitrary SS-matching $M$ with single colors $s$ and $s'$. Pick a matching $M_1$ with colors $a$ and $b$ from $C_1$ such that none of the pre-colored edges of $M_1$ is parallel to an edge colored $s$ or $s'$ in $M$. Pick a matching $M_2$ with colors $a'$ and $b'$ from $C_2$ with the same property. These matchings exist because both cycles have length at least 3 and do not contain parallel pre-colored edges. Consequently, we have $|\{r(s), r(a), r(a')\}| = 3$ and $|\{l(s'), l(b), l(b')\}| = 3$.

We consider the 3-regular subgraph $H = M \cup M_1 \cup M_2$. Remove the vertices $x_0$ and $X_0$ from $H$ and add dummy edges $(r(a), l(b'))$, $(r(a'), l(b))$, and $(r(s), l(s'))$. The obtained graph $H'$ is again 3-regular and can be partitioned into three perfect matchings. If each of the matchings contains exactly one dummy edge, $H$ can be split into an SS-matching with colors $s$ and $s'$, a TT-matching with colors $a$ and $b'$, and a TT-matching with colors $a'$ and $b$. Exchanging the new matchings for the old ones turns the cycles $C_1$ and $C_2$ into a single cycle of even length, which can then be colored as shown in the previous section.

If one of the matchings obtained from $H'$ does not contain a dummy edge, this matching gives a partitioning of $H$ into gadget $H_1$ and matching $H_2$. In that case, we can color $H_1$ using only its old colors such that no row except the top row sees more than 3 colors. This is possible because we have excluded degenerate cases by ensuring that pre-colored edges are not parallel. In addition, double colors must be reused carefully in order to make sure that no conflict with the pre-colored edges colored $a$, $b$, $a'$, and $b'$ outside $H_1$ is introduced. To exhibit the details of the coloring, a thorough case analysis is required (see Lemma 3.7). $H_2$ is colored using a single new color. Therefore, $H$ is colored such that no row except the top row sees more than 4 colors.

When $H$ has been colored, the remaining sequences $C_1 \setminus M_1$ and $C_2 \setminus M_2$ have even length and the property that consecutive matchings share a double color. Hence, they can be colored as shown in Section 3.1, reusing only single colors and double colors not in $\{a, b, a', b'\}$. □

Since this lemma allows us to combine pairs of cycles of odd length with SS-matchings, we are left with SS-matchings, at most one PP-matching, at most one cycle of odd length, and at most one chain of even length. In addition, we note that the number of edges with double colors is divisible by 4 since the number of double colors is even ($D = 2\ell$ by assumption). Hence, only the following cases can occur (each with the appropriate number of SS-matchings):

**(a)** one PP-matching and one cycle of odd length
**(b)** one PP-matching and one chain of even length
**(c)** one cycle of odd length and one chain of even length

These are handled by the following three lemmas.

LEMMA 3.4. *Given a PP-matching, a cycle of length $2i + 1$ ($i \geq 1$) without parallel pre-colored edges, and $i+1$ SS-matchings, these $n = 3(i+1)$ matchings can be colored using at most $n/3$ new colors such that no row sees more than $\frac{4}{3}n$ colors.*

PROOF. Denote the PP-matching by $P$ and its double color by $p$. Pick an arbitrary SS-matching $M$ with colors $s$ and $s'$. Pick a matching $M'$ with double colors $a$ and $b$ from the cycle such that none of its pre-colored edges is parallel to an edge colored $s$ or $s'$ in $M$. Consider the 3-regular subgraph $H$ obtained as the union of the PP-matching $P$, the SS-matching $M$, and the TT-matching $M'$. We color $H$ using only colors $p$, $s$, $s'$, and at most one new color such that no row except the top row sees more than 4 colors.

More precisely, we color the PP-matching $P$ using its double color $p$ and view $M \cup M'$ as a cycle cover $C$. Since $C$ does not contain parallel pre-colored edges, Lemma 3.8 implies that $C$ can be colored using only one new color such that every pair of opposite vertices shares a color. Hence, every pair of opposite vertices in $H$ shares at least two colors altogether, and the obtained coloring satisfies the requirements.

