A Note on Triangle-Free and Bipartite Graphs

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Dedicated to Daniel J. Kleitman on the occasion of his 65th birthday.

Abstract

Using a clever inductive counting argument Erdős, Kleitman and Rothschild showed that almost all triangle-free graphs are bipartite, i.e., the cardinality of the two graph classes is asymptotically equal.

In this paper we investigate the structure of the few triangle-free graphs which are not bipartite. Using similar techniques as Erdős, Kleitman and Rothschild we prove that with high probability these graphs can be made bipartite by removing a single vertex. In this sense these graphs are almost bipartite.

Key words: Random graphs, asymptotic graph theory, inductive counting, triangle-free graphs.

1 Introduction

In a sequence of two papers ([KR70] and [KR75]) Kleitman and Rothschild succeeded in counting the number of different partial orders on a set of n (labeled) elements. The method they invented for proving this result, subsequently also called the Kleitman-Rothschild method, is based on a clever inductive counting argument. Even though the Kleitman-Rothschild method was first used for partial orders, its highlights were within graph theory. Here the starting paper was by Erdős, Kleitman and Rothschild [EKR76] who proved that almost all

[1] Supported by DFG-grant Pr 296/4-2.
[2] Supported by DFG-grant Ste 464/3-1.

triangle-free graphs are bipartite (see below for a precise statement of this result). In the following years this result was generalized in many respects, e.g. [LR84], [KPR87], [PS92b], [PS92a]. For details and more references we refer the interested reader to [Ste93] or [PST99].

In this paper we reconsider the original Erdős, Kleitman, Rothschild result on triangle-free graphs and study the structure of the (negligible) subclass of those triangle-free graphs which are *not* bipartite. As it turns out, the result from [EKR76] is 'almost true' for those graphs. Namely, almost all triangle-free graphs which are not bipartite can be made bipartite by removing a single vertex.

2 Main Result

In order to state our main result precisely, we first introduce some notation.

Let $\mathcal{B}_0(n)$ denote the set of bipartite graphs on n (labeled) vertices. Furthermore, we define $\mathcal{B}_1(n)$ as the set of all triangle-free graphs $G = (V, E)$ which can be made bipartite by removing a single vertex, i.e., we can find a vertex $v \in V$ such that the graph $G' := G[V \setminus \{v\}] \in \mathcal{B}_0(n-1)$. In the sequel we call these graphs *quasibipartite*.

$\mathcal{F}_0(n)$ is defined as the set of all triangle-free graphs and $\mathcal{F}_1(n) := \mathcal{F}_0(n) \backslash \mathcal{B}_0(n)$ contains all triangle-free graphs which are not bipartite.

Using this terminology the original result by Erdős, Kleitman and Rothschild can be stated as follows:

Theorem 1 [EKR76] *Almost all triangle-free graphs are bipartite. Or, more precisely,*

$$|\mathcal{F}_0(n)| = |\mathcal{B}_0(n)| \cdot (1 + o(1)).$$

We prove an analogous result for the negligible set of graphs in $\mathcal{F}_1(n)$:

Theorem 2 *Almost all triangle-free graphs which are not bipartite, are quasibipartite, i.e.,*

$$|\mathcal{F}_1(n)| = |\mathcal{B}_1(n)| \cdot (1 + o(1)).$$

3 The Proof

In this section we give the proof of Theorem 2. Before we do so we fix some notations. Let $G = (V, E)$ be a graph. $\Gamma^i(S)$ denotes the *ith neighborhood* of a set $S \subseteq V$, i.e., $\Gamma^0(S) = S$ and

$$\Gamma^{i+1}(S) = \{v \in V \setminus [\Gamma^0(S) \cup \ldots \cup \Gamma^i(S)] \mid \exists x \in \Gamma^i(S), \{x, v\} \in E\}$$

for $i \geq 0$. With $\log n := \log_2 n$ we denote the logarithm to base 2.

The following definition specifies some properties which we expect to hold for a 'typical' bipartite graph. Based on the negation of these properties we will later define sets of 'strange' graphs, i.e., graphs with unusual properties. The cardinality of these sets can then be shown to be negligible in comparison to $|\mathcal{B}_1(n)|$.

