Boolean NP-Partitions and Projective Closure

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Abstract. When studying complexity classes of partitions we often face the situation that different partition classes have the same component classes. The projective closures are the largest classes among these with respect to set inclusion. In this paper we investigate projective closures of classes of boolean NP-partitions, i.e., partitions with components that have complexity upper-bounds in the boolean hierarchy over NP. We prove that the projective closures of these classes are represented by finite labeled posets. We give algorithms for calculating these posets and related problems. As a consequence we obtain representations of the set classes $\mathrm{NP}(m) \cap \mathrm{coNP}(m)$ by means of finite labeled posets.

Keywords. NP, partitions, boolean hierarchy, projective closure.

1 Introduction

In many areas of life there are more than two alternatives to consider, e.g., a teacher has not only to worry about whether a student should pass a test but also which mark is appropriate, or when playing chess, a current configuration you are facing with on the board can evolve into a win, loss, or tie situation depending on the next moves. In such cases evaluating the situation is rather a classification problem than simply a decision problem, or, speaking set-theoretically, rather a partition (into more than two parts) than simply a set (partition into two parts). In complexity theory, classification problems have long been investigated by encodings into sets, but more recently partitions themselves have become a matter of research [7, 6].

A classification problem A is given by its characteristic function: for any A the characteristic function c_A says for every x, to which component of A this x belongs. A typical point of view is to consider the complexity of the components of A, i.e., what the complexity of deciding whether $c_A(x) = i$ is. However, having these informations for all components of A is certainly not enough to get the right idea of the complexity of the partition itself. For instance, let us consider the canonical example from [7]: the classification problem that is induced by the entailment relation $\models$ for formulas of propositional logic. For formulas H and H', the relation $\models$ is defined as

$$H \models H' \iff_{\mathrm{def}} \text{each satisfying assignment for } H \text{ also satisfies } H'.$$

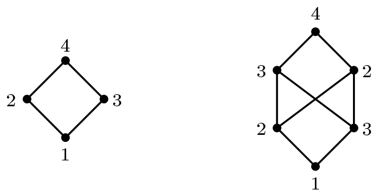

Fig. 1. Labeled posets being candidates for the complexity of ENTAILMENT.

Deciding whether $H \not\models H'$ (H does not entail H') holds true is clearly NP-complete (note that H is satisfiable if and only if $H \not\models H \wedge \neg H$). We define the characteristic function of a corresponding classification problem ENTAILMENT, for any pair (H, H') as

$$c_{\text{ENTAILMENT}}(H, H') =_{\text{def}} \begin{cases} 1 & \text{if } H \not\models H' \text{ and } H' \not\models H, \\ 2 & \text{if } H \not\models H' \text{ and } H' \models H, \\ 3 & \text{if } H \models H' \text{ and } H' \not\models H, \\ 4 & \text{if } H \models H' \text{ and } H' \models H. \end{cases}$$

Regarding the component-wise complexities it is fairly easy to see that deciding whether a pair (H, H') belongs to

- the first component is NP-complete,
- the second component is NP(2)-complete where NP(2) is the class of all set differences of NP sets with NP sets,
- the third component is, by symmetry, also NP(2)-complete,
- the fourth component is coNP-complete.

Thus, on the one hand, ENTAILMENT belongs to the class of classification problems that can be described by $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$. But, on the other hand, is ENTAILMENT also complete[1] for that class, as might be indicated by the completeness of all of its components?

This question can be translated into the framework proposed in [7, 6] where partition classes have been studied that are generated by finite labeled posets. Roughly, the posets represent NP sets and the relations among them with respect to set inclusion, and the labels assign sets to the components (precise definitions can be found in Section 3). For instance, ENTAILMENT is known to be complete for the partition class generated by the left lattice in Fig. 1. The membership can be easily seen (without knowing exact definitions) as follows: Choose A be the set of all pairs (H, H') such that H does not entail H', choose B be the set of all pair (H, H') such that H' does not entail H, and assign $A \cap B$ to the node with label one, A to the node with label two, B to the node with label three, and the basic set to the node with label 4. The set portions remaining with the

[1] With respect to many-one reductions on partitions: for partitions A and B (into k parts), $A \leq_m^p B$ if and only if there is function f computable in polynomial time such that for all x, $c_A(x) = c_B(x)$.

nodes after removing all elements that belong to sets assigned to lower nodes in the poset yield Entailment according to its characteristic function.