When $H$ is colored as required, we are left with a sequence of matchings of even length such that consecutive matchings share a double color, and SS-matchings. These can be colored as shown in Section 3.1. □

LEMMA 3.5. *Given a PP-matching, a chain of length $2i$ ($i \geq 2$) without parallel pre-colored edges, and $i-1$ SS-matchings, these $n = 3i$ matchings can be colored using at most $n/3$ new colors such that no row sees more than $\frac{4}{3}n$ colors.*

PROOF. This proof is similar to the proof of Lemma 3.4. Pick an arbitrary SS-matching $M$ with colors $s$ and $s'$. Let $M'$ be the second matching of the chain and let $a$ and $b$ be its double colors. If $M \cup M'$ contains parallel pre-colored edges, exchanging these two edges turns the original chain and the SS-matching into a 2-chain and a chain of length $2i-1$. As shown in the previous section, the $(2i-1)$-chain can be colored as required, and the 2-chain can be combined with the PP-matching.

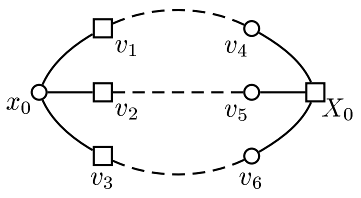
Hence, we can assume that $C = M \cup M'$ does not contain parallel pre-colored edges. Like in the proof of Lemma 3.4, we color the PP-matching using its double color and the cycle cover $C$ according to Lemma 3.8. Furthermore, we combine the first matching of the chain and its last two matchings to obtain a KS-subgraph. This leaves us with an even-length sequence of matchings such that consecutive matchings share a double color, and this sequence can be combined with the SS-matchings as shown in Section 3.1. □

LEMMA 3.6. *Given a cycle of length $2i+1$ ($i \geq 1$) and a chain of length $2j$ ($j \geq 2$), each without parallel pre-colored edges, and $i+j-1$ SS-matchings, these $n = 3(i+j)$ matchings can be colored using at most $n/3$ new colors such that no row sees more than $\frac{4}{3}n$ colors.*

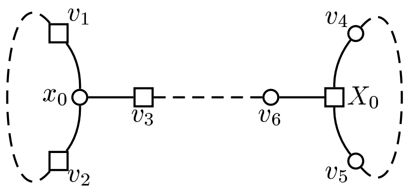
PROOF. This proof is similar to the proof of Lemma 3.3. Denote the cycle by $C$ and the chain by $N$. If $C \cup N$ contains parallel pre-colored edges, exchanging two such edges turns $C \cup N$ into a chain of length $2(i+j)+1$. This chain has odd length and can be colored as shown in Section 3.1. Hence, we assume that $C \cup N$ does not contain parallel pre-colored edges.

Pick an arbitrary SS-matching $M$ with single colors $s$ and $s'$. Pick a matching $M'$ with double colors $a$ and $b$ from the cycle such that none of its pre-colored edges is parallel to an edge colored $s$ or $s'$ in $M$. Let $M''$ be the second matching of the chain $N$ and let $a'$ and $b'$ be its double colors. Let $H$ be the subgraph $M \cup M' \cup M''$. If $H$ contains parallel pre-colored edges, these edges must either be colored $s$ and $a'$ or $s'$ and $b'$. In both cases, exchanging the parallel pre-colored edges turns $N \cup M$ into a 2-chain and a chain of length $2j-1$. As shown in Section 3.1, the 2-chain can be combined with $C$ and the $(2j-1)$-chain can be combined with SS-matchings.