Definition 3 *We introduce the following abbreviations for properties of a graph $G = (V, E)$ with $|V| = n$:*

(P1) Minimum degree is not too small: $|\Gamma(v)| \geq 2 \log n$ for all $v \in V$.

(P2) Large second neighborhood: $|\Gamma(Q)| \geq (\frac{1}{2} - \frac{2}{\log \log n})n$ for all $Q \subseteq V$ and $|Q| = \log n$.

The following lemmas show some results on the structure of graphs for which the specified properties hold. The proofs of Lemma 4 and Lemma 6 are implicit in [Ste93] and [PST99] but we also briefly include them here for clarity and completeness.

Lemma 4 *Let $G = (V, E)$ be a graph on n vertices, where n is sufficiently large, and $C \subseteq V$ be a cycle with $|C| \leq 9$. If it holds that $|\Gamma^2(v)| \geq (\frac{1}{2} - \frac{2}{\log \log n})n$ for all $v \in C$ and that $|\Gamma^2(x) \cap \Gamma^2(y)| \leq \frac{1}{100}n$ for all edges $\{x, y\}$ of C then the cycle cannot be odd.*

PROOF. For simplicity we will only show that there is no cycle C_3. The proof for C_5, C_7 and C_9 is similar.

Let $G = (V, E)$ denote a graph and let $C = (v_1, v_2, v_3)$ be a cycle in G which satisfies the conditions of the lemma. In the sequel we write $R_i := \Gamma^2(v_i)$ and $\alpha := \frac{2}{\log \log n}$, $\beta := \frac{1}{100}$ for short. We directly obtain the following estimates:

$$\begin{aligned} |R_1 \cup R_2| &= |R_1| + |R_2| - |R_1 \cap R_2| \geq (1 - 2\alpha - \beta)n \\ |\overline{R}_1 \cap \overline{R}_2| &= n - |R_1 \cup R_2| \leq (2\alpha + \beta)n. \end{aligned}$$

Hence,

$$|R_3| \le |\overline{R}_1 \cap \overline{R}_2| + |R_3 \cap R_1| + |R_3 \cap R_2| \le (2\alpha + 3\beta)n,$$

contradicting the assumption on the minimum size of R_3.

Now we define a property for sets of edges. If this property and also the properties from Definition 3 hold for a graph then this graph must be bipartite.

Definition 5 *Given a graph $G = (V, E)$ with $|V| = n$ we define the following property for a set of edges $F \subseteq E$:*

(P3) Few cycles C_5: $|\Gamma^2(x) \cap \Gamma^2(y)| \le \frac{1}{100}n$ for all $\{x, y\} \in F$.

Lemma 6 *If a graph $G = (V, E)$ satisfies the properties (P1), (P2) and (P3) for all edges and $|V| = n$ is sufficiently large then G is bipartite.*

PROOF. We construct a 2-coloring of G as follows: By Lemma 4 G contains neither a C_3 nor a C_5 nor a C_7 nor a C_9.

Now choose an arbitrary edge $\{x, y\} \in E$. For brevity let $Q_x := \Gamma(x)$, $Q_y := \Gamma(y)$, $R_x := \Gamma^2(x)$ and $R_y := \Gamma^2(y)$. R_x and R_y are stable because otherwise we could find a C_5. Similarly we conclude that $R_x \cap R_y = \emptyset$.

We denote all other vertices by $S := V \setminus [Q_x \cup Q_y \cup R_x \cup R_y]$ and partition them in two classes:

$$S_x = \{v \in S \mid R_v \cap R_x \neq \emptyset\} \quad \text{and} \quad S_y = \{v \in S \mid R_v \cap R_y \neq \emptyset\}.$$

Observe that $S_x \cup S_y = S$, since R_x and R_y cover almost the whole graph and R_v is big for all v. Hence, for every vertex v the intersection $R_v \cap (R_x \cup R_y)$ cannot be empty. Moreover, $R_x \cup S_x$ and $R_y \cup S_y$ are stable and $S_x \cap S_y = \emptyset$ because there are no cycles C_7 and C_9 (see Figure 1).

Hence, we have found a 2-coloring with the color classes $R_x \cup S_x \cup Q_y$ and $R_y \cup S_y \cup Q_x$.