The problem above now has changed to determine whether the partition class generated by the left lattice is equal to $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$. We will prove that the latter is not the case unless NP = coNP. In order to do this, we address the more general question of how to describe classes of partitions with components from classes in the boolean hierarchy over NP. Such partitions are called *boolean* NP*-partitions*. The main result of this paper is an algorithm for calculating a representation of these classes by finite labeled posets. In the particular case of $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$ we obtain the right poset in Fig. 1. Since the right poset cannot be mapped to the left lattice such that orders and labels are preserved, it follows from results in [7] that both generated classes are equal if and only if NP = coNP, and consequently, Entailment is not complete for $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$ unless NP = coNP. Classes of boolean NP-partitions can be regarded as projective closures since they are, with respect to set inclusion, the largest classes among all classes having the same component classes (projection classes). And it is rather surprising result that the projective closure for classes of boolean NP-partitions can be represented by a labeled poset. Our algorithm thus also provides a criterion for deciding whether a given labeled poset (with a least element) represents a projectively closed class.

2 Preliminaries

We adopt the reader to be familiar with basic complexity theory (see, e.g., [4]). Throughout the paper we use the finite alphabet $\Sigma = \{0, 1\}$. Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of subsets of Σ^*. We define $\mathrm{co}\mathcal{K} =_{\mathrm{def}} \{\, \overline{A} \mid A \in \mathcal{K} \,\}$, $\mathcal{K} \vee \mathcal{K}' =_{\mathrm{def}} \{\, A \cup B \mid A \in \mathcal{K}, B \in \mathcal{K}' \,\}$, and $\mathcal{K} \oplus \mathcal{K}' =_{\mathrm{def}} \{\, A \triangle B \mid A \in \mathcal{K}, B \in \mathcal{K}' \,\}$. The classes $\mathrm{NP}(i)$ and $\mathrm{coNP}(i)$ defined by $\mathrm{NP}(0) =_{\mathrm{def}} \{\emptyset\}$ and $\mathrm{NP}(i+1) = \mathrm{NP}(i) \oplus \mathrm{NP}$ build the boolean hierarchy over NP that has many equivalent definitions (see [8, 2, 5, 1]).[2] BC(NP) is the boolean closure of NP, i.e., the smallest class which contains NP and which is closed under intersection, union, and complements. It is well known that $\mathrm{NP}(k) = \mathrm{coNP}(k)$ implies $\mathrm{BC}(\mathrm{NP}) = \mathrm{NP}(k)$.

We need some notions from lattice theory and order theory (see, e.g., [3]). A pair $(G, \leq)$ is a poset if $\leq$ is a partial order on the set G. Usually, we talk about the poset G. A finite poset G is a lattice if for all $x, y \in G$ there exist (a) exactly one maximal element $z \in G$ such that $z \leq x$ and $z \leq y$, and (b) exactly one minimal element $z \in G$ such that $z \geq x$ and $z \geq y$. A poset G with $x \leq y$ or $y \leq x$ for all x, y is a chain. We will only consider finite posets. Any pair (G, f) of a poset G and a function $f : G \to \{1, 2, \ldots, k\}$ is a k-poset. A k-poset which is a lattice (chain) is a k-lattice (a k-chain).

[2] Usually, a level 0 is not considered in the way we do. The zero-level there is P. However for our purposes it is more helpful to regard P not as an element of the boolean hierarchy (unless P = NP).

3 Representing Partitions and Partition Classes

In this section we introduce the notions on partition classes we need throughout the paper. First, let us make some conventions about partitions. A k-tuple $A = (A_1, \ldots, A_k)$ with $A_i \subseteq \Sigma^*$ for each $i \in \{1, \ldots, k\}$ is said to be a *k-partition of Σ^** if and only if $A_1 \cup A_2 \cup \cdots \cup A_k = \Sigma^*$ and $A_i \cap A_j = \emptyset$ for all i, j with $i \neq j$. The set A_i is said to be the *i-th component of A*. For two k-partitions A and B to be equal it is sufficient that $A_i \subseteq B_i$ for all $i \in \{1, \ldots, k\}$. Let $c_A : \Sigma^* \to \{1, \ldots, k\}$ be the characteristic function of a k-partition $A = (A_1, \ldots, A_k)$, that is, $c_A(x) = i$ if and only if $x \in A_i$.

For set classes $\mathcal{K}_1, \ldots, \mathcal{K}_k$ define

$$(\mathcal{K}_1, \ldots, \mathcal{K}_k) =_{\text{def}} \left\{ A \mid A \text{ is } k\text{-partition of } \Sigma^* \text{ and } A_i \in \mathcal{K}_i \text{ for } i \in \{1, \ldots, k\} \right\}.$$

We say that $(\mathcal{K}_1, \ldots, \mathcal{K}_k)$ is a *bound representation* of a partition class. Note that a partition class can have infinitely many bound representations. For instance, $(\mathrm{P}, \mathcal{K}) = (\mathcal{K}, \mathrm{P}) = (\mathrm{P}, \mathrm{P})$ for all $\mathrm{P} \subseteq \mathcal{K}$.