If $H$ does not contain parallel pre-colored edges, we try to partition $H$ into a gadget $H_1$ and a matching $H_2$. As in the proof of Lemma 3.3, either such a partitioning exists and $H$ can be colored using Lemma 3.7, or $H$ can be split into three new matchings $M_1$, $M_2$, and $M_3$ such that $M_1$ is a TT-matching with colors $a$ and $b'$, $M_2$ is a TT-matching with colors $a'$ and $b$, and $M_3$ is an SS-matching with colors $s$ and $s'$. In the latter case, replacing the old matchings by the new ones turns $C \cup N$ into a single chain of length $2(i+j)+1$, which can be colored as shown in Section 3.1. In the former case, what remains is an even-length sequence such that consecutive matchings share a double color and an even-length chain without its second matching. The sequence can be colored as shown in Section 3.1, and the incomplete chain can be colored as in the proof of Lemma 3.5. □



(a) football-shaped gadget



(b) dumbbell-shaped gadget

FIGURE 2. The two kinds of gadgets

**3.3. Two helpful lemmas.** The following lemma, which was used in Section 3.2, shows that under certain conditions a gadget $H_1$ can be colored without any new color even if there is no double color that appears twice in $H_1$.

LEMMA 3.7. *Let $H_1$ be a subgraph of $G_v$ that is a gadget with single color $s$ and double colors $a$ and $a'$ incident to $x_0$, and with single color $s'$ and double colors $b$ and $b'$ incident to $X_0$. Assume that $H_1$ does not contain parallel pre-colored edges. Furthermore, assume that the set of pre-colored edges of $G_v$ with colors in $\{a, a', b, b'\}$ does not contain parallel edges. Then $H_1$ can be colored without using any new color such that opposite vertices (except $x_0$ and $X_0$) share at least one color.*

SKETCH OF PROOF. As $H_1$ is a gadget, it can be seen as a collection of three paths, each running from $x_0$ to $X_0$ or looping back from $x_0$ or $X_0$ to its starting point, and a certain number of cycles involving only vertices other than $x_0$ and $X_0$. The cycles have even length and are colored by alternating colors $s$ and $s'$. There are two possibilities for the remaining paths: either all three of them run from $x_0$ to $X_0$, or only one runs from $x_0$ to $X_0$ while the other two loop back to their starting point. Kumar and Schwabe refer to such gadgets as *football-shaped* gadgets and *dumbbell-shaped* gadgets, respectively [**5**]. Figure 2 shows the gadget shapes and our naming of vertices adjacent to $x_0$ and $X_0$. Note that dumbbell-shaped gadgets are non-degenerate (i.e., $v_1 \neq v_2$ and $v_4 \neq v_5$) because there are no parallel pre-colored edges in $H_1$.

We can show that it is always possible to color the paths such that opposite vertices (except $x_0$ and $X_0$) share at least one color. More precisely, we color the

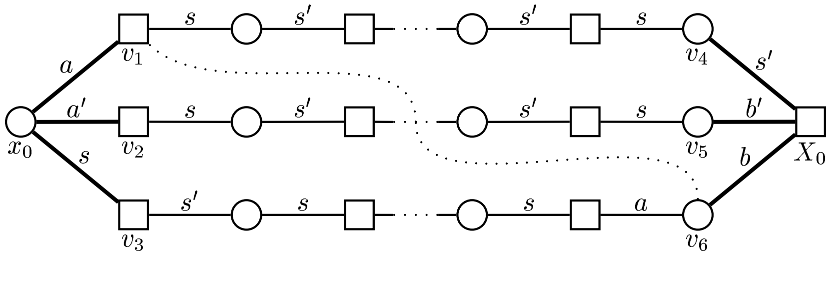
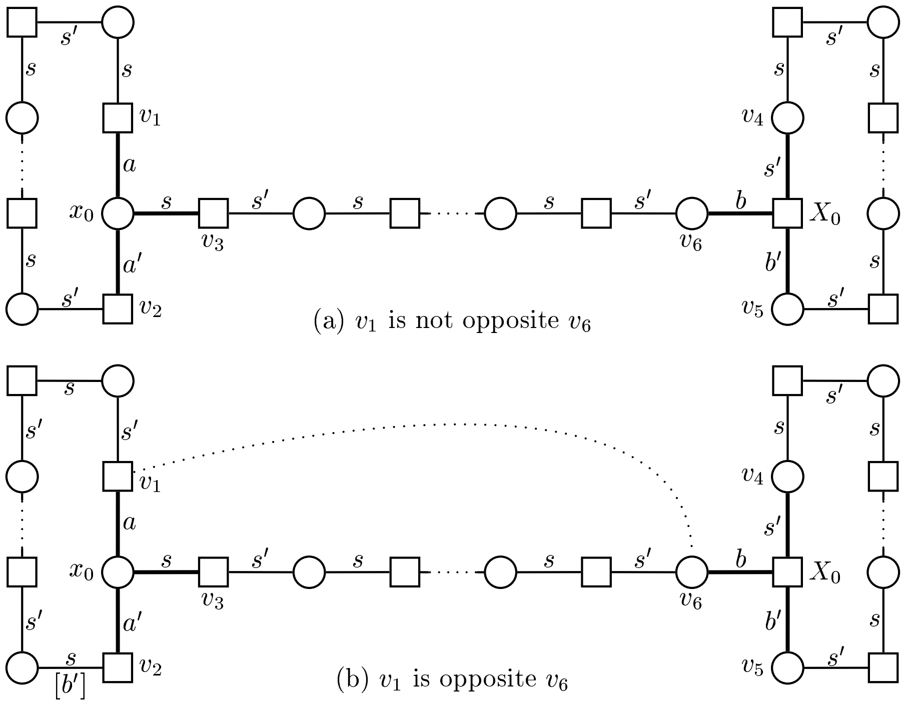