Consider a graph G which satisfies properties (P1) and (P2). Furthermore, assume that the graph is composed of a bipartite subgraph on $n - 2$ vertices and two additional vertices. The following lemma shows how to find two short vertex disjoint odd cycles in G. Later in the paper we will apply the lemma

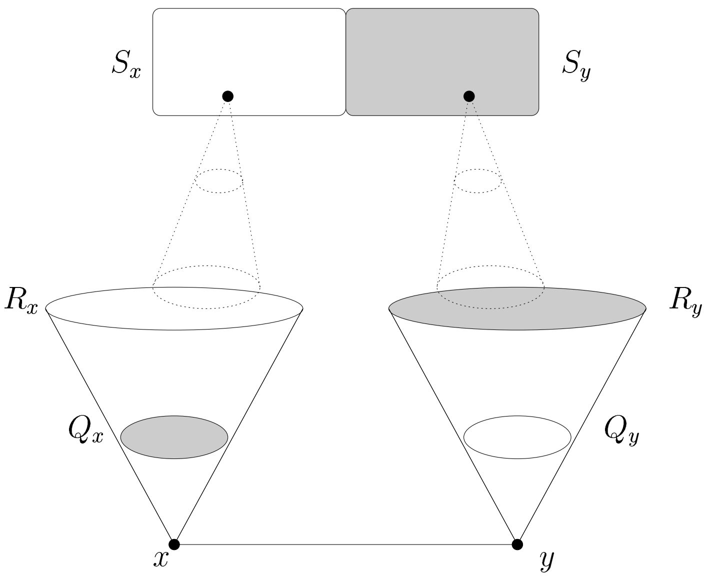

Fig. 1. How the color classes in Lemma 6 are found.

in order to exploit the fact that a graph with two such cycles remains non-bipartite even if an edge of a cycle is removed.

Lemma 7 *Let $G = (V, E)$ be a graph with two vertices $v_1, v_2 \in V$ such that $G' := G[V \setminus \{v_1, v_2\}] \in \mathcal{B}_0(n)$. Consider an arbitrary 2-coloring of G' with color classes S_1, S_2 and assume that for $j \in \{1, 2\}$ there are vertices $w_j^{(1)} \in (S_1 \cap \Gamma(v_j))$ and $w_j^{(2)} \in (S_2 \cap \Gamma(v_j))$ such that $w_1^{(1)}, w_1^{(2)}, w_2^{(1)}, w_2^{(2)}$ are pairwise different.*

If G' satisfies the properties (P1) and (P2) and n is sufficiently large we can find two vertex disjoint cycles C_7 in G.

PROOF. We look for vertex disjoint paths P_j in G' of length five that connect $w_j^{(1)}$ and $w_j^{(2)}$ for $j \in \{1, 2\}$: By (P1) we can find sets $Q_j^{(1)} \subseteq \Gamma(w_j^{(1)}) \cap S_2$ and $Q_j^{(2)} \subseteq \Gamma(w_j^{(2)}) \cap S_1$ of size at least $\log n$. Since $\Gamma(Q_j^{(1)})$ and $\Gamma(Q_j^{(2)})$ are almost as large as one partition of V they are obviously connected by an edge and a path can be found.

Assume that the cycle at v_1 is fixed first. By (P1) the degrees of $w_j^{(1)}$ and $w_j^{(2)}$ are large and, hence, we are able to choose the set $Q_2^{(1)}$ and $Q_2^{(2)}$ in such a way that the resulting cycle at v_2 is vertex disjoint from the cycle at v_1.

We also need a few results on the growth of $|\mathcal{B}_0(n)|$ and $|\mathcal{B}_1(n)|$. Later we will use them to show that the size of the 'bad sets' grows asymptotically slower than $|\mathcal{B}_1(n)|$ and, thus, these sets contain only a negligible number of graphs.