Definition 1. *Let $\mathcal{C}$ be any class of k-partitions.*

1. *For $i \in \{1, \ldots, k\}$, the set $\{A_i \mid A \in \mathcal{C}\}$ is called the* i-th projection class $\mathcal{C}_i$.
2. *The partition class $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ is called the* projective closure of $\mathcal{C}$.
3. *$\mathcal{C}$ is said to be* projectively closed *if and only if $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$.*

These terms are justified since the operator Π defined as $\Pi(\mathcal{C}) = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ certainly satisfies $\mathcal{C} \subseteq \Pi(\mathcal{C})$, $\mathcal{C} \subseteq \mathcal{C}' \Rightarrow \Pi(\mathcal{C}) \subseteq \Pi(\mathcal{C}')$, and $\Pi(\Pi(\mathcal{C})) = \Pi(\mathcal{C})$, thus all conditions of a closure operator. Note that only projectively closed classes of k-partitions can have a bound representation.

In many cases it suffices to specify $k-1$ components of a class of k-partitions. This leads to *free representations* of partition classes. For classes $\mathcal{K}_1, \ldots, \mathcal{K}_{k-1}$ of subsets of Σ^* define

$$(\mathcal{K}_1, \ldots, \mathcal{K}_{k-1}, \cdot) =_{\text{def}} (\mathcal{K}_1, \ldots, \mathcal{K}_{k-1}, \mathcal{P}(\Sigma^*)).$$

Note that only for the sake of convenience we define free representations with respect to the last component. Each freely represented partition class can be boundly represented since clearly, $(\mathcal{K}_1, \ldots, \mathcal{K}_{k-1}, \cdot)_k \subseteq \mathrm{co} \bigvee_{i=1}^{k-1} \mathcal{K}_i$.

Following [7] we identify a set A with the 2-partition $(A, \overline{A})$ and a class of sets $\mathcal{K}$ is, boundly represented, identified with the class $(\mathcal{K}, \mathrm{co}\mathcal{K})$ of 2-partitions or is, freely represented, identified with the class $(\mathcal{K}, \cdot) = (\cdot, \mathrm{co}\mathcal{K})$ of 2-partitions. For instance, $\mathrm{NP} = (\mathrm{NP}, \mathrm{coNP}) = (\mathrm{NP}, \cdot)$. Classes of 2-partitions are always projectively closed since for every set A its complement $\overline{A}$ is uniquely determined.

We are particularly interested in partition classes defined by posets.

Definition 2. [7, 6] *Let G be a poset and let $f : G \to \{1, 2, \ldots, k\}$ be a function.*

1. *A mapping $S : G \to \mathrm{NP}$ is said to be an* NP-homomorphism on G *if and only if $\bigcup_{a \in G} S(a) = \Sigma^*$ and $S(a) \cap S(b) = \bigcup_{c \leq a, c \leq b} S(c)$ for all $a, b \in G$.*

2. *For an* NP*-homomorphism S on G and $a \in G$ let* $T_S(a) =_{\mathrm{def}} S(a) \setminus \bigcup_{b<a} S(b)$.
3. *For an* NP*-homomorphism S on G,* the k-partition defined by (G,f) and S *is given by* $(G,f,S) =_{\mathrm{def}} \left(\bigcup_{f(a)=1} T_S(a)\ ,\ldots,\ \bigcup_{f(a)=k} T_S(a)\right)$.
4. $\mathrm{NP}(G,f) =_{\mathrm{def}} \left\{ (G,f,S) \,\middle|\, S \textit{ is an } \mathrm{NP}\textit{-homomorphism on } G \right\}$ *is the* class of k-partitions defined by the k-poset (G,f).

Proposition 3. [6] *Let $k \geq 2$. For all k-posets (G,f) with $f : G \to \{1,2,\ldots,k\}$ surjective, it holds* $(\mathrm{NP},\ldots,\mathrm{NP}) \subseteq \mathrm{NP}(G,f) \subseteq (\mathrm{BC}(\mathrm{NP}),\ldots,\mathrm{BC}(\mathrm{NP}))$.