FIGURE 3. Coloring a football-shaped gadget if $v_1$ is opposite $v_6$




(a) $v_1$ is not opposite $v_6$

(b) $v_1$ is opposite $v_6$

[(c) $v_2$ is opposite $v_5$]

FIGURE 4. Coloring a dumbbell-shaped gadget that has one single color on the path from $x_0$ to $X_0$

three paths by alternating colors $s$ and $s'$ in an appropriate way. The uncolored edges at the beginning or end of a path are an exception, because we sometimes color them by carefully reusing colors $a$, $a'$, $b$, or $b'$.

For example, Figure 3 shows the coloring we use for one case of a football-shaped gadget if $v_1$ is opposite $v_6$. Note that the uncolored edge incident to $v_6$ can be colored with color $a$ because neither $v_6$ nor its other endpoint sees $a$. (Furthermore, $H_1$ is the only subgraph of $G_v$ for whose coloring we reuse color $a$.) Now, $v_1$ and $v_6$ share color $a$, and all other pairs of opposite vertices (except $x_0$ and $X_0$) share color $s$.

Figure 4 shows the various cases for coloring a dumbbell-shaped gadget if the two pre-colored edges on the path from $x_0$ to $X_0$ are colored with one single color and one double color. Without loss of generality we assume that $v_1$ is not opposite $v_5$. (Otherwise, we rename $v_1$ and $v_2$.) If $v_1$ is not opposite $v_6$, the coloring shown in part (a) of Figure 4 is valid. All vertices (we ignore $x_0$ and $X_0$ in this discussion) except $v_1$ see color $s'$. $v_1$ sees color $s$. The only vertices that do not see $s$ are $v_2$, $v_5$, and $v_6$. $v_2$ cannot be opposite $v_1$ because both are right vertices. $v_5$ and $v_6$ are not opposite $v_1$ by assumption. Hence, $v_1$ shares $s$ with its opposite vertex, and all other pairs of opposite vertices share $s'$.

If $v_1$ is opposite $v_6$ but $v_2$ is not opposite $v_5$, we use the coloring from part (b) of Figure 4. $v_2$ shares $s$ with its opposite vertex, and all other pairs of opposite