Lemma 8 *For $i \in \{0,1\}$ and all sufficiently large n*

$$\log \frac{|\mathcal{B}_i(n-1)|}{|\mathcal{B}_i(n)|} \leq -\frac{n-i-1}{2}.$$

PROOF. We construct pairwise different graphs in $\mathcal{B}_i(n)$ as follows. First we choose a graph $G \in \mathcal{B}_i(n-1)$ for the first $n-1$ vertices. By definition, this graph G contains a stable set of size at least $\frac{n-i-1}{2}$. The n-th vertex can be connected to this stable set in at least $2^{\frac{n-i-1}{2}}$ many ways. This shows that

$$|\mathcal{B}_i(n)| \geq 2^{\frac{n-i-1}{2}} \cdot |\mathcal{B}_i(n-1)|,$$

which is equivalent to the claimed inequality.

In [PS95] a precise estimate of $|\mathcal{B}_0(n)|$ has been given. We will use it to obtain a rough bound on the relation between $|\mathcal{B}_0(n-1)|$ and $|\mathcal{B}_1(n)|$ and state it without proof.

Theorem 9 *[PS95]*

$$|\mathcal{B}_0(n)| = \Theta(2^{\frac{1}{4}n^2 - \frac{1}{2}\log n}).$$

Of course, Theorem 9 implies the case $i = 0$ of Lemma 8, but we don't need a better estimate there. We will use Theorem 9 only to obtain a short proof for the following rather obvious technical lemma.

Lemma 10 *There is a constant $\rho > 0$ such that for all sufficiently large n*

$$\log \frac{|\mathcal{B}_0(n-1)|}{|\mathcal{B}_1(n)|} \leq \rho.$$

PROOF. The following procedure yields pairwise different graphs in $|\mathcal{B}_1(n)|$: We construct a cycle $C = (v_1, \ldots, v_5)$ on the first five vertices and choose a graph $G' \in \mathcal{B}_0(n-5)$ for the remaining $n-5$ vertices. Let S_1, S_2 denote an arbitrary 2-coloring of G'. We connect v_1 and v_3 to S_1 and v_2 and v_4 to S_2. There are $2^{2|S_1|+2|S_2|} = 2^{2(n-5)}$ possibilities for this. Note that the resulting graph contains no triangles because the neighbors in G' of adjacent vertices on the cycle are disjoint. Hence, we conclude that

$$|\mathcal{B}_1(n)| \geq |\mathcal{B}_0(n-5)| \cdot 2^{2(n-5)}.$$

Using Theorem 9 one easily checks that

$$\frac{|\mathcal{B}_0(n-1)|}{|\mathcal{B}_0(n-5)|} = \Theta\left(\frac{2^{\frac{1}{4}n^2-\frac{1}{2}n+\frac{1}{4}-\frac{1}{2}\log(n-1)}}{2^{\frac{1}{4}n^2-\frac{5}{2}n+\frac{25}{4}-\frac{1}{2}\log(n-5)}}\right) = \Theta(2^{2n}).$$

Then the lemma follows immediately.

Remark 11 *Note that $|\mathcal{B}_0(n-1)|/|\mathcal{B}_1(n)|$ is actually much smaller than the bound in Lemma 10. Comparing Lemma 10 to Lemma 8 one would expect an exponentially small expression. But since a rough estimate suffices for our proofs we only state this almost trivial bound.*

For the proof of Theorem 2 we will employ a similar strategy as in [EKR76]. We partition $\mathcal{F}_1(n)$ into $\mathcal{B}_1(n)$ and several 'bad' sets of graphs with 'unlikely' properties based on (P1) up to (P3). Then we show that the cardinality of those bad sets is negligible.

$\mathcal{X}(n)$ The set of all graphs in $\mathcal{F}_1(n) \setminus \mathcal{B}_1(n)$ which contain a vertex v such that $|\Gamma(v)| \leq 3\log n$,

$\mathcal{Y}(n)$ The set of all graphs in $\mathcal{F}_1(n) \setminus [\mathcal{B}_1(n) \cup \mathcal{X}(n)]$ which contain a set Q of size $\log n$ such that $|\Gamma(Q)| \leq (\frac{1}{2} - \frac{1}{\log\log n})n$,

$\mathcal{Z}(n)$ The set of all graphs G in $\mathcal{F}_1(n) \setminus [\mathcal{B}_1(n) \cup \mathcal{X}(n) \cup \mathcal{Y}(n)]$ which contain an edge $\{x, y\}$ and sets $Q_x \subseteq \Gamma(x)$ and $Q_y \subseteq \Gamma(y)$ of size $|Q_x| = |Q_y| = \log n$ such that $G - \{x, y\} \in \mathcal{F}_1(n-2)$ and $|\Gamma(Q_x) \cap \Gamma(Q_y)| > \frac{1}{100}n$,

First we have to show that $\mathcal{B}_1(n)$ and the 'bad sets' cover $\mathcal{F}_1(n)$.