In order to decide whether $\mathrm{NP}(G,f) \subseteq \mathrm{NP}(G',f')$ for k-posets (G,f) and (G',f'), we introduce a relation $\leq$ on finite labeled posets. Let (G,f) and (G',f') be k-posets with $k \geq 2$. Define $(G,f) \leq (G',f')$ if and only if there is a monotonic mapping $\varphi : G \to G'$ such that for every $x \in G$, $f(x) = f'(\varphi(x))$. Equivalence of labeled posets is defined by $(G,f) \equiv (G',f')$ if and only if $(G,f) \leq (G',f')$ and $(G',f') \leq (G,f)$. Note that among all equivalent k-posets there is always (up to isomorphism) a unique poset having the minimal number of elements [7]. For the relation $\leq$ the following Embedding Lemma holds.

Lemma 4. (Embedding Lemma.) [6] *Let (G,f) and (G',f') be two k-posets. If $(G,f) \leq (G',f')$, then* $\mathrm{NP}(G,f) \subseteq \mathrm{NP}(G',f')$.

Whether it is possible to redirect the Embedding Lemma (under complexity-theoretic assumptions) is not completely cleared (see [7, 6]). However, with respect to relativizable inclusions, it is possible [6]: $\mathrm{NP}^A(G,f) \subseteq \mathrm{NP}^A(G',f')$ for all oracles A if and only if $(G,f) \leq (G',f')$.

4 Mind-Change Complexity of Projection Classes

In this section we make an attempt to determine the component-wise descriptions of the projective closure of classes of NP-partitions generated by labeled posets. From Proposition 3 we know that all components of partition classes generated by k-posets belong to the boolean closure BC(NP) of NP. Theorem 6 below will tell us what exactly the complexity of the components is.

For $m \in \mathbb{N}$ let $\mathfrak{D}_m$ be the 2-poset represented in Fig. 2.

Lemma 5. $\mathrm{NP}(\mathfrak{D}_m) = (\mathrm{NP} \cap \mathrm{coNP}) \oplus \mathrm{NP}(m)$ *for all $m \in \mathbb{N}$.*

For a k-poset (G,f) and $i \in \{1,2,\ldots,k\}$ let $\mu_i(G,f)$ denote the maximum number of alternations between f-labels i and f-labels different to i in a chain of G whose minimum has the label i, and let $\mu_{\bar{i}}(G,f)$ denote the maximum number of alternations between f-labels i and f-labels different to i in a chain of G whose minimum has a label different to i. If it is clear from the context which k-poset is considered then we may write μ_i (resp., $\mu_{\bar{i}}$) instead of $\mu_i(G,f)$ (resp., $\mu_{\bar{i}}(G,f)$).

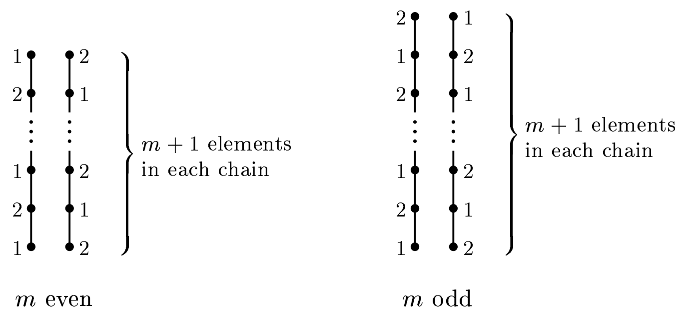

Fig. 2. The 2-poset $\mathfrak{D}_m$ for $m \in \mathbb{N}$

Theorem 6. (Mind-Change Theorem.) *Let (G, f) be a k-poset with at least two different labels, and let $i \in \{1, 2, \ldots, k\}$. Then the following holds:*

$$\mathrm{NP}(G,f)_i = \begin{cases} \mathrm{NP}(\mu_i) & \text{if } \mu_i > \mu_{\bar{i}} \text{ and } \mu_i \text{ is odd,} \\ \mathrm{coNP}(\mu_i) & \text{if } \mu_i > \mu_{\bar{i}} \text{ and } \mu_i \text{ is even,} \\ \mathrm{NP}(\mu_{\bar{i}}) & \text{if } \mu_{\bar{i}} > \mu_i \text{ and } \mu_{\bar{i}} \text{ is even,} \\ \mathrm{coNP}(\mu_{\bar{i}}) & \text{if } \mu_{\bar{i}} > \mu_i \text{ and } \mu_{\bar{i}} \text{ is odd,} \\ (\mathrm{NP} \cap \mathrm{coNP}) \oplus \mathrm{NP}(\mu_i) & \text{if } \mu_i = \mu_{\bar{i}}. \end{cases}$$