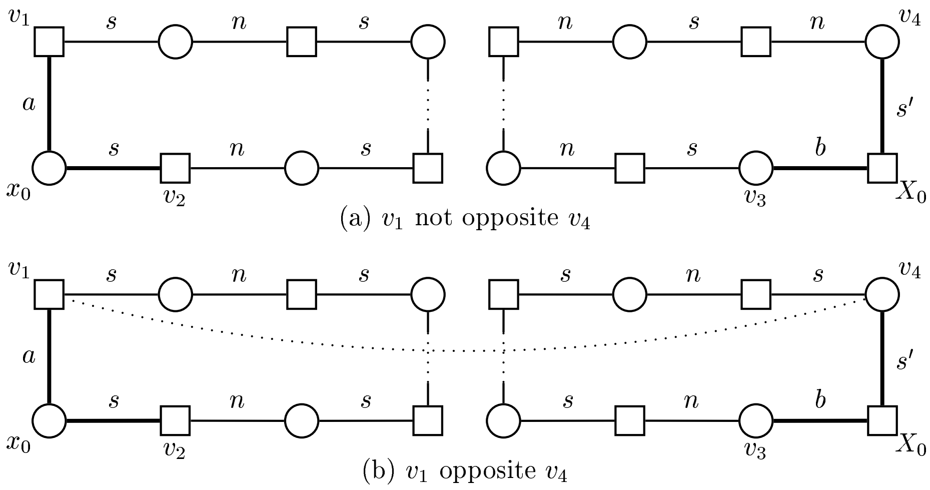



FIGURE 5. $x_0$ and $X_0$ are in separate cycles

vertices share $s'$. If $v_1$ is opposite $v_6$ and $v_2$ is opposite $v_5$, the same coloring becomes valid if we replace color $s$ by $b'$ on the edge incident to $v_2$: $v_2$ and $v_5$ share $b'$, and all other pairs of opposite vertices share $s'$. Using $b'$ to color the uncolored edge incident to $v_2$ does not introduce a conflict with the pre-colored edge colored $b'$ outside $H_1$ because that edge is not parallel to the edge $(x_0, v_2)$ by assumption.

Similar arguments can be used to deal with all possible arrangements of colors on the pre-colored edges of football-shaped gadgets and non-degenerate dumbbell-shaped gadgets. The details are omitted. □

Finally, the following lemma shows that non-degenerate cycle covers obtained from an SS-matching and a TT-matching can be colored with a single new color and at most three colors per row.

LEMMA 3.8. *Let $C$ be a cycle cover obtained as the union of an SS-matching and a TT-matching without parallel pre-colored edges. Denote its single colors by $s$ and $s'$, and its double colors by $a$ and $b$. $C$ can be colored using only $s$ and a new color $n$ such that every pair of opposite vertices (except $x_0$ and $X_0$) shares at least one color.*

PROOF. All cycles that do not contain $x_0$ or $X_0$ have even length and are colored by alternating colors $s$ and $n$. As $C$ does not contain any parallel pre-colored edges, cycles containing $x_0$ or $X_0$ are non-degenerate. If $x_0$ and $X_0$ are contained in the same cycle, both paths from $x_0$ to $X_0$ are colored by alternating $n$ and $s$, starting and ending with $n$. In this case, all vertices except $x_0$ and $X_0$ share color $n$.

If $x_0$ and $X_0$ are contained in different cycles, let the vertices adjacent to $x_0$ be $v_1$ and $v_2$ and those adjacent to $X_0$ be $v_3$ and $v_4$. Assume that the edge $(x_0, v_1)$ is colored $a$ and $(X_0, v_3)$ is colored $b$. Color both cycles by alternating colors $s$ and $n$, starting with $s$ at $v_1$ and $v_3$. Every vertex except $v_4$ (and $X_0$) sees color $s$. $v_4$ sees color $n$. The only vertex that could be opposite $v_4$ and that does not see $n$ is $v_1$. Therefore, if $v_1$ is not opposite $v_4$, the coloring is valid: $v_4$ shares $n$ with its opposite vertex, and all other pairs of opposite vertices (except $x_0$ and $X_0$) share $s$, see part (a) of Figure 5. If $v_1$ is opposite $v_4$, exchange colors $s$ and $n$ on the cycle including $X_0$. Now, $v_1$ and $v_4$ share $s$, and all other pairs of opposite vertices (except $x_0$ and $X_0$) share $n$, see part (b) of Figure 5. □

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