Lemma 12 *It holds that*

$$\mathcal{F}_1(n) \subseteq \mathcal{B}_1(n) \cup \mathcal{X}(n) \cup \mathcal{Y}(n) \cup \mathcal{Z}(n)$$

for all sufficiently large n.

PROOF. Consider a graph $G = (V, E)$ in $\mathcal{F}_1(n)\setminus[\mathcal{B}_1(n)\cup\mathcal{X}(n)\cup\mathcal{Y}(n)\cup\mathcal{Z}(n)]$. Then the properties (P1) and (P2) hold for G by the definitions of $\mathcal{X}(n)$ and $\mathcal{Y}(n)$, and are still satisfied if one or two vertices are deleted from G.

Assume that there is an edge $\{x, y\}$ such that $G' := G - \{x, y\} \notin \mathcal{F}_1(n-2)$. Since $G' \in \mathcal{F}_0(n-2) = \mathcal{F}_1(n-2) \cup \mathcal{B}_0(n-2)$ we may conclude that $G' \in \mathcal{B}_0(n-2)$.

Consider an arbitrary 2-coloring of G'. Note that $\Gamma(x) \cap \Gamma(y) = \emptyset$ because G is triangle-free. Furthermore, x and y must have neighbors in both color classes

S_1 and S_2 since they are part of odd cycles in G. Otherwise we could deduce that $G \in \mathcal{B}_0(n) \cup \mathcal{B}_1(n)$ which contradicts our choice of G.

Assume that there is one color class, say S_1, and a vertex w such that $\Gamma(x) \cap \Gamma(y) \cap S_1 = \{w\}$. Then the vertices x, y, w would form a triangle and we get a contradiction. Thus, x and y have at least to disjoint neighbors in both color classes and we can find two vertex disjoint odd cycles C_1 and C_2 with $|C_1|, |C_2| \leq 7$ using Lemma 7. It follows by the definition of $\mathcal{Z}(n)$ that (P3) holds for all edges of C_1 and C_2. Lemma 4 then shows that C_1 and C_2 cannot be odd and we obtain a contradiction. Therefore, we may conclude that $G - \{x, y\} \in \mathcal{F}_1(n-2)$ for all edges $\{x, y\}$.

Hence, by the definition of $\mathcal{Z}(n)$ it holds for all $\{x, y\} \in E$ that $|\Gamma^2(x) \cap \Gamma^2(y)| \leq \frac{1}{100}n$ and Lemma 6 proves that G is bipartite, which once again yields a contradiction.

The following lemmas help us to estimate the cardinality of the bad sets.

Lemma 13 *For all sufficiently large n*

$$\log \frac{|\mathcal{X}(n)|}{|\mathcal{F}_1(n-1)|} \leq 4(\log n)^2.$$

PROOF. Consider a graph $G = (V, E) \in \mathcal{X}(n)$ and a vertex $x \in V$. By definition of $\mathcal{X}(n)$ we know that $G[V \setminus \{x\}] \notin \mathcal{B}_0(n-1)$. Hence, all graphs in $\mathcal{X}(n)$ can be constructed as follows. First choose the vertex v and a graph $G \in \mathcal{F}_1(n-1)$ on $V \setminus \{v\}$ (in at most $n \cdot |\mathcal{F}_1(n-1)|$ ways). Then choose the set $\Gamma(v)$. As there are at most

$$\sum_{j=0}^{3\log n} \binom{n-1}{j} \leq n^{3\log n}$$

ways to do this, it follows that

$$|\mathcal{X}(n)| \leq |\mathcal{F}_1(n-1)| \cdot n \cdot n^{3\log n}.$$

The lemma is an immediate consequence of this.