As an example consider the 4-poset (G, f) represented as described in the right part of Fig. 1. Then we easily calculate:

- $\mu_1(G, f) = 1$ and $\mu_{\bar{1}}(G, f) = 0$. Thus, $\mathrm{NP}(G, f)_1 = \mathrm{NP}(1) = \mathrm{NP}$.
- $\mu_2(G, f) = 1$ and $\mu_{\bar{2}}(G, f) = 2$. Thus, $\mathrm{NP}(G, f)_2 = \mathrm{NP}(2)$.
- $\mu_3(G, f) = 1$ and $\mu_{\bar{3}}(G, f) = 2$. Thus, $\mathrm{NP}(G, f)_3 = \mathrm{NP}(2)$.
- $\mu_4(G, f) = 0$ and $\mu_{\bar{4}}(G, f) = 1$. Thus, $\mathrm{NP}(G, f)_4 = \mathrm{coNP}(1) = \mathrm{coNP}$.

Hence, $\mathrm{NP}(G, f) \subseteq (\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$. Thus the upper bound is already shown. It remains to show that in fact both classes are equal. This will be done in the next sections.

5 Partition Classes Given in Free Representations

We now turn to the problem of how to determine which k-posets correspond to freely represented classes $(\mathcal{K}_1, \ldots, \mathcal{K}_{k-1}, \cdot)$ with $\mathcal{K}_i$ a class in the boolean hierarchy over NP. Note that in this case the projection class $\mathcal{K}_k$ is contained in (but not necessarily equal to) classes from the boolean hierarchy over NP. Without loss of generality we suppose that none of the $\mathcal{K}_i$'s ($i \leq k-1$) is $\mathrm{coNP}(0)$. This is justified since $\mathrm{coNP}(0) = \{\Sigma^*\}$, thus $\mathcal{K}_i = \{\Sigma^*\}$ implies $\mathcal{K}_j = \{\emptyset\}$ for all $j \neq i$, and so the whole partition class can be considered a freely represented class with the dot in component i. An appropriate permutation of the components

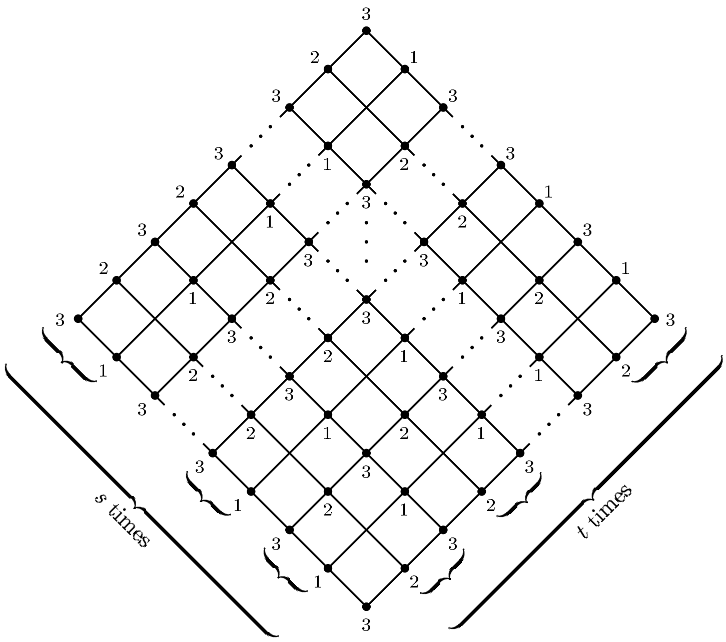

Fig. 3. The 3-poset $\mathfrak{B}_{2s,2t}$ for $s, t \in \mathbb{N}$

yields a partition class as assumed. Admitting trivial components at all becomes extremely useful when considering classes given in bound representation.

For labeled posets we choose a representation as sets of vectors over natural numbers. We consider an n-tuple $r = (r_1, \ldots, r_n)$ of integers that later on will be used to describe partition classes. For r let $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ be an adjoint n-tuple given by

$$m_i = \begin{cases} \frac{r_i}{2} & \text{if } r_i \geq 0 \text{ and } r_i \text{ is even,} \\ \frac{|r_i|-1}{2} & \text{if } r_i \text{ is odd,} \\ \frac{-r_i-2}{2} & \text{if } r_i < 0 \text{ and } r_i \text{ is even.} \end{cases} \tag{1}$$

Define an $(n+1)$-poset $\mathfrak{B}_r = (B_r, f)$ depending on the tuple r as follows:

$$\begin{aligned} B_r =_{\mathrm{def}} & \left\{ x \in \mathbb{N}^n \;\middle|\; 1 \leq x_\nu \leq 2m_\nu + 1 \text{ for all } \nu, \text{ and } \|\{\nu \mid x_\nu \text{ is even}\}\| \leq 1 \right\} \\ & \cup \left\{ x \in \mathbb{N}^n \;\middle|\; r_\nu \geq 0, r_\nu \text{ is odd}, x_\nu = 0, \text{ and } x_\mu = 1 \text{ for } \nu \neq \mu \right\} \\ & \cup \left\{ x \in \mathbb{N}^n \;\middle|\; r_\nu < 0, r_\nu \text{ is even}, x_\nu = 0, \text{ and } x_\mu = 1 \text{ for } \nu \neq \mu \right\} \\ & \cup \left\{ x \in \mathbb{N}^n \;\middle|\; r_\nu < 0, x_\nu = 2m_\nu + 2, \text{ and } x_\mu = 2m_\mu + 1 \text{ for } \nu \neq \mu \right\} \end{aligned}$$

Consider B_r to be partially ordered by the vector-ordering. For each $x \in B_r$ let

$$f(x) =_{\mathrm{def}} \begin{cases} \nu & \text{if } x_\nu \text{ is even and } x_\mu \text{ is odd for } \nu \neq \mu, \\ n+1 & \text{if } x_\mu \text{ is odd for all } \mu. \end{cases}$$

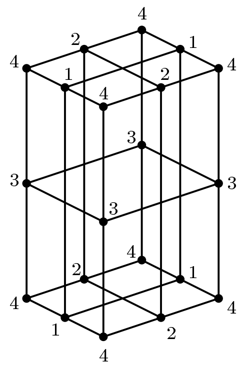

Fig. 4. The 4-poset $\mathfrak{B}_{2,2,2}$

Although the representation seems technically involved $\mathfrak{B}_r$ is easy to handle for several cases. For instance, if $n = 1$ then $\mathfrak{B}_{2s}$ is simply a 2-chain with $2s + 1$ elements alternately labeled with one and two, and the least element has label two. For $n = 2$ the 3-poset $\mathfrak{B}_{2s,2t}$ can be drawn as the grid in Figure 3. For $n = 3$ the 4-posets $\mathfrak{B}_{2s,2t,2u}$ are depicted by 3-dimensional cuboids as done for $\mathfrak{B}_{2,2,2}$ in Figure 4.

The next proposition compares labeled posets with respect to tuples.

Proposition 7. *Let r and r' be n-tuples of integers such that $|r_i| \leq |r'_i|$ for all $1 \leq i \leq n$. Then $\mathfrak{B}_r \leq \mathfrak{B}_{r'}$.*

The following lemma shows the connection between a labeled poset $\mathfrak{B}_r$ and its mind-change complexity.

Lemma 8. *Let $r = (r_1, \ldots, r_n)$ be an n-tuple of integers and let $j \in \{1, \ldots, n\}$.*

1. *If $r_j = 0$ then $\mu_j(\mathfrak{B}_r) = \mu_{\overline{j}}(\mathfrak{B}_r) = 0$.*
2. *If $r_j > 0$ and r_j is even then $\mu_j(\mathfrak{B}_r) < \mu_{\overline{j}}(\mathfrak{B}_r) = r_j$.*
3. *If $r_j > 0$ and r_j is odd then $\mu_{\overline{j}}(\mathfrak{B}_r) < \mu_j(\mathfrak{B}_r) = r_j$.*
4. *If $r_j < 0$ and r_j is even then $\mu_{\overline{j}}(\mathfrak{B}_r) < \mu_j(\mathfrak{B}_r) = -r_j$.*
5. *If $r_j < 0$ and r_j is odd then $\mu_j(\mathfrak{B}_r) < \mu_{\overline{j}}(\mathfrak{B}_r) = -r_j$.*

The lemma is useful, e.g., in proving the following Embedding Theorem for a subclass of labeled posets.

Theorem 9. *Assume the boolean hierarchy over* NP *is infinite. Let r and r' n-tuples of integers. Then $\mathrm{NP}(\mathfrak{B}_r) \subseteq \mathrm{NP}(\mathfrak{B}_{r'})$ if and only if $\mathfrak{B}_r \leq \mathfrak{B}_{r'}$.*

Connections between posets $\mathfrak{B}_r$ and partition classes are given by boolean characteristics.