Lemma 14 *For all sufficiently large n*

$$\log \frac{|\mathcal{Y}(n)|}{|\mathcal{F}_0(n-\log n)|} \leq \left(\frac{1}{2} - \frac{1}{2\log\log n}\right) n \log n.$$

PROOF. Construct all graphs in $\mathcal{Y}(n)$ as follows. First choose the set Q and a triangle-free graph G on $V \setminus Q$. This can be done in at most $\binom{n}{\log n} \cdot |\mathcal{F}_0(n - \log n)|$ ways. Then, we have less than 2^n possibilities to fix the set $R = \Gamma(Q)$. Additionally, there are at most $2^{|Q|\cdot|R|} \leq 2^{\log n \cdot (\frac{1}{2} - \frac{1}{\log\log n})n}$ possible choices for the edges between Q and R and less than $2^{(\log n)^2}$ possibilities for the edges inside Q. All in all we get

$$\begin{aligned}\log \frac{|\mathcal{Y}(n)|}{|\mathcal{F}_0(n - \log n)|} &\leq 2(\log n)^2 + n + \left(\frac{1}{2} - \frac{1}{\log\log n}\right) n \log n \\ &\leq \left(\frac{1}{2} - \frac{1}{2\log\log n}\right) n \log n\end{aligned}$$

for n sufficiently large.

Lemma 15 *For all sufficiently large n*

$$\log \frac{|\mathcal{Z}(n)|}{|\mathcal{F}_1(n-2)|} \leq \left(1 - \frac{1}{2000}\right) n.$$

PROOF. Construct all graphs in $\mathcal{Z}(n)$ as follows. First choose two vertices x and y, a triangle-free graph $G' \in \mathcal{F}_1(n-2)$ on $V \setminus \{x, y\}$, and appropriate sets Q_x, Q_y in less than $n^2 \cdot |\mathcal{F}_i(n-2)| \cdot n^{2\log n}$ ways. Let for conciseness of notation $R_x = \Gamma(Q_x)$ and $R_y = \Gamma(Q_y)$ and observe that R_x and R_y are determined by the choice of Q_x and Q_y. Finally, connect x and y to $V \setminus \{x, y\}$. As no vertex in R_x (R_y) may be connected to x (y) and no vertex in $V \setminus (R_x \cup R_y)$ may be connected to both x and y there are at most

$$\begin{aligned}&2^{|R_x \setminus R_y| + |R_y \setminus R_x|} \cdot 3^{n - |R_x \cup R_y|} \\ &\leq 2^{|R_x| + |R_y| - 2|R_x \cap R_y| + \frac{7}{4}(n - |R_x| - |R_y| + |R_x \cap R_y|)} \\ &\leq 2^{\frac{7}{4}n - \frac{3}{4}(|R_x| + |R_y|) - \frac{1}{4}|R_x \cap R_y|} \leq 2^{n - \frac{1}{1000}n}\end{aligned}$$

ways to do this. Recall that by definition of $\mathcal{Y}(n)$ we have $|R_x|, |R_y| \geq (\frac{1}{2} - \frac{1}{\log\log n})n$ and that by assumption $|R_x \cap R_y| \geq \frac{1}{100}n$. Furthermore, one immediately checks that $\log 3 \leq \frac{7}{4}$. Putting everything together we obtain

$$\log \frac{|\mathcal{Z}(n)|}{|\mathcal{F}_i(n-2)|} \leq 2\log n + 2(\log n)^2 + n - \frac{1}{1000}n \leq \left(1 - \frac{1}{2000}\right) n$$

for n sufficiently large.

Now that we have estimated the cardinality of all bad sets, we are in a position to prove Theorem 2.

Proof of Theorem 2 Set $\gamma := 2^{\frac{1}{3000}}$. We will show that there is a constant $c \geq 1$ such that

$$|\mathcal{F}_1(n)| \leq (1 + c\gamma^{-n})|\mathcal{B}_1(n)| \tag{1}$$

holds for all $n \in \mathbb{N}$. Choose n_0 large enough so that all lemmas above and all asymptotic estimates below hold for $n \geq n_0$. Subsequently, choose $c \geq 1$ such that (1) is satisfied for all $n \leq n_0$.