Definition 10. *Let $\mathcal{C}$ be a class of k-partitions that can be freely represented by $k-1$ classes from the boolean hierarchy over* NP*. A $(k-1)$-tuple β of integers is said to be a* boolean characteristic of $\mathcal{C}$ *if and only if for all $i \in \{1, \dots, k-1\}$,*

$$\mathcal{C}_i = \begin{cases} \mathrm{NP}(\beta_i) & \text{if } \beta_i \geq 0, \\ \mathrm{coNP}(-\beta_i) & \text{if } \beta_i < 0. \end{cases}$$

Note that a freely represented partition class can have either exactly one boolean characteristic (e.g., if the boolean hierarchy over NP is infinite) or infinitely many boolean characteristics.

Theorem 11. *Let $\mathcal{C}$ be any class of k-partitions that can be freely represented by $k-1$ classes from the boolean hierarchy over* NP*. Let β be any boolean characteristic of $\mathcal{C}$. Then $\mathcal{C} = \mathrm{NP}(\mathfrak{B}_\beta)$.*

As a consequence of this theorem we conclude:

Corollary 12. $\mathrm{NP}(\mathfrak{B}_r)$ *is projectively closed for all n-tuples r of integers.*

6 Partition Classes Given in Bound Representations

In the previous section we have solved the problem of characterizing freely represented k-partition classes that are explicitly given by $k-1$ components of the boolean hierarchy over NP. Now we are going to solve the same problem for boundly represented partition classes. The key for this is already given by Theorem 11 because we clearly have the following bridge between freely and boundly represented partition classes.

Proposition 13. *Let $(\mathcal{K}_1, \dots, \mathcal{K}_{k-1}, \cdot)$ be a partition class and let (G, f) be a k-poset such that $\mathrm{NP}(G, f) = (\mathcal{K}_1, \dots, \mathcal{K}_{k-1}, \cdot)$. Then $(\mathcal{K}_1, \dots, \mathcal{K}_{k-1}, \{\emptyset\}) = \mathrm{NP}(G', f')$ with $G' = G \setminus \{a \in G \mid f(a) = k\}$.*

From this proposition we easily conclude:

Theorem 14. *Let $\mathcal{K}_1, \dots, \mathcal{K}_k$ be classes from the boolean hierarchy over* NP*. Let β be any boolean characteristic of the corresponding freely represented class $(\mathcal{K}_1, \dots, \mathcal{K}_k, \cdot)$. Then $(\mathcal{K}_1, \dots, \mathcal{K}_k) = \mathrm{NP}(\mathfrak{B}'_\beta)$ where $\mathfrak{B}'_\beta$ is the k-poset that emerges from the $(k+1)$-poset $\mathfrak{B}_\beta$ by eliminating all elements having label $k+1$.*

As an example, applying Theorem 14 easily leads to the right 4-poset in Fig. 1 as a representation of the class $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$. The steps to verify this are as follows:

- Take the boolean characteristic $\beta = (1, 2, 2, -1)$ of the freely represented class $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP}, \cdot)$ and calculate the corresponding vector $m = (0, 1, 1, 0)$ following Equation (1).

- The 5-poset $\mathfrak{B}_\beta$ is now given by set

$$B_\beta = \{1111, 1121, 1131, 1211, 1231, 1311, 1321, 1331\} \cup \{0111\} \cup \{1332\}$$

equipped with the vector-ordering (note that we have identified vectors and words) and the labels

$$\begin{aligned} f(0111) &= 1, \\ f(1211) = f(1231) &= 2, \\ f(1121) = f(1321) &= 3, \\ f(1332) &= 4, \\ f(1111) = f(1131) = f(1311) = f(1331) &= 5. \end{aligned}$$

- Eliminating all elements with label 5 gives the 4-poset $\mathfrak{B}'_\beta$ on the six elements $0111, 1121, 1211, 1231, 1321, 1332$. However, it is easy to see that this labeled poset is just the right 4-poset in Fig. 1.

Since we can find for each class of boolean NP-partitions, which is projectively closed, a boolean characteristic (for the corresponding freely represented class) we conclude from Theorem 14:

Corollary 15. *Each projectively closed class of boolean* NP*-partitions is generated by a labeled poset.*

7 Algorithmic Issues Related to the Projective Closure

The results of previous three sections (in particular, the Mind-Change Theorem and Theorem 14) can be used to address algorithmic issues concerning projective closures. In this section we discuss their relevance to the three issues that appear naturally in our setting.

> *Issue 1*: Given the boolean characteristic of some NP-partition class, compute a labeled poset representing the projective closure of that class.

> *Issue 2*: Given a labeled poset representing some NP-partition class, compute a labeled poset representing the projective closure of that class.

> *Issue 3*: Given a labeled poset representing some NP-partition class, decide whether the class represented is projectively closed.

Issue 1 can be solved directly using Theorem 14: Given β compute the $(k+1)$-poset $\mathfrak{B}_\beta$ and eliminate all elements with label $k+1$. Note that relative to the input which is a tuple of integers, the computed labeled poset has, in general, exponential size. Using this method we observe an interesting characterization of classes in the boolean hierarchy over NP. We can describe each class $\mathrm{NP}(m) \cap \mathrm{coNP}(m)$ as a class generated by labeled posets. For $m \geq 1$ let $\mathfrak{H}_m$ be the 2-poset presented in Figure 5.

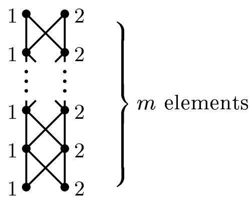

Fig. 5. The 2-poset $\mathfrak{H}_m$ for $m \geq 1$

Corollary 16. $\mathrm{NP}(m) \cap \mathrm{coNP}(m) = \mathrm{NP}(\mathfrak{H}_m)$ *for all* $m \geq 1$.

For Issue 2 we obtain the following algorithm which takes as input a k-poset (G, f) on n elements (represented as a k-colored directed acyclic graph):

1. For all $i \in \{1, \ldots, k\}$ compute $\mu_i(G, f)$ and $\mu_{\bar{i}}(G, f)$ (e.g., by dynamic programming in polynomial time in the number n of elements of G).
2. Compute the boolean characteristic β from the set of mind changes (using the Mind-Change Theorem and the definition of the boolean characteristic).
3. Compute the $(k+1)$-poset $\mathfrak{B}_\beta$ and eliminate all elements with label $k+1$.

However, since the boolean characteristic reflects only those mind-change cases where $\mu_i(G, f) \neq \mu_{\bar{i}}(G, f)$ for all $i \in \{1, \ldots, k\}$, the algorithm cannot be applied to compute the projective closure of all poset-defined NP-partition classes. But it works well at least on all labeled posets with a least element, i.e., on all posets that can represented by a connected graph.

Proposition 17. *Let (G, f) be any k-poset with $k \geq 2$. If G has a least element then for all $i \in \{1, \ldots, k\}$, $\mu_i(G, f) \neq \mu_{\bar{i}}(G, f)$.*

Note again that relative to the size of the given labeled poset, the projective closure of the class represented by the poset can have exponential size. In case of families of labeled posets on n elements and with at least εn labels (for some $0 < \varepsilon \leq 1$), the poset $\mathfrak{B}_\beta$ has up to $n^{\varepsilon n}$ elements. However, in case of a fixed number k of labels, there can be at most $O(n^{k+1})$ elements in $\mathfrak{B}_\beta$. Thus the following theorem is true.

Theorem 18. *Let $k \geq 2$. Restricted to k-posets with a least element, Issue 2 can be solved in time polynomial in the size of the posets.*

The theorem gives as a special case that, if we focus on classes defined by k-lattices, i.e., classes from the boolean hierarchy of k-partitions over NP [7], for a fixed k, projective closures of such classes can be computed in polynomial time.

As to Issue 3, we can use the algorithm for Issue 2 to compute the projective closure, and check whether the resulting labeled poset describes the same partition class, i.e., whether both posets are isomorphic. Besides that it is unknown whether graph isomorphism for colored directed acyclic graphs is in P, we should note that the k-posets we obtain by the algorithm are, in general, not

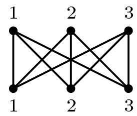

Fig. 6. The 3-poset generating $(\mathrm{NP}(2), \mathrm{NP}(2), \mathrm{NP}(2))$

minimal k-posets opposite to those constructed in Theorem 11. After calculating the k-poset for a given boundly represented classes of k-partitions we thus have to minimize the labeled poset. Remind that there is up to isomorphism a unique minimal equivalent k-poset. E.g., the k-poset in Figure 6 is the minimal k-poset which is equivalent to the k-poset we obtain from the theorem above to represent $(\mathrm{NP}(2), \mathrm{NP}(2), \mathrm{NP}(2))$. Note that $(\mathrm{NP}(2), \mathrm{NP}(2), \mathrm{NP}(2), \cdot) = \mathrm{NP}(\mathfrak{B}_{2,2,2})$ (see Figure 4). In general, it is however an open issue to determine the complexity of the minimization problem for labeled posets:

> *Open Issue*: Given any labeled poset, compute the minimal equivalent labeled poset.

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