We conclude the proof by induction on n. So assume (1) holds for all graphs on n' vertices with $n' < n$. By Lemma 12 we deduce that

$$|\mathcal{F}_1(n)| \leq |\mathcal{B}_1(n)| + |\mathcal{X}(n)| + |\mathcal{Y}(n)| + |\mathcal{Z}(n)|.$$

Hence, it suffices to show that the ratio of $|\mathcal{X}(n)|$, $|\mathcal{Y}(n)|$ and $|\mathcal{Z}(n)|$ to $|\mathcal{B}_1(n)|$ is at most $\frac{c}{3}\gamma^{-n}$.

By Lemma 13, Lemma 8 and the induction hypothesis we conclude that

$$\begin{aligned}
\frac{|\mathcal{X}(n)|}{|\mathcal{B}_1(n)|} &\leq \frac{|\mathcal{X}(n)|}{|\mathcal{F}_1(n-1)|} \cdot \frac{|\mathcal{F}_1(n-1)|}{|\mathcal{B}_1(n-1)|} \cdot \frac{|\mathcal{B}_1(n-1)|}{|\mathcal{B}_1(n)|} \\
&\leq 2^{4(\log n)^2} \cdot \underbrace{(1 + c\gamma^{-n+1})}_{\leq 2c} \cdot 2^{-\frac{1}{2}(n-2)} \leq \frac{c}{3}\gamma^{-n}.
\end{aligned}$$

Similarly, using Lemma 14, Lemma 8, Lemma 10 and Theorem 1 we obtain

$$\begin{aligned}
\frac{|\mathcal{Y}(n)|}{|\mathcal{B}_1(n)|} &\leq \frac{|\mathcal{Y}(n)|}{|\mathcal{F}_0(n-\log n)|} \cdot \frac{|\mathcal{F}_0(n-\log n)|}{|\mathcal{B}_0(n-\log n)|} \cdot \\
&\quad \frac{|\mathcal{B}_0(n-\log n)|}{|\mathcal{B}_1(n-\log n+1)|} \cdot \prod_{j=0}^{\log n-2} \frac{|\mathcal{B}_1(n-j-1)|}{|\mathcal{B}_1(n-j)|} \\
&\leq 2^{(\frac{1}{2}-\frac{1}{2\log\log n})n\log n} \cdot 2c \cdot \rho \cdot 2^{\sum_{j=0}^{\log n-2}[-\frac{1}{2}(n-j-2)]} \\
&\leq 2^{(\frac{1}{2}-\frac{1}{2\log\log n})n\log n} \cdot 2c \cdot \rho \cdot 2^{-\frac{1}{2}(\log n-1)(n-\log n-4)} \\
&\leq \frac{c}{3} \cdot 2^{-\frac{1}{2\log\log n}\cdot n\log n+n} \leq \frac{c}{3}\gamma^{-n}.
\end{aligned}$$

Finally, we use Lemma 15, Lemma 8 and the induction hypothesis to show that

$$\begin{aligned}
\frac{|\mathcal{Z}(n)|}{|\mathcal{B}_1(n)|} &\leq \frac{|\mathcal{Z}(n)|}{|\mathcal{F}_1(n-2)|} \cdot \frac{|\mathcal{F}_1(n-2)|}{|\mathcal{B}_1(n-2)|} \cdot \frac{|\mathcal{B}_1(n-2)|}{|\mathcal{B}_1(n-1)|} \cdot \frac{|\mathcal{B}_1(n-1)|}{|\mathcal{B}_1(n)|} \\
&\leq 2^{(1-\frac{1}{2000})n} \cdot (1 + c\gamma^{-n+2}) \cdot 2^{-\frac{1}{2}(n-3)-\frac{1}{2}(n-2)} \leq \frac{c}{3}\gamma^{-n}.
\end{aligned}$$

This completes the proof.

4 Conclusion

Using techniques which are known as the Kleitman-Rothschild method we have investigated the relationship between triangle-free and quasibipartite graphs.

As shown in [EKR76] almost all triangle-free graphs are bipartite. We extended this result by considering the triangle-free graphs which are not bipartite. It turned out that almost all of these graphs can be made bipartite by removing just one vertex.